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Contents of Volume 12

Number 1, February 1976

David Cass and Karl Shell. Introduction to Hamiltonian Dynamics in Economics 1

José A. Scheinkman. On Optimal Steady States of *n*-Sector Growth Models when Utility is Discounted 11

David Cass and Karl Shell. The Structure and Stability of Competitive Dynamical Systems 31

R. Tyrrell Rockafellar. Saddle Points of Hamiltonian Systems in Convex Lagrange Problems Having a Nonzero Discount Rate 71

Robert E. Gaines. Existence of Solutions to Hamiltonian Dynamical Systems of Optimal Growth 114

Lawrence J. Lau. A Characterization of the Normalized Restricted Profit Function 131

William A. Brock and José A. Scheinkman. Global Asymptotic Stability of Optimal Control Systems with Applications to the Theory of Economic Growth 164

R. Tyrrell Rockafellar. A Growth Property in Concave-Convex Hamiltonian Systems 191

Number 2, April 1976

Rolf R. Mantel. Homothetic Preferences and Community Excess Demand Functions 197

Menahem E. Yaari. A Law of Large Numbers in the Theory of Consumer's Choice under Uncertainty 202

Jack Schechtman. An Income Fluctuation Problem 218

Avinash Dixit. Public Finance in a Keynesian Temporary Equilibrium 242

Donald E. Campbell. Democratic Preference Functions 259

M. Ali Khan. Oligopoly in Markets with a Continuum of Traders: An Asymptotic Interpretation 273

Milton Harris. Optimal Planning under Transaction Costs: The Demand for Money and Other Assets 298

Anthony A. Sampson. A Model of Optimal Depletion of Renewable Resources 315

CONTENTS

LAWRENCE M. BENVENISTE. A Complete Characterization of Efficiency for a General Capital Accumulation Model 325

NOTES, COMMENTS, AND LETTERS TO THE EDITOR

LAWRENCE M. BENVENISTE. Two Notes on the Malinvaud Condition for Efficiency of Infinite Horizon Programs . . 338

MIKIO NAKAYAMA. A Note on the Core of an Income Redistribution Game 347

NUMBER 3, JUNE 1976

MARTIN L. WEITZMAN. The Optimal Development of Resource Pools 351

STEVEN A. LIPPMAN AND JOHN J. MCCALL. Job Search in a Dynamic Economy 365

ROBERT D. WILLIG. Integrability Implications for Locally Constant Demand 391

MASAHIRO OKUNO. General Equilibrium with Money; Indeterminacy of Price Level and Efficiency 402

SERGE-CHRISTOPHE KOLM. Unequal Inequalities. I 416

DANIEL A. GRAHAM, L. PETER JENNERGREN, DAVID W. PETERSON, AND E. ROY WEINTRAUB. Trader-Commodity Parity Theorems 443

J. S. JORDAN. Temporary Competitive Equilibrium and the Existence of Self-Fulfilling Expectations 455

RICHARD D. MCKELVEY. Intransitivities in Multidimensional Voting Models and Some Implications for Agenda Control . 472

NOTES, COMMENTS, AND LETTERS TO THE EDITOR

JEAN-JACQUES LAFFONT. Risk, Stochastic Preference, and the Value of Information: A Comment 483

EDI KARNI AND DAVID SCHMEIDLER. Independence of Nonfeasible Alternatives, and Independence of Nonoptimal Alternatives 488

C. A. FIELD. A Note on Shepherd's Duality Theorem 494

EDWIN BURMEISTER. The Factor-Price Frontier and Duality with Many Primary Factors 496

AUTHOR INDEX FOR VOLUME 12 504

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Introduction to Hamiltonian Dynamics in Economics

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Economics during the fifties and sixties was marked by a substantial resurgence of interest in the theory of capital. While the advances during this period were very impressive, there was also an uneveness in the development of the subject. One-good models were studied in detail, as were many-good models of production–maximal growth and many-good models of consumption–optimal growth for the special case in which there is *no* social impatience. When treating heterogeneous capital, the literatures on decentralized or descriptive growth and consumption–optimal growth with positive time discounting were dominated by special cases and examples.

Reliance on examples and special cases proved to have some unfortunate consequences. The Battle of the Two Cambridges, ostensibly an argument over approaches to modeling distribution and accumulation, often seemed to focus on the robustness (or lack of robustness) of certain "fundamental" properties of the one-sector model and other worked-out examples when extended to more general heterogeneous-capital models. Furthermore, in large part because growth theory appeared to be an enterprise based only on proliferating special cases, the attention of the young able minds in the profession turned elsewhere, for example, to the—at least seemingly—more evenly-developed general equilibrium tradition.

This is a shame. Intertemporal allocation and its relationship with the wealth of societies is one of the most important problems in our discipline. Growth models are natural vehicles for the study of what is called "temporary equilibrium." Dynamic models of multi-asset accumulation provide the theoretically most satisfactory environment for modeling the macroeconomics of income determination, employment, and inflation.

The papers in this volume can be thought of as attempts at providing some unification of the theory of heterogeneous capital. The major

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topic, although by no means the only one, which we investigate here is the stability of long-run steady-state equilibrium in models of heterogeneous capital accumulation. However, the basic techniques used in our investigations have wide application to economic dynamics in all its manifestations. This follows from the observation that the economic growth models which we treat each belong to a class of general intertemporal economic models which is essentially representable by what we call a Hamiltonian dynamical system.

The static technology of our growth models can in general be described by an instantaneous technology set, T, with feasible production satisfying

$$(c, z, -k, -l) \in T \subset \{(c, -k, -l): (c, k, l) \geqslant 0\},$$

where c denotes the vector of consumption-goods outputs, z the vector of net investment-goods outputs, k the vector of capital-goods inputs, and l the vector of primary-goods inputs. There is an alternative representation of static technological opportunities that is more congenial to dynamic analysis, i.e., the representation of the technology by its Hamiltonian function.

Let p be the vector of consumption-goods prices and q the vector of investment-goods prices. Define the Hamiltonian function,

$$H(p, q, k, l) = \sup_{(c', z')} \{pc' + qz': (c', z', -k, -l) \in T\}.$$

H is typically defined on the nonnegative orthant $\{(p, q, k, l): (p, q, k, l) \geqslant 0\}$, and is precisely interpreted as the maximized value of net national product at output prices (p, q) given input endowments (k, l).

If the set T is closed, convex, and permits free disposal of outputs, then technology is also completely characterized by a (unique) continuous Hamiltonian function which is convex and linear homogeneous in output prices (p, q), and concave in input stocks (k, l). For each such T-representation, there is a unique H-representation of technology and, conversely, for each such H-representation there is a unique T-representation of technology. While we focus on macroeconomic dynamics, the Hamiltonian function is of more general usefulness. Thus, in his paper in this volume, relating properties of the set T to properties of the function $H(p, q, k, l)$, Lau quite properly refers to the Hamiltonian function as the "restricted profit function."

Lau studies the case where technology can be described by production functions. For the case in which the production functions are twice differentiable, Lau develops identities linking the Hessians of the production function with those of the Hamiltonian (or restricted-profit) function. Lau also analyzes some special cases in which increasing returns

to scale are allowed. For these cases, the Hamiltonian function may not be concave but is quasiconcave in the input stocks (k, l).

Representation of static technology by the Hamiltonian function allows us to describe all the growth models in this volume as *Hamiltonian dynamical systems*. In continuous time, $t \in [0, \infty)$, the system follows the laws of motion

$$\begin{cases} \dot{k}(t) \in \partial_q H(p(t), q(t), k(t), l(t)), \\ \dot{q}(t) \in -\partial_k H(p(t), q(t), k(t), l(t)) \end{cases} \tag{HDS}$$

where $\dot{k}(t)$ and $\dot{q}(t)$ are time derivatives of $k(t)$ and $q(t)$, respectively; $\partial_q H$ and $\partial_k H$ are respectively the subdifferentials (or generalized gradients) with respect to q and k. The first correspondence in (HDS) follows from the definition of net investment since it is equivalent to the equation $\dot{k}(t) = z(t)$. The second correspondence in (HDS) is somewhat more subtle and can be rewritten (employing static duality theory) as $\dot{q}(t) + r(t) = 0$, where $r(t)$ is the vector of competitive capital–goods rental rates. Thus, the second line in (HDS) states that the return to asset holders (including rentals and capital gains) are equal across capital goods. If there is (short-run) perfect foresight about price changes, this is the asset-market-clearing equation that follows from a simple arbitrage argument for competitive economies. The existence of prices $(p(t), q(t))$ satisfying (HDS) also follows from the necessary conditions in the optimal and maximal growth problems.

For discrete time, $t = 0, 1, \ldots$, the Hamiltonian dynamical system can be represented by

$$\begin{cases} k_{t+1} \in k_t + \partial_q H(p_t, q_t, k_t, l_t), \\ q_{t+1} \in q_t - \partial_k H(p_{t+1}, q_{t+1}, k_{t+1}, l_{t+1}). \end{cases} \tag{HDS}'$$

Line one of (HDS)′ follows from the definition of net investment, since it is equivalent to $k_{t+1} = k_t + z_t$. Line two is the (short-run) perfect-foresight, competitive, asset-market-clearing equation, since it is equivalent to $q_{t+1} - q_t + r_{t+1} = 0$, where r_{t+1} is the vector of capital–goods rentals in period $t + 1$. (HDS)′ can also be derived from the discrete-time necessary conditions in optimal and maximal growth problems.

From (HDS) and (HDS)′, we see that the properties of static technology as characterized by the Hamiltonian function will be basic to the dynamic analysis of competitive and optimizing economic dynamical systems. So far, as can be seen in (HDS) or (HDS)′, the development of consumption goods prices, $p(t)$ or p_t, and of primary factors, $l(t)$ or l_t, has been left as exogenous to the model.

The Hamiltonian dynamical system is said to be autonomous if H depends on time solely through q and k. The system would be autonomous

if, for example, p and l are constant through time, i.e., $p(t) = \bar{p}$ and $l(t) = l$, or $p_t = \bar{p}$ and $l_t = l$. Autonomous Hamiltonian dynamical systems have special structure that has been exploited in the mathematics and physics literature. (For references, see [8, 10, 19]).

Let (q^*, k^*) be a rest point to the autonomous version of the Hamiltonian dynamical system so that

$$0 \in \partial_q H(\bar{p}, q^*, k^*, l),$$

$$0 \in \partial_k H(\bar{p}, q^*, k^*, l).$$

Of course, (q^*, k^*) is a rest point for the autonomous version of (HDS) if and only if it is a rest point for the parallel autonomous version of (HDS)'. Assume further that H is quadratic in (q, k) so that the (HDS) is a system of linear differential equations and (HDS)' is a system of linear difference equations. In this case, a theorem of Poincaré tells us that if λ is a root to the characteristic equation for the autonomous version of the system (HDS) at (q^*, k^*) then $-\lambda$ is also a root. For the autonomous version of (HDS)', the analogous theorem tells us that if λ is a characteristic root, so also is $1/\lambda$.

The Poincaré theorem, although extremely simple, is suggestive of deep fundamental results. If, for example, in the continuous-time case we could rule out λ's with zero real parts ($\text{Re}\ \lambda \neq 0$), then we would have established a saddlepoint result for the autonomous version of the linear Hamiltonian dynamical system: In (q, k)-phase space, the dimension of the manifold of (forward) solutions tending to (q^*, k^*) as $t \to +\infty$ is equal to the dimension of the manifold of (backward) solutions tending to (q^*, k^*) as $t \to -\infty$, and each manifold is equal in dimension to half the dimension of the phase space.

Since this "saddlepoint property" and related properties turn out to be of basic interest in dynamic economic analyses, a natural question is: Are there interesting restrictions that can be imposed on the Hamiltonian function (i.e., on technology) such that the general (nonlinear) autonomous Hamiltonian dynamical system will possess the saddlepoint property (or related properties)? Rockafellar [12], in a paper motivated by problems of economic growth (see [18]), established the saddlepoint property of the autonomous version of (HDS) for cases in which H is *strictly* concave in k and *strictly* convex in q. Rockafellar goes on to consider the (HDS) as derived from an intertemporal optimization model which can be interpreted as the problem of consumption-optimal growth with zero discounting. Here $c(t)$ is interpreted as the (scalar) current instantaneous utility at time t. Because of zero-discounting, the (scalar) price of "utility" $p(t)$ must be constant. $l(t)$ is also considered to be a constant scalar.

Global stability is established for the optimal trajectory: Given initial capital stocks $k(0) = k_0$, $\lim_{t\to\infty} k(t) = k^*$. Rockafellar's stability result had been foreshadowed, in one form or other, in the earlier literature on maximal growth and optimal growth. For continuous-time versions, see, e.g., [15, 16]. For discrete-time versions, see, e.g., [3, 9, 11].

Consider next the problem of consumption-optimal growth with positive (constant) rate of time discount, $\rho > 0$. Again, $c(t)$ or c_t must be interpreted as the (scalar) instantaneous utility at time t. Assume, for simplicity, a constant scalar fixed factor, $l(t) = 1$ or $l_t = 1$. Because of discounting, $-\dot{p}(t)/p(t) = \rho$ or $-(p_t - p_{t-1})/p_t = \rho$. In this case, (HDS) and (HDS)′ are no longer autonomous and further analysis is required. (HDS) can now be rewritten as

$$\begin{cases} k \in \partial_Q H(Q, k), \\ \dot{Q} \in -\partial_k H(Q, k) + \rho Q \end{cases} \tag{PHDS}$$

where $Q \equiv q/p$ is the vector of current capital-goods prices and $H(Q, k) \equiv H(1, q/p, k, 1)$ is the current value Hamiltonian. The above system can be thought of as a perturbation (by the term ρQ) of a Hamiltonian dynamical system. In discrete time, the perturbed Hamiltonian dynamical system is

$$\begin{cases} k_{t+1} \in k_t + \partial_Q H(Q_t, k_t), \\ Q_{t+1} \in Q_t - \partial_k H(Q_{t+1}, k_{t+1}) + \rho Q_t. \end{cases} \tag{PHDS)′}$$

Let (Q^*, k^*) be a steady state, or rest point to (PHDS) and (PHDS)′. Then

$$0 \in \partial_Q H(Q^*, k^*),$$
$$0 \in -\partial_k H(Q^*, k^*) + \rho Q^*.$$

One can ask the question: Do the properties of models based on (HDS) and (HDS)′, such as the stability of optimal economic growth, carry over to models based on the perturbed systems, (PHDS) and (PHDS)′, for the case of ρ positive? The simple answer is, of course, "no," as Kurz's local analysis [6] shows. Two approaches to modifying this question have been studied: (1) Papers such as that by Samuelson [16] and José Scheinkman's contribution in this volume take technology as fixed and investigate whether or not global stability properties of the optimal growth model are preserved as the discount rate ρ is changed from zero to a small positive number. (2) The approach taken by the contributions in this volume by Cass and Shell, Rockafellar, and Brock and Scheinkman can be thought of as generalizations of the Samuelson-Scheinkman approach. Conditions are sought on the geometry of the Hamiltonian function (i.e.,

on static technology) that suffice to preserve under (not necessarily small) perturbation the basic properties of the Hamiltonian dynamical system.

Scheinkman's paper treats local and global stability in a discrete-time, optimal growth model with an indirect, current, instantaneous utility function which is discounted at some small rate $\rho > 0$. His model therefore satisfies (PHDS)′ although he does not work directly with the Hamiltonian formalism. A critical assumption is that linearization of the Euler equations in the neighborhood of the "optimal steady state" possesses no characteristic roots with $|\lambda| = 1$. Using Radner's bounded-value loss technique, a turnpike property is established for $\rho > 0$ sufficiently small. Scheinkman's proof is in two steps. First, an ingenious argument shows that optimal paths "visit" neighborhoods of the "modified optimal steady state," (Q^*, k^*). Second, local stability follows directly from his assumption about characteristic roots.

The contribution by Cass and Shell treats optimal growth and decentralized or descriptive growth models in both continuous and discrete time as applications of Hamiltonian dynamics. They review the problem of optimal growth with zero discounting and show that a steepness condition on the Hamiltonian function (a condition weaker than Rockafellar's [12] strict convexity–concavity of the Hamiltonian) suffices to insure uniqueness of the steady-state capital vector, k^*. Furthermore, a uniform strengthening of the steepness condition insures global stability of k^* and can be thought of as a generalization of Radner's assumption of bounded value-loss. This approach relies crucially on the property that (q^*, k^*) is a saddlepoint of the Hamiltonian function $H(1, q, k, 1)$.

In the optimal-growth model with discount rate $\rho \neq 0$, the rest point (Q^*, k^*) is not a saddlepoint of $H(Q, k)$. Therefore, the Cass–Shell steepness condition must be modified by a term dependent on ρ. The modified steepness condition establishes the uniqueness of k^*. A uniform strengthening of the steepness condition insures global stability of k^* for the continuous-time, optimal growth model. Because of different effects of "interest compounding," a different steepness condition is required to establish stability in the discrete-time, optimal growth model.

While optimal growth with time-discounting yields a very simple perturbation of a Hamiltonian dynamical system, there are other perturbed Hamiltonian dynamical systems that arise in economic theory. Cass and Shell discuss the general problem of decentralized growth with instantaneously adjusted expectations about price changes. In its general form, the model allows for the interpretation of competitive growth with utility-maximizing agents. However, in such a model the form of the perturbation from Hamiltonian dynamics can be very subtle. The formal analysis in the Cass–Shell paper is thus restricted to the continuous-time

version of the model in which demand for consumption is Marxian. It is shown that if in addition to the strengthened steepness assumption from zero-discount optimal growth, the Hamiltonian is such that a "capital-intensity" condition is satisfied, then decentralized growth in the Marxian decentralized model is stable.

In his first contribution to this volume, Rockafellar studies the basic mathematics of the problem which arises in continuous-time, optimal growth with positive discounting. His paper is thus a generalization of his earlier work [12], where his results follow from the strict convexity–concavity of the Hamiltonian function. In the present paper, he relies on a strengthened convexity–concavity: The Hamiltonian function, $H(Q, k)$, is assumed to be α-concave in k and β-convex in Q with $\alpha > 0, \beta > 0$, and $4\alpha\beta > \rho^2$. This convexity–concavity hypothesis is substantially stronger than the related Cass–Shell steepness condition (because Rockafellar's hypothesis ignores cross effects between Q and k), but Rockafellar goes substantially beyond the Cass–Shell paper in the range of results established for this model. In addition to local and global stability results, Rockafellar establishes the existence of solutions to optimizing programs satisfying (PHDS). He shows directly that the saddlepoint property holds when his convexity–concavity hypothesis is satisfied: The dimension of the manifold of solutions to (PHDS) tending to (Q^*, k^*) as $t \to +\infty$ is equal to the dimension of the manifold of solutions to (PHDS) tending to (Q^*, k^*) as $t \to -\infty$. Each manifold has dimension equal to half the dimension of the (Q, k) phase space. Rockafellar goes on to develop deep results on intertemporal duality, which have important bearing on the economic theory of the asymptotic behavior of $(Q(t), k(t))$.

Gaines' paper in this volume is a study of the existence of solutions to the full dynamical system arising in the time-discounted, optimal growth problem. He follows the Cass–Shell formulation and appends to (PHDS) appropriate nonnegativity conditions and boundary conditions, an initial capital-stock endowment condition and a transversality condition. The full system is thus

$$\begin{cases} k \in \partial_Q H(Q, k), \\ \dot{Q} \in -\partial_k H(Q, k) + \rho Q, \\ Q(t) \geqslant 0, \\ k(t) \geqslant 0, \\ k(0) = k_0 \geqslant 0, \\ \lim\limits_{t \to +\infty} Q(t)\, e^{-\rho t} k(t) = 0. \end{cases} \tag{FPHDS}$$

Gaines assumes only that H is convex in Q and concave in k, a much weaker assumption than Rockafellar's curvature assumption, but his methods require exploitation of other structural features of the Hamiltonian function. In the spirit of the Cass–Shell paper, he assumes (1) the existence of a stationary point (Q^*, k^*), (2) limitation of feasible output by primary factors, (3) productivity of technology, (4) bounded rate of depreciation, and (5) free disposal in allocation. From these conditions, a priori bounds are developed for solutions to a finite-time problem associated with (FPHDS). The proof of existence relies on a continuation principle from the theory of fixed points for multivalued mappings in infinite-dimensional spaces.

Both the Cass–Shell and Rockafellar papers in this volume make heavy analytic use of monotonicity properties of the Lyapunov function $V = (Q - Q^*)(k - k^*)$ in studying optimal growth (as did, for example, the Samuelson paper [16]). The contribution to this volume by Brock and Scheinkman also investigates stability of optimal growth in continuous-time models and develops "local" properties of the Hamiltonian function that insure monotonicity of V or some alternative Lyapunov function, and thus insure stability of optimal growth for bounded trajectories. They assume throughout that the Hamiltonian function is twice-continuously differentiable. Much of their analysis is based on what they call the "curvature matrix"

$$C = \begin{bmatrix} H_{QQ} & (\rho/2)I \\ (\rho/2)I & -H_{kk} \end{bmatrix},$$

where H_{QQ} is the matrix of cross partial derivatives with respect to current prices and H_{kk} is the matrix of cross partial derivatives with respect to capital stocks.

Brock and Scheinkman show that if the quadratic $(\dot{Q}, \dot{k})\, C(\dot{Q}, \dot{k})$ is positive except when $(\dot{Q}, \dot{k}) = 0$, then every bounded trajectory satisfying (PHDS) converges to a rest point. Their result is obtained by using the Lyapunov function $\dot{Q}\dot{k}$. In their paper, it is also shown that if $(Q - Q^*)\dot{k} + (k - k^*)\dot{Q} = 0 \Rightarrow (Q - Q^*, k - k^*)C(Q - Q^*, k - k^*) > 0$ and if the matrix C evaluated at (Q^*, k^*) is positive definite, then every bounded trajectory satisfying (PHDS) converges to (Q^*, k^*). The last conditions imply the Cass–Shell steepness condition, and thus insure that the Lyapunov function $V = (Q - Q^*)(k - k^*)$ is monotonically increasing for trajectories from (PHDS).

By convexity–concavity of H, the matrices H_{QQ} and $-H_{kk}$ are positive semidefinite. Let α be the smallest eigenvalue of H_{QQ} and β be the smallest eigenvalue of $-H_{kk}$. For the special case of twice continuous differentiability, the Rockafellar curvature condition is that $4\alpha\beta > \rho^2$, which

insures the positive definiteness of the matrix C. While the matrix C, when it is calculable, does provide a measure of "curvature" of H, it does not allow for the more general interpretation of Hamiltonian steepness since information about the interaction terms between the Q's and the k's, representable for the differentiable case by the matrix of cross partials H_{Qk}, are ignored in the C matrix.

In his second contribution to this volume (and our final paper), Rockafellar offers an (apparently weaker) alternative to the Cass-Shell steepness condition. With the Rockafellar growth condition, which is very much like the steepness condition, trajectories satisfying (PHDS) and the transversality condition $\lim_{t\to\infty}(Q(t) - Q^*)(k(t) - k^*) = 0$ are shown to converge to (Q^*, k^*).

The present volume has been in planning and preparation for more time than we would like to admit. Extensive (and often intense) communication among the authors was stimulated by two fruitful conferences—one at the Minary Center, Squam Lake, New Hampshire, the other at the University of Pennsylvania, Philadelphia. We are grateful to the Mathematical Social Science Board for its generous and understanding support of these opportunities for scholarly interaction.

While the topics treated here have mostly to do with the theory of economic growth, in particular, with existence and stability of macro-economic growth, we hope you will be able to read between the lines and appreciate the potential usefulness to economic theory of what we call the Hamiltonian approach to economic dynamics.

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On Optimal Steady States of n-Sector Growth Models when Utility is Discounted*

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This paper contains results on local and global stability of n-sector growth models when utility is discounted mostly for small rates of discount. It is well known that when future utility is not discounted one can prove precise results about optimal steady states (OSS's) under fairly general assumptions. In particular, existence, uniqueness, and turnpike properties have been established by several authors. The counter examples presented by Kurz, Sutherland, and Weitzman, however, show that when utility is discounted, additional assumptions are required to obtain turnpike results. In general, it would be interesting to know how the submanifolds of stability change as δ changes. One hopes that certain conditions on the utility function would be sufficient to "classify" the submanifolds of stability and instability. Such a question is apparently very difficult to answer, but we think that the results obtained here will help in this task.

The proof that the turnpike theorem holds for discount factors near one is divided in two parts. First, we prove that optimal paths "visit" neighborhoods of the modified OSS's. Then, we prove that local stability holds for such neighborhoods.

In order to show this fact, we must prove that the local "stable manifold" varies continuously with the discount factor. This roundabout method is necessary since our problem is similar to proving uniform continuity with respect to a parameter of solutions of a differential equation in a noncompact interval of time.

Other problems analyzed here include uniqueness and continuity of OSS's. We also discuss the relation between the saddle-point property and the local stability of infinite horizon optimal growth paths.

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I. The Visit Lemma, Continuity, and Local Properties of Optimal Steady States

1. *Introduction*

Part I contains basic results that will later be used in the study of the behavior of paths of optimal growth in n-sector models. We first prove that any optimal path "visits" a neighborhood of the undiscounted optimal steady state (OSS) when the discount factor δ is sufficiently near 1. As a corollary, we obtain the continuity of the OSS correspondence, which associates to each δ a set $K(\delta)$ of OSS's when the rate of discount is δ at $\delta = 1$. Both results are proven for general models such as the ones presented by Gale [3] and McKenzie [11]. We then use those results to prove that uniqueness and the saddle-point property also hold for δ near 1, for a model like the one used by Samuelson and Solow in [17]. We also show how the saddle-point property "implies" uniqueness.

2. *The Visit Lemma and Continuity*

In this section we use the value-loss technique of Radner [12] to prove that given any ϵ-neighborhood of the undiscounted OSS k^*, we can find $\delta(\epsilon) < 1$ such that if the discount factor δ is at least as large as $\delta(\epsilon)$, optimal paths have to visit such ϵ-neighborhoods. The model used is essentially McKenzie's [11] although an assumption similar to Gale's strict concavity of the utility function at the "turnpike" (cf. [3]) is used. For completeness, we repeat here the basic assumptions made in [11].

A.1. There exists a closed convex set $Y \subset R \times R_+^n \times R_-^n$ of "triples" $(u, k^1, -k)$ where u is the utility achievable with initial capital stocks k and final capital stocks k^1. Also $(u, k^1, -k) \in Y$ implies $(u(1), k^1(1), -k(1)) \in Y$ if $u(1) \leqslant u$, $0 \leqslant k^1(1) \leqslant k^1$, and $k(1) \geqslant k$.

Definition 1. A sequence $\{k_t\}$ is called a *growth path* if there exists sequence $\{u_t\}$ such that $(u_t, k_t, -k_{t-1}) \in Y$ for all t.

A.2. There exists $\tilde{k}$ with $(u, k, -\tilde{k}) \in Y$ and $k > \tilde{k}$, i.e., $\tilde{k}$ is *expansible*.

A.3. (a) For any r, $\exists \eta$ such that $\| k \| < r$ and $(u, k^1, -k) \in Y$ implies $u < \eta$ and $\| k^1 \| < \eta$. (b) There exists R and $\gamma < 1$ such that $\| k \| \geqslant R$ and $(u, k^1, -k) \in Y$ implies $\| k^1 \| < \gamma \| k \|$.

Lemma 1. *There exists* $(u^*, k^*, -k^*)$ *such that* $u^* \geqslant u$ *for any* $(u, k^1, -k)$ *with* $k^1 \geqslant k$.

Proof. McKenzie [11].

A.4. k^* is expansible. From A-4, there exists $k^1 > k^*$ such that $(u, k^1, -k^1) \in Y$, i.e., k^* is not *saturated.*

LEMMA 2. *There exists* p^* *such that* $p^* \geqslant 0$ *and for* $(u, k^1, -k) \in Y$ *we have* $u + p^*(k^1 - k) \leqslant u^*$.

Proof. McKenzie [11].

A.5. There exists a unique "triple" $(u^*, k^*, -k^*)$ that satisfies Lemma 1. Furthermore, $u + p^*(k^1 - k) < u^*$ for any $(u, k^1, -k) \neq (u^*, k^*, -k^*)$. This assumption is similar to Gale's strict concavity of $u(k^1, k)$ at the turnpike (k^*, k^*).

LEMMA 3. k^0 *expansible and* k *not saturated imply that there exists* $T < \infty$ *and sequence* $(u_t, k_t, -k_{t-1})$, $t = 1, \ldots, T$ *with* $k_T = k$.

Proof. McKenzie [11].

LEMMA 4. (*Radner–Atsumi*). *For a fixed* M *suppose* $\| k \| < M$ *and* $(u, k^1, -k) \in Y$. *Then, for any* $\epsilon > 0$, *there exists* $\rho > 0$ *such that if* $d((u^*, k^*, -k^*), (u, k^1, -k)) > \epsilon$ (*with* $d(x, y) = \| x - y \|$ *for any* $x \in R^n$, $y \in R^n$), *we have* $u + p^*(k^1 - k) < u^* - \rho$.

Proof. If the assertions were false, there would exist a sequence $y(s) = (u(s), k^1(s), -k(s)) \in Y$ with $d((u^*, k^*, -k^*), y(s)) > \epsilon$ and $u(s) + p^*(k^1(s) - k(s)) \to u^*$. However, by assumption A.3(a), since $\| k(s) \| < 2M$, we have $\| k^1(s) \| < M'$, $| u(s) | < M'$ for some $M' > 0$. Hence, $y(s)$ lies in a bounded region for all s. Thus, there is a point of accumulation $\bar{y} = (\bar{u}, \bar{k}^1, -\bar{k})$, and $\bar{u} + p^*(\bar{k}^1 - \bar{k}) = u^*$ although $d(\bar{y}, (u^*, k^*, -k^*)) > \epsilon$. Q.E.D.

DEFINITION 2. $\underline{k} \in R^n$ is said to be sustainable if there is $u \in R$ such that $(u, \underline{k}, -\underline{k}) \in Y$.

DEFINITION 3. Given $0 < \delta < 1$ and $k_0 > \underline{k}$, where $\underline{k}$ is sustainable, a growth path $\{k_t\}$ is called optimal if there exists $\{u_t\}$ associated with $\{k_t\}$ and $\lim_{N \to \infty} \sum_1^N \delta^t u_t \geqslant \sum_1^\infty \delta^t u_t'$ for any other utility sequence $\{u_t'\}$ associated with any growth path k_t' with $k_0' = k_0$. We will also write $k(t, k_0, \delta)$ to indicate the tth element of such an optimal path.

We use "lim" in Definition 3 instead of "lim sup" because assumption A.3 assures us that for any path, either the limit exists, or the sum converges to $-\infty$. Also, there always exists a path, namely $(u_0, k_0, -k_0)$ for any sustainable k_0, such that $\sum_1^\infty \delta^t u_0 = u_0/(1 - \delta)$. If k_0 is not sustainable, by free disposal, one can consider the path $(\underline{u}, \underline{k}, -\underline{k})$.

We do not discuss here the problem of existence of an optimal path since this subject is extensively treated elsewhere (cf. [18]).

DEFINITION 4. k^δ is an optimal steady state if $(u^\delta, k^\delta, -k^\delta)$ is an optimal solution to

$$\text{Max} \sum_{t=1}^{\infty} \delta^t u_t$$
$$\text{s.t. } k_0 = k^\delta \qquad (u_t, k_{t+1}, -k_t) \in Y.$$

We will write $K(\delta)$ for the set of all such optimal steady states k^δ.

LEMMA 5 (THE VISIT LEMMA). *Let $\underline{k}$ be expansible. Given any $k_0 \geqslant k$ and given $\epsilon > 0$, there exists $\delta(\epsilon) < 1$ such that if $1 \geqslant \delta \geqslant \delta(\epsilon)$, any optimal path $k(t, k_0, \delta)$ with $k(0, k_0, \delta) = k_0$ has the property that there exists a $\bar{t}$ such that $\| k(\bar{t}, k_0, \delta) - k^* \| < \epsilon$.*

Proof. Since k_0 is fixed, write $k(t, k_0, \delta) = k_t(\delta)$ and suppose

$$\| k_t(\delta) - k^* \| > \epsilon \text{ for all } t.$$

First assume that $\| k_0 \| < R$. Then by A.3(a), $\| k_1 \| < M$ for some $M > 0$. Suppose $\| k_t \| < \max(R, M)$. If $\| k_t \| < R$, then $\| k_{t+1} \| < M$. If

$$R \leqslant \| k_t \| < M,$$

then by A.3(b), $\| k_{t+1} \| < \| k_t \|$. Hence, $\| k_{t+1} \| < \max(R, M)$ for all t. By the Radner–Atsumi lemma, there exists $\rho > 0$ such that

$$\delta^t u_t(\delta) + \delta^t p^*(k_t(\delta) - k_{t-1}(\delta)) < \delta^t u^* - \delta^t \rho \qquad \text{for } t = 1, 2, \ldots.$$

Summing, one gets

$$\sum_1^\infty \delta^t u_t(\delta) + \sum_1^\infty \delta^t p^*(k_t(\delta) - k_{t-1}(\delta)) < \sum_1^\infty \delta^t u^* - \sum_1^\infty \delta^t \rho.$$

Hence,

$$\sum_1^\infty \delta^t u_t(\delta) + \sum_1^\infty (\delta^t - \delta^{t+1}) p^* k_t(\delta) - p^* k_0 < \sum_1^\infty \delta^t u^* - \sum_1^\infty \delta^t \rho,$$

or,

$$\sum_1^\infty \delta^t u_t(\delta) - p^* k_0 < \sum_1^\infty \delta^t u^* - \sum_1^\infty \delta^t \rho, \tag{1.1}$$

since

$$p^*k_t \geqslant 0 \qquad \text{and} \qquad \delta^{t-1} \geqslant \delta^t. \tag{1.1}$$

By Lemma 3, there exists a path $\underline{k}_t$ with $\underline{k}_0 = \underline{k}$ and $\underline{k}_N = k^*$ for some $N < \infty$. Let $\underline{u}_t$ be the associated utility sequence and $\lambda(\delta) = \sum_1^N \delta^t \underline{u}_t$. Since $\lambda(\delta)$ is a continuous function of δ, there exists $\hat{\lambda}$ such that $\hat{\lambda} \leqslant \sum_1^N \delta^t \underline{u}_t$ for $0 \leqslant \delta \leqslant 1$. By free disposal (A.1), there exists a path $\{\underline{k}_t\}_1^N$ leading from k_0 to k^*, and such that $\sum_1^N \delta^t \underline{u}_t \geqslant \hat{\lambda}$, where $\underline{u}_t$ is the associated utility sequence.

Consider the following alternative path.

$$k_t = \underline{k}_t \qquad t = 1, \ldots, N,$$
$$k_t = k^* \qquad t = N, N+1, \ldots.$$

Since $k_t(\delta)$ is optimal, we must have

$$\sum_N^\infty \delta^t u^* + \hat{\lambda} \leqslant \sum_1^\infty \delta^t u_t(\delta). \tag{1.2}$$

Equations (1.1) and (1.2) imply

$$\sum_N^\infty \delta^t u^* + \hat{\lambda} - p^*k_0 < \sum_1^\infty \delta^t u^* - \sum_1^\infty \delta^t \rho$$

or,

$$-\hat{\lambda} + p^*k_0 + \sum_1^{N-1} \delta^t u^* > \sum_1^\infty \delta^t \rho.$$

Since $\sup_{\|k_0\| < R} p^*k_0 \leqslant \max_{\|k_0\| \leqslant R} p^*k_0 = \bar{\lambda}$, we get by setting $\lambda = \bar{\lambda} - \hat{\lambda}$, $\lambda + ((\delta^{N-1} - 1)\delta u^*/(\delta - 1)) + \rho > \rho/(1 - \delta)$. This, of course, is a contradiction for δ close to 1. Also, if $\| k_0 \| \geqslant R$, then there exists finite $\underline{t}$ such that $\| k_{\underline{t}} \| < R$ (by A.3(b)). Hence, one can apply the previous proof to the path $k(t, k_{\underline{t}}, \delta)$. Since $k(t - \underline{t}, k_{\underline{t}}, \delta) = k(t, k_0, \delta)$ the result follows. Q.E.D.

Before starting our result concerning continuity of OSS's, we need

A.6. There exists $\delta^* < 1$ such that if $\delta \geqslant \delta^*$ then $\underline{k} < k^\delta$, where $\underline{k}$ is expansible.

COROLLARY 1. *For any $\epsilon > 0$ there exists $\delta(\epsilon)$ such that $\| k^\delta - k^* \| < \epsilon$, for $1 \geqslant \delta \geqslant \delta(\epsilon)$ for any $k^\delta \in K(\delta)$.*

Proof. By A.6, $k^\delta \geqslant \underline{k}$, where $\underline{k}$ is expansible, for $\delta \geqslant \delta^*$. By the visit lemma, given $\epsilon > 0$, there exists $\delta(\epsilon) < 1$ such that if $1 \geqslant \delta \geqslant \delta(\epsilon)$ any

optimal path $k(t, k^\delta, \delta)$ with $k(0, k^\delta, \delta) = k^\delta$ has the property that there exists $\bar{t}$ such that $\| k(\bar{t}, k^\delta, \delta) - k^* \| < \epsilon$. Since $k(t, k^\delta, \delta) \equiv k^\delta$, this means $\| k^\delta - k^* \| < \epsilon$. Hence, for any $\delta \geq \delta(\epsilon)$, $\| k^\delta - k^* \| < \epsilon$.

COROLLARY 2. *Let $\underline{k}$ be expansible. Given any $k_0 \geq \underline{k}$ and given $\epsilon > 0$, there exists $\bar{\delta}(\epsilon) < 1$ such that if $1 \geq \delta \geq \bar{\delta}(\epsilon)$, any optimal path $k(t, k_0, \delta)$ with $k(0, k_0, \delta) = k_0$ has the property that there exists a $\bar{t}$ such that*

$$\| k(\bar{t}, k_0, \delta) - k^\delta \| < \epsilon.$$

Proof. By the visit lemma, there exists $\delta^1(\epsilon/2)$ such that if $1 \geq \delta \geq \delta^1(\epsilon/2)$, there exists $\underline{t}$ such that $\| k(\underline{t}, k_0, \delta) - k^* \| < \epsilon/2$. By Corollary 1, one can find $\delta^2(\epsilon/2)$ such that $\| k^\delta - k^* \| < \epsilon/2$ for $1 \geq \delta \geq \delta^2(\epsilon/2)$. Hence, if $\bar{\delta}(\epsilon) = \max(\delta^1(\epsilon/2), \delta^2(\epsilon/2))$, we have that there exists $\underline{t}$ such that $\| k(\underline{t}, k_0, \delta) - k^\delta \| < \epsilon$. Q.E.D.

3. *Local Properties*

We need now to make use of differentiability properties of the utility function. For this purpose we postulate a discrete version of the Samuelson and Solow in model (cf. 17, p. 542]). Here one maximizes

$$\sum_{t=1}^{\infty} \delta^t V(k_t, k_{t+1}),$$

where $V : A \subset R_+^{2n} \to R$, A a closed convex set with $(k^*, k^*) \in \mathring{A}$. We will also assume.

A.7. (a) V is C^3 (i.e., at least three times continuously differentiable) in $\mathring{A}$ and concave. (b) The matrix

$$\begin{bmatrix} V_{k_t k_t}(k^\delta, k^\delta) & V_{k_t k_{t+1}}(k^\delta, k^\delta) \\ V_{k_{t+1} k_t}(k^\delta, k^\delta) & V_{k_{t+1} k_{t+1}}(k^\delta, k^\delta) \end{bmatrix} = \begin{bmatrix} A^\delta & B^\delta \\ B^{\delta'} & C^\delta \end{bmatrix}$$

is negative definite at (k^*, k^*).

A.8. All solutions are interior solutions so that the Euler "difference" equation system

$$\frac{\delta \, \partial V(k_t, k_{t+1})}{\partial k_t} + \frac{\partial V(k_{t-1}, k_t)}{\partial k_t} = 0 \tag{1.3}$$

is satisfied at any optimal path.

A.7 and A.8 are going to be used only in a fixed neighborhood of (k^*, k^*).

The linearization of Eq (1.3) at an OSS k^δ is given by

$$B^{\delta\prime}X_{t+1} + [\delta A^\delta + C^\delta]\, X_t + \delta B^\delta X_{t-1} = 0.$$

Hence, the characteristic equation of the Euler difference system becomes

$$| B^{\delta\prime}\lambda^2 + (\delta A^\delta + C^\delta)\,\lambda + \delta B^\delta | = 0. \tag{1.4}$$

We will assume

A.9. $\lambda = 0$ is not a root for $\delta = 1$.

A.9 is clearly a "generic" assumption. This assumption is equivalent to $\det B = | B | = | B' | \neq 0$ at $\delta = 1$.

The following lemmas are proved in Levhari and Leviatan [8].

LEMMA 6. *If $\lambda \neq 0$ solves* (1.4) *then $1/\delta\lambda$ is also a solution.*

LEMMA 7. *If $\delta = 1$ and λ solves* (1.4), *then $\| \lambda \| \neq 1$.*

We can now prove

THEOREM 1. *There exists ϵ such that if $\delta > 1 - \epsilon$, then k^δ possesses the saddle-point*[1] *property.*

(Samuelson [16] contains a similar result.)

Proof. From Lemmas 6 and 7, there exists a set of n roots for $\delta = 1$, e.g., $\lambda_1^1,\ldots, \lambda_n^1$, where each root is counted with its multiplicity such that $\| \lambda_1^1 \| < 1$. Since (1.4) is a polynomial equation of degree $2n$ with coefficients, say p_i^δ, there exists by Rouché's theorem a μ such that if $\| p_i^\delta - p_i^1 \| < \mu$, then there exists n roots λ_i^δ with $\| \lambda_i^\delta \| < 1$. Since p_i is a continuous function of δ and the elements a_{ij}^δ, b_{ij}^δ, and c_{ij}^δ of A^δ, B^δ, and C^δ, respectively, one can find η such that if $\| \delta - 1 \| < \eta$ and $\| \gamma_{ij}^\delta - \gamma_{ij}^1 \| < \eta$, where $\gamma = a, b, c$, then $\| p_i^\delta - p_i^1 \| < \mu$. Because V is C^2, there exists η' such that if $\| k^\delta - k^* \| < \eta'$, then $\| \gamma_{ij}^\delta - \gamma_{ij}^1 \| < \eta$. But by Corollary 1 of the visit lemma there exists $\eta^{1\prime}$ such that $\| \delta - 1 \| < \eta^{1\prime}$ implies $\| k^\delta - k^* \| < \eta'$. Choosing $\epsilon = \min\{\eta, \eta^{1\prime}\}$ and using Lemma 6, the proof is complete. Q.E.D.

The saddle-point property established above for δ near 1 makes one expect that uniqueness should also prevail in a neighborhood of $\delta = 1$. We now prove that this is so. Our strategy here consists in formulating an implicit function problem such that if k^δ is an OSS, then k^δ solves our

[1] We use here the word saddle point for a fixed point of a difference equation such that half of the roots have absolute value greater than one and half of the roots are nonzero, and with absolute value less than one.

implicit function problem. In the end we indicate how the proofs presented here can be generalized for the Gale model with (twice) differentiable utility function.

Recall that if k^δ is an OSS, then it must solve

$$\delta \frac{\partial V}{\partial x}(k^\delta, k^\delta) + \frac{\partial V}{\partial y}(k^\delta, k^\delta) = 0, \tag{1.5}$$

where $V = V(x, y)$. Let us define $f: \mathring{R}_+ \times B(k^*, \eta) \to R^n$[2]

$$f(\delta, k) = -\delta \frac{\partial V}{\partial x}(k, k) - \frac{\partial V}{\partial y}(k, k);$$

then if k^δ is an OSS for some $0 < \delta \leq 1$, we have $f(\delta, k^\delta) = 0$ with $0 < \delta \leq 1$. We can now prove that uniqueness prevails near $\delta = 1$.

THEOREM 2. *There exists* $\epsilon > 0$ *such that if* $1 - \epsilon < \delta \leq 1$, *there exists only one* OSS k^δ.

Proof. Note that $D_k f(\delta, k^\delta) = -B^{\delta'} - \delta A^\delta - C^\delta - \delta B^\delta$. By assumption A.7

$$\begin{bmatrix} A^1 & B^1 \\ B^{1'} & C^1 \end{bmatrix}$$

is negative definite, hence, $-A^1 - B^1 - B^{1'} - C^1$ is positive definite. By the implicit function theorem, there exists neighborhoods V of $(1, k^*)$, and U of 1 such that for each $\delta \in U$, there exists a unique $k(\delta) \in R_+^n$ with $(\delta, k(\delta)) \in V$ and $f(\delta, k(\delta)) = 0$. By Corollary 1 of the visit lemma, we have that k^δ is unique for $1 \geq \delta > 1 - \epsilon$ for some $\epsilon > 0$. Q.E.D.

The next lemma is very similar to Brock's Theorem 1 [2, Section 3.1].

LEMMA 8 *If* $|B^{\delta'} - \delta A^\delta - C^\delta - \delta B^\delta| \neq 0$ *for* $\delta \leq \delta < 1$, *there exists at most one* OSS k^δ *for* $\delta \leq \delta \leq 1$.

Proof. The proof of 2, Section 3.1, Theorem 1 applies with the obvious modifications.

COROLLARY. *If the saddle-point property holds for all* OSS *corresponding to* δ *for* $\delta \geq \delta > 1$, *then uniqueness holds.*

[2] $B(k^*, \eta) = \{k \in R^n : |k - k^*| < \eta\}$.

Proof. Since there exists no λ with $|\lambda| = 1$ such that

$$|B^\delta\lambda^2 + (\delta A^\delta + C^\delta)\lambda + \delta B^\delta| \neq 0,$$

we have, in particular, $|B^{\delta'} + \delta A^\delta + C^\delta + \delta B^\delta| \neq 0$.

Although we used here the model of Samuelson and Solow, we note that except for the corollary, we actually only used (1.5). In the Gale model, Sutherland has shown than under certain conditions every OSS has the so-called support property, i.e., an OSS (k^δ, k^δ) maximizes

$$\Phi_\delta(k_{t-1}, k_t) = u(k_{t-1}, k_t) - p^\delta k_{t-1} + \delta p^\delta k_t,$$

for an appropriate vector p^δ. If an OSS k^δ is interior, one has

$$\frac{\partial u}{\partial x}(k^\delta, k^\delta) - p^\delta = 0$$

and

$$\frac{\partial u}{\partial y}(k^\delta, k^\delta) + \delta p^\delta = 0,$$

or

$$\delta\frac{\partial u}{\partial x}(k^\delta, k^\delta) + \frac{\partial u}{\partial y}(k^\delta, k^\delta) = 0,$$

i.e., (1.5). There is no need of assuming that all optimal paths do in fact satisfy (1.3).

II. The Continuity of the Stable Manifold

1. *Introduction*

In Part II we establish continuity properties of the local stable manifolds associated with the Euler difference equations generated by optimal growth problems. For this purpose, we start by transforming these second-order difference equations into first-order difference equations around the steady states. We then use the "stable manifold theorem for a point" of Hirsch and Pugh to show that the stable manifold varies continuously with the parameter δ.[3] We also study the relationship between the existence of such local stable manifolds and the convergence of

[3] If one is willing to assume that the linearization of the Euler equation at the undiscounted optimal steady state has no multiple eigenvalues, one can furnish a proof of the "continuity" of the stable manifold using only elementary results in linear algebra, and a result on upper-semicontinuous correspondences from Berge [1]. The proof is long, however, and quite messy. It may be found in my paper [13].

optimal paths that start sufficiently close to the optimal steady state, the so-called local turnpike result.

The proof of the local turnpike theorem usually consists in showing that a system of difference equations on the capital stocks variable and some dual variable possesses a saddle point singularity, together with the fact that a convergent path is optimal. This reasoning assumes that the "stable manifold" S is such that there exists $\epsilon > 0$ such that given any capital stock k' with $\| k' - k^* \| < \epsilon$, where k^* is the singularity, one can find a value for the dual variable, say y', such that $(k', y') \in S$. There is no *a priori* reason for that to be true. Suppose, for instance, that $n = 2$ and that $S = \{k^1, k^2, y^1, y^2)/k^1 = (k^*)^1,\ y^2 = \Phi(k^2, y^1),\ \| k^2 - (k^*)^2 \| < \epsilon,\ \| y^1 - y^* \| < \epsilon\}$. Then S is a two dimensional manifold but given $\bar{k} = (\bar{k}^1, \bar{k}^2)$ with $\bar{k}^1 \neq (k^*)^1$, there exists no y such that $(\bar{k}, y) \in S$. What needs to be shown is that $(k^*, y^*) \in \text{Int}(\Pi_1(S))$, where $\Pi_1(S)$ is the projection of S on the k subspace. The only author, to our knowledge, who took care of this problem was McKenzie (cf. [10]). There he assumes, essentially, that the projection on the k subspace of the stable manifold S' associated with the linear system that approximates the original system at (k^*, y^*) has a nonempty interior. Since S' and S are tangent, this ensures the result. It turns out, however, that because our $2n$-dimensional system is generated from a second-order n-dimensional system with concavity properties, if all eigenvalues are nonzero, $(k^*, y^*) \in \text{Int}(\Pi_1(S'))$.

2. *Reduction to a Standard Form*

In this section, we show how one can reduce the Euler "difference" equations that are satisfied by optimal paths to a standard form in a neighborhood of $\delta = 1$ when these equations are given by

$$\delta \frac{\partial V}{\partial k_t}(k_t, k_{t+1}) + \frac{\partial V}{\partial k_t}(k_{t-1}, k_t) = 0. \qquad (*)$$

More precisely, we prove that one can write $(*)$ as $z_{t+1} = F_\delta(z_t)$, where F_δ is a diffeomorphism in a neighborhood of a fixed point z^δ. We will also prove some continuity results that will be needed later.

LEMMA 9. *There exists $\epsilon^* > 0$ such that if $1 \geqslant \delta \geqslant 1 - \epsilon^*$ then, Eq. $(*)$ can be written as $k_{t+1} = g(\delta, k_t, k_{t-1})$ in an ϵ^* neighborhood of $(1, k^*, k^*, k^*)$. Furthermore, g is C^2.*

Proof. Let $f(\delta, k_{t-1}, k_t, k_{t+1}) = \delta(\partial V/\partial k_t)(k_t, k_{t+1}) + (\partial V/\partial k_t)(k_{t-1}, k_t)$. Hence, $| D_{k_{t+1}} f(1, k^*, k^*, k^*)| = |B| \neq 0$, by A.9. By the implicit function theorem [7, Theorem 8, p. 362], there exists function g

such that in a ϵ^* neighborhood of $(1, k^*, k^*, k^*)$, one has $f(\delta, k_{t-1}, k_t, k_{t+1}) = 0$ iff $k_{t+1} = g(\delta, k_{t-1}, k_t)$. Also, since $f \in C^2$, we have $g \in C^2$.

Let us call U the neighborhood obtained in the above lemma. By defining functions $g_\delta : R^{2n} \to R^n$ by $g_\delta(k_{t-1}, k_t) \equiv g(\delta, k_{t-1}, k_t)$, one can rewrite $(*)$ as

$$k_{t+2} = g_\delta(k_t, k_{t+1}). \tag{2.1}$$

We can then define $y_t = k_{t+1}$ and rewrite (2.1) as

$$k_{t+1} = y_t,$$
$$y_{t+1} = g_\delta(k_t, y_t),$$

or

$$z_{t+1} = F_\delta(z_t), \tag{2.2}$$

where $z_t = (k_t, y_t)$ and $F_\delta(z_t) = [\Pi_2(z_t), g_\delta(z_t)]$ where Π_2 is the projection on the y plane. Since Π_2 is C^∞, we have $F_\delta \in C^2$. Alternatively, one can define a function F by $F(\delta, z_t) = F_\delta(z_t)$ and write

$$z_{t+1} = F(\delta, z_t). \tag{2.3}$$

Our theorems here will depend partly on local analysis around the OSS k^δ. Since F is C^2, we may write around any point x that

$$F(\delta, z) = F(\delta, x) + D_z F(\delta, x)(z - x) + r(\delta, z),$$

where given $\alpha > 0$, there exists $\beta = \beta(\alpha, \delta, x)$ such that if $\| z - x \| < \beta$, then $\| r(\delta, z) \| \leq \alpha \| z - x \|$.

Since $F \in C^2$, the application

$$D_z = U \subset R^{2n+1} \to M(R^{2n}, R^{2n}),$$
$$(\delta, x) \to D_z F(\delta, x),$$

i.e., the application which gives us the partial derivative matrix with respect to z calculated at (δ, x), is continuous. Here $M(R^{2n}, R^{2n})$ indicates the set of all $2n \times 2n$ matrices with real entries.

By Theorem 2 of Part 1, there exists function K: $[\underline{\delta}, 1] \to R^n$ such that $K(\delta) = k^\delta$ is the (unique) OSS when the rate of discount is δ. Furthermore, $K \in C^2$.

Consider the function $r = R \times R^{2n} \times R^{2n} \to R^{2n}$ defined by $r(\delta, z, x) = F(\delta, z) - F(\delta, x) - D_z F(\delta, x)(z - x)$. Since $F \in C^2$, we have $r \in C^1$. Also $D_z r(\delta, z, x) = D_z F(\delta, z) - D_z F(\delta, x)$. Hence, $\lim_{z \to x} D_z r(\delta, z, x) = 0$. Since $D_z r$ is continuous, it is uniformly continuous in any compact set.

Hence, given ϵ, there exists $\beta(\epsilon)$ such that $\| D_z r(\delta, z, x)\| < \epsilon$, if $\|(\delta, z, x) - (\delta, x, x)\| < \beta(\epsilon)$. Hence, if z^δ is a fixed point of F_δ, we have $\| r(\delta, z, z^\delta) - r(\delta, z', z^\delta)\| \leqslant \sup_{s\in[z,z']} \| D_z(r, z, x^\delta)\| \| z - z' \| \leqslant \epsilon \| z - z' \|$. Let $r^\delta(z - z^\delta) = r(\delta, z, z^\delta)$.

We now summarize in Lemma 10 the results obtained above.

LEMMA 10. *There exists* $\bar\epsilon$ and $\bar{\bar\epsilon}$ *such that if* $1 \geqslant \delta \geqslant 1 - \bar\epsilon$ *and* $\| z - z^\delta \| < \bar{\bar\epsilon}$, *where* $z^\delta = (k^\delta, k^\delta)$ *we have*

(a) *The difference equation* $(*)$ *is equivalent to* $z_{t+1} = F_\delta(z_t)$.

(b) *Given* δ, *one has* $F_\delta(z) = z^\delta + D_z F_\delta(z^\delta)(z - z^\delta) + r^\delta(z - z^\delta)$ *where given* $\alpha > 0$; *there exists* $\beta(\alpha)$ *such that if* $\| z - z^\delta \| < \beta$, $\| \tilde z - z^\delta \| < \beta$, *then* $\| r^\delta(z - z^\delta) - r^\delta(\tilde z - z^\delta)\| < \alpha \| z - \tilde z \|$.

(c) *The application*

$$\bar D_z : [1 - \bar\epsilon, 1] \to M(R^{2n}, R^{2n}),$$
$$\delta \to D_z F_\delta(z^\delta),$$

is continuous.

(d)

$$D_z F_\delta(z) = \left[\begin{array}{c|c} 0 & I \\ \hline M & N \end{array}\right]$$

where all blocks are $n \times n$.

(e)

$$D_z F_1(z^*) = \left[\begin{array}{c|c} 0 & I \\ \hline -B^{-1}B' & -B^{-1}(A + C) \end{array}\right]$$

3. *The Stable Manifold Theorem*

The stable manifold theorem of Hirsh and Pugh [4], stated below, establishes the existence and continuity properties of the stable manifold that are needed to prove our results. Before stating the theorem, we need certain definitions and preliminary results:

DEFINITION 1. A map $f: X \to Y$ between metric spaces in Lipschitz if there exists number l such that $d(f(x), f(y)) \leqslant l d(x, y)$ for all $x, y \in X$. The smallest such l is the Lipschitz constant $L(f)$.

Observation. Every continuously differentiable map $f: U \subset R^m \to R^n$ is Lipschitz in a compact set $K \subset U$.

DEFINITION 2. Let $T: R^m \to R^m$ be an isomorphism. T is hyperbolic if T has no complex eigenvalue of unit norm.

Observation. Given any hyperbolic operator T, there exists decomposition of $R^m = E_1 \oplus E_2$ such that $T \mid E_i = E_i$ and all eigenvalues of $T_1 = T \mid E_1$ have absolute value less than one, and of $T_2 = T \mid E_2$ have absolute value greater than one. Hence, for every $x \in R^m$, x can be written uniquely as $x = x_1 + x_2$ with $x_i \in E_i$. Also one can renorm $E_1(E_2)$ such that $T_1(T_2^{-1}$ resp.) is a contraction. We can then choose a norm for R^m as $\| x \|' = \max\{\| x_1 \|', \| x_2 \|'\}$ for $x_1 \in E_1$, $x_2 \in E_2$. The quantity

$$\tau = \max(\| T_1 \|', \| T_2^{-1} \|') < 1$$

is called the skewness of T. In what follows, we consider R^m with the norm $\| \|'$. Let $B(r)$ be the closed ball of radius r about the origin. Let $B_i(r) = B(r) \cap E_i$. Define $f^{-n}(B(r)) = \{x \in B(r) / f(x), \ldots, f^n(x) \in B(r)\}$, and $W(f) = \bigcap_{n \geq 0} f^{-n}(B(r))$.

DEFINITION 3. The space $C^i(B(r), R^m)$ is the set of all functions $f: B(r) \subset R^m \to R^m$ of class C^i with bounded i-norm $\| f \|_i' = \sup_{x \in B(r)} \operatorname{Max}(\| f(x) \|', \| (Df)(x) \|', \ldots, \| (D^i f)(x) \|')$.

DEFINITION 4. For $r > 0$ fixed, a "Lipschitz neighborhood" of $T \mid B(r)$ in $C^i(B(r), R^m)$ is the set $N_\epsilon^i(T) = \{f \in C^i(B(r), R^m) \mid L(f - T) < \epsilon, \| f(0) \|' < \epsilon\}$.

DEFINITION 5. Let $f: B(r) \subset R^m \to R^m$ be such that $f(0) = 0$. The stable manifold of f is the set $S(f) = \{x \in B(r) \mid \lim_{n \to \infty} f^n(x) = 0\}$.

We are now ready to state

LEMMA 11 (THE STABLE MANIFOLD THEOREM FOR A POINT) *Let $0 < \tau < 1$ and $r > 0$ be given. There exists $\epsilon > 0$ independent of r and $0 < c < \epsilon$ with the following property. If T is a hyperbolic operator in $R^m = E_1 \oplus E_2$ of skewness τ and $f: B(r) \to R^m$ is Lipschitz and satisfies $L(f - T) < \epsilon$, $\| f(0) \|' < c$, then $W^r(f) = W$ is the graph of a unique function $g_f: B_1(r) \to B_2(r)$. Moreover, $L(g_f) \leqslant 1$ and g_f is of class C^i if f is. The assignment $f \to g_f$ is continuous as a map $N_\epsilon^i(T) \to C^i(B_1(r), B_2(r))$. The map $f \mid W: W \to W$ contracts W into its interior.*

Proof. [4, pp. 146–147]. Note that the theorem actually says that $W(f) \subset S(f)$ the stable manifold of f since $f \mid W(f)$ is a contraction.

We will apply Lemma 11 to the diffeomorphisms F_δ defined in Section 2. For this purpose, let $T = DF_1(z^*)$.

LEMMA 12. *T is a hyperbolic operator and, furthermore, T has n nonzero eigenvalues (counting multiplicity) with absolute value less than one, and n eigenvalues with absolute value greater than one.*

Proof. Let λ be an eigenvalue of T. Hence, there exists $x \in \mathbb{C}^{2n}$ [4] such that $Tx = \lambda x$. Write $x = (x_1, x_2)$, $x_i \in \mathbb{C}^n$, $i = 1, 2$. By Lemma 10(e), $Tx = (x_2, -B^{-1}B'x_1 - B^{-1}(A + C)x_2)$. Since $Tx = \lambda x$, we must have $x_2 = \lambda x_1$. Hence, $-B^{-1}B'x_1 - B^{-1}(A + C)\lambda x_1 = \lambda^2 x_1$. By A.9, $|B| \neq 0$, and, hence, $B'x_1 + (A + C)\lambda x_1 + \lambda^2 B x_1 = 0$, i.e., $|B' + \lambda(A + C) + \lambda^2 B| = 0$. Again using A.9, we have that $\lambda = 0$ is not an eigenvalue of T. Hence, if λ is an eigenvalue of T, $1/\lambda$ solves Eq. (1.4) of Part I, Section 3. The result follows from Lemmas 6 and 7. Q.E.D.

Let us define $G_\delta(x) = F_\delta(z^\delta + x) - z^\delta$. By Lemma 10, there exists $\bar{\epsilon} > 0$ and $\bar{\bar{\epsilon}} > 0$ such that if $1 \geq \delta \geq 1 - \bar{\epsilon}$ and $\|x\| < \bar{\bar{\epsilon}}$, then G_δ is well defined. Since $\|\ \|'$ is equivalent to any of the standard norms of R^m, there exists η such that $B(0, \eta) \subset \{x \in R^m / \|x\| < \bar{\epsilon}\}$, where $B(0, \eta)$ is taken with reference to $\|\ \|'$.

LEMMA 13. *Let $T: R^{2n} \to R^{2n}$ be defined by $Tx = DF_1(z^*) \cdot x$. Then given any $\epsilon > 0$, there exists $\rho(\epsilon)$ and $\delta^1 < 1$ such that if we consider $G_\delta: B(0, \rho) \to R^{2n}$, then $\|G_\delta - T\|_1' < \epsilon$ and $L(G_\delta - T) < \epsilon$.*

Proof. $\|(G_\delta - T)(x) - (G_\delta - T)(y)\|' = \|G_\delta(x) - Tx - G_\delta(y) + Ty\|' = \|DF_\delta(z^\delta)x + r^\delta(x) - Tx - DF_\delta(z^\delta)y - r^\delta(y) + Ty\|' < \|DF_\delta(z^\delta) - T)(x - y)\|' + \|r^\delta(x) - r^\delta(y)\|'$. By Lemma 10, given $\epsilon > 0$, there exists $\rho^1 > 0$ such that if $\|x\|', \|y\|' < \rho^1$, then $\|r^\delta(x) - r^\delta(y)\|' < \epsilon/2\|x - y\|'$, for $1 \geq \delta \geq \delta^2$. Also, by the same lemma, there exists $1 > \delta \geq \delta^2$ such that if $1 > \delta > \delta$, then $\|DF_\delta(z^\delta) - T\|' < \epsilon/2$. Hence, $L(G_\delta - T) < \epsilon$. Also, $\|G_\delta(x) - T(x)\|' = \|G_\delta(x) - Tx - G_\delta(0) - T0\|' < \epsilon/2\|x\|'$, and $\|DG_\delta(x) - T\|' = \|DF_\delta(z^\delta + x) - T\|' = \|D_zF(\delta, z^\delta + x) - D_zF(1, z^*)\|'$. Since F is C^1 and the function given by $z(\delta) = z^\delta$, $1 \geq \delta \geq \delta^2$ is continuous, there exists $\rho^2 \geq 0$ and $1 > \bar{\delta} > \delta^2$ such that if $x \in B(\rho^2)$ and $\delta > \bar{\delta}$, then $\|D_zF(\delta, z^\delta + x) - D_zF(1, z^*)\|' < \epsilon/2$. Let $\rho = \min(\rho^1, \rho^2)$ and $\delta^1 = \max\{\bar{\delta}, \delta\}$. Then $\|G_\delta - T\|_1' < \epsilon$ and $L(G_\delta - T) < \epsilon$, for $\delta \geq \delta^1$.

Lemmas 11, 12, and 13 allow us to construct a local stable manifold around $z^\delta = (k^\delta, k^\delta)$ for δ sufficiently close to one which varies continuously with δ. However, as mentioned in Section 1, we are interested in showing that given any k_0 sufficiently close to k^δ, one can find a k_1 such that (k_0, k_1) belong to the stable manifold. The following lemma is needed to show such result.

[4] $C^{2n} = \{(x_1, \ldots, x_n) / x_i \in C\}$ where C is the field of complex numbers.

LEMMA 14. *Let $T: R^{2n} \to R^{2n}$ be defined by $Tx = DF_1(z^*)\,x$. Assume also that T has $2n$ eigenvalues, $\lambda_1, \ldots, \lambda_{2n}$ with $|\lambda_i| < 1$ for $i = 1, \ldots, n$, $|\lambda_i| > 1$ for $i = n+1, \ldots, 2n$. Let $E_1 \oplus E_2$ be a decomposition of R^{2n} such that $T_1 = T \mid E_1$ has eigenvalues λ_i, $i = 1, \ldots, n$, and $T_2 = T \mid E_2$ has eigenvalues λ_i, $i = n+1, \ldots, 2n$. Then, there exists $b > 0$ and $\mu > 0$ such that given any $g: B_1(r) \to B_2(r)$ with $g(0) = 0$ and $L(g) < b$, there exists $\tilde{g}$: $B(0, \mu) \subset R^n \to R^n$ such that if $y \in$ graph g and $\| \Pi_1(y) \|' < \mu$, then $\Pi_2(y) = \tilde{g}(\Pi_1(y))$.*[5]

Proof. By the primary decomposition theorem [5, p. 180], there exists a real matrix P such that

$$P^{-1}AP = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} = L,$$

where each L_i is $n \times n$, and the eigenvalues of L_1 are those of A with absolute value less than one, and the ones of L_2 are those of A with absolute value greater than one. Hence, $E_1 = \{v \in R^{2n} / v = Qw, w \in R^n \times \{0\}\}$. Write $v = (v_1, v_2)$. Then if

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix},$$

where all blocks are $n \times n$, and if Q_1 is invertible, $E_1 = \{v \in R^{2n} / v = (v_1, Q_3 Q_1^{-1} v_1)\}$ for any $v_1 \in R^n$. We will show that this is, in fact, so. For if $\det Q_1 = 0$, then there exists $z \in R^n \times \{0\}$, $z \neq 0$, such that $Qz = (0, v_2)$ and $v_2 \neq 0$ since Q is invertible. Hence, $(0, v_2) \in E_1$, and, hence,

$$\lim_{t \to \infty} A^t \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = 0.$$

Consider now the problem (P)

$$\text{Max} \sum_{t=0}^{\infty} (z_t, z_{t+1})^T D^2 V(k^*, k^*)(z_t, z_{t+1})$$

subject to $z_0 = 0$. It is easy to see that since D^2V is negative definite that $z_t = 0$ is the optimal solution to (P). The Euler equation associated with (P) is exactly $(z_{t+2}, z_{t+1}) = A(z_t, z_{t+1})$. Since

$$\lim_{t \to \infty} A^t \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = 0,$$

there exists solution $\{z_t\}$ of the Euler equation system with $z_0 = 0$ and $z_1 = v_2 \neq 0$ such that $\lim_{t \to \infty} z_t = 0$. This solution must yield a negative

[5] If $v \in R^n$, $\| v \|' = \| (v, 0) \|'$.

value for (P) since $(0, v_2)^T D^2V(k^*, k^*)(0, v_2) < 0$. Let $w(z_t, z_{t+1}) = (z_t, z_{t+1})^T D^2V(k^*, k^*)(z_t, z_{t+1})$. By concavity of w,

$$\begin{aligned}\lim_{T\to\infty} \sum_{t=0}^{T} w(z_t, z_{t+1}) &= \lim_{T\to\infty} \sum_{t=0}^{T} w(z_t, z_{t+1}) - w(0, 0) \\ &\geqslant \lim_{T\to\infty} \sum_{t=0}^{T} \frac{\partial w}{\partial y_1}(z_t, z_{t+1})\, z_t + \frac{\partial w}{\partial y_2}(z_t, z_{t+1})\, z_{t+1} \\ &= \lim_{T\to\infty} \frac{\partial w}{\partial y_2}(z_T, z_{T+1})\, z_{T+1},\end{aligned}$$

since $(\partial w/\partial y_2)(z_t, z_{t+1})\, z_{t+1} + (\partial w/\partial y_1)(z_{t+1}, z_{t+2})\, z_{t+1} = 0$ and $z_0 = 0$. But since $\lim_{t\to\infty} z_t = 0$, we must have

$$\lim_{T\to\infty} \frac{\partial w}{\partial y_2}(z_T, z_{T+1})\, z_{T+1} = 0.$$

Hence, $\sum_{t=0}^{\infty} w(z_t, z_{t+1}) = 0$. This is a contradiction. Hence Q_1 must be invertible. Therefore, given any $v_1 \in R^n$, there exists unique $v_2 = Q_2Q_1^{-1}v_1 \in R^n$ such that $(v_1, v_2) \in E_1$. Since $\|v_2\|' \leqslant \alpha \|v_1\|'$ for some $\alpha > 0$, there exists $\bar{r}$ such that if $\|v_1\|' < \bar{r}$, then $\|v\|' < r$.

Given any $y \in$ graph g, write $y = (y_1, y_2)$. Then, there exists $z \in E_1$ such that $y_1 = g_1(z) + z_1$ where $g_1(z) = \Pi_1(g(z))$, $z_1 = \Pi_1(z)$, etc., $y_2 = g_2(z) + z_2$. Let $\bar{g}_1: B(0, r) \subset R^n \to R^n$ be given by $\bar{g}_1(v_1) = g_1(v_1, Q_2Q_1^{-1}v_1)$. Then, there exists $b > 0$ such that if $L(g) < b$, $L(\bar{g}_1) < 1$. Hence, $h_1: B(0, \bar{r}) \to R^n$ defined by $h_1(v_1) = v_1 + \bar{g}_1(v_1)$ is an homeomorphism over $B(0, \mu) \subset R^n$ for some $\mu > 0$. Given $y \in$ graph g with $\|y_1\|' < \mu$, we can define $v_1 = h_1^{-1}(y_1)$. Hence,

$$y_1 = v_1 + \bar{g}_1(v_1) = v_1 + g_1(v_1, Q_2Q_1^{-1}v_1).$$

Let $z = (v_1, Q_2Q_1^{-1}v_1)$. Then,

$$\begin{aligned}y_2 &= g_2(v_1, Q_2Q_1^{-1}v_1) + Q_2Q_1^{-1}v_1 = \bar{g}_2(v_1) + Q_2Q_1^{-1}v_1 \\ &= \bar{g}_2 \circ h_1^{-1}(y_2) + Q_2Q_1^{-1}h_1^{-1}(y_1) = \tilde{g}(y_1).\end{aligned}$$

Hence, $y_2 = \tilde{g}(y_2)$ for $\|y_1\|' < \mu$. Q.E.D.

LEMMA 15. *Under assumptions* A.1–A.9, *there exists* δ^* *and* $\mu > 0$ *such that if* $\bar{y} \in B(0, \mu) \cap R^n$, *there exists* $\bar{\mu}, \bar{\bar{\mu}} > 0$ and $\tilde{y} = \tilde{y}(\delta, \bar{y}) \in R^n \cap B(0, \bar{\mu})$ *such that* $\tilde{y}_0 = (\bar{y}, \tilde{y}) \in W^{\bar{\bar{\mu}}}(G_\delta) \subset S(G_\delta)$ *for any* $1 \geqslant \delta \geqslant \delta^*$. *Furthermore, if* $y_t(\tilde{y}_0)$ *solves* $y_{t+1} = G_\delta(y_t)$ *with* $y_0 = \tilde{y}_0$, *we have* $\|y_t(\tilde{y}_0)\|' < \bar{\bar{\mu}}$ and $\lim_{t\to\infty} y_t(\tilde{y}_0) = 0$.

Proof. Choose $r = \eta/2$ in Lemma 14. Then there exists $\epsilon > 0$ *independent of* r such that if $L(f - T) < \epsilon$ and $\|f(0)\|' = 0$ and $f: B(r) \to R^m$

is Lipschitz, then there exists a unique map $g_f: B_1(r) \to B_2(r)$ whose graph is $W^r(f)$. Now choose $\rho = \rho(\epsilon)$ as in Lemma 13 and $\bar{\rho} = \min(r, \rho)$. Then there exists $\bar{g}_f = g_f \mid B_1(\bar{\rho})$ whose graph is $W^o(f)$. Note that $\bar{g}_f(B_1(\bar{\rho})) \subset B_2(\bar{\rho})$ since g_f is a contraction. Note that $g_T \equiv 0$. Since the assignment $f \to g_f$ is continuous in the C^1 topology, we have $\| Dg_f \|' < c/2$ for any $c > 0$, for f sufficiently near T. Write $g_\delta = g_{G_\delta}$. Then for $\delta \geqslant \delta^*$ for some $\delta^* < 1$, we have $L(g_\delta) \leqslant \sup \| Dg_\delta \|' \leqslant c/2 < c$. Choose $c = b$ given in Lemma 14. Then there exists $\mu > 0$ such that given any $\bar{y}$ with $\| \bar{y} \|' < \mu$, then $\bar{y} = \bar{g}(\tilde{y})$ for some C^1 function $\bar{g}$. Since $\bar{g}(0) = 0$ and $\bar{g}$ is continuous, we have $\| \tilde{y} \|' < \bar{\mu}$ for some $\bar{\mu} > 0$. Hence, $\| \tilde{y}_0 \|' < \bar{\mu}$ for some $\bar{\mu} > 0$. Since G_δ is a contraction in $B(\bar{\rho})$, we have $\| y_t(\tilde{y}_0) \|' \leqslant \| \tilde{y}_0 \|' < \bar{\mu}$ and $\lim_{t\to\infty} y_t(\tilde{y}_0) = 0$.

LEMMA 16. *Under assumptions* A.1–A.9, *there exists* δ^* *and* $\mu, \gamma > 0$, *such that if* $\| k_0 - k^\delta \|' < \mu$ *and* $1 \geqslant \delta \geqslant \delta^*$, *there exists an optimal path* $k_t(k_0, \delta)$ *such that* $\| k_t(k_0) \|' < \gamma$ *and* $\lim_{t\to\infty} k_t(k_0, \delta) = k^\delta$.

Proof. Let $\bar{y} = k_0 - k^\delta$. By the previous lemma, there exists $\bar{\mu} > 0$ and $\tilde{y}$ with $\| (\tilde{y}, \bar{y}) \|' < \bar{\mu}$, such that $\lim_{t\to\infty} y_t(\tilde{y}_0) = 0$ and $\| y_t(\tilde{y}_0) \|' < \bar{\mu}$ where $\tilde{y}_0 = (\tilde{y}, \bar{y})$. Let $k_t(k_0) = \Pi_1(y_t) + k^\delta$. Then for some $\gamma > 0$, $\| k_t(k_0, \delta) - k^\delta \|' < \gamma$ and $\lim_{t\to\infty} k_t(k_0, \delta) = k^\delta$.

Now suppose there exists $\bar{k}_t$ with $\bar{k}_0 = k_0$ such that

$$\limsup_{T\to\infty} \sum_0^T \delta^t V(\bar{k}_t, \bar{k}_{t+1}) > \sum_{t=0}^{\infty} \delta^t V(k_t, k_{t+1}),$$

where $k_t = k_t(k_0, \delta)$.[6] Hence, given any $T > 0$, there exists $\bar{t} > T$ such that

$$\sum_{t=0}^{\bar{t}} \delta^t V(\bar{k}_t, \bar{k}_{t+1}) > \sum_{t=0}^{\bar{t}} \delta^t V(k_t, k_{t+1}) + \epsilon$$

for some $\epsilon > 0$ independent of T. However,

$$\begin{aligned}
&\sum_{t=0}^{\bar{t}} \delta^t V(k_t, k_{t+1}) - \sum_{t=0}^{\bar{t}} \delta^t V(\bar{k}_t, \bar{k}_{t+1}) \\
&\quad = \sum_{t=0}^{\bar{t}} \delta^t [V(k_t, k_{t+1}) - V(\bar{k}_t, \bar{k}_{t+1})] \\
&\quad > \sum_{t=0}^{\bar{t}} \delta^t \left[\frac{\partial V}{\partial y_1}(k_t, k_{t+1}) k_t + \frac{\partial V}{\partial y_2}(k_t, k_{t+1}) k_{t+1} \right. \\
&\qquad \left. - \frac{\partial V}{\partial y_1}(k_t, k_{t+1}) \bar{k}_t - \frac{\partial V}{\partial y_2}(k_t, k_{t+1}) \bar{k}_{t+1} \right],
\end{aligned}$$

by concavity of V.

[6] What follows is basically the argument contained in Mangasarian [9].

Since

$$\delta \frac{\partial V}{\partial y_1}(k_t, k_{t+1}) + \frac{\partial V}{\partial y_2}(k_{t-1}, k_t) = 0,$$

we get

$$\begin{aligned} -\epsilon &> \sum_{t=0}^{\bar{t}} \delta^t V(k_t, k_{t+1}) - \sum_{t=0}^{\bar{t}} \delta^t V(\bar{k}_t, \bar{k}_{t+1}) \\ &\geq \frac{\partial V}{\partial y_1}(k_0, k_1)\, k_0 + \delta^{\bar{t}} \frac{\partial V}{\partial y_2}(k_{\bar{t}}, k_{\bar{t}+1})\, k_{\bar{t}+1} \\ &\quad - \frac{\partial V}{\partial y_1}(k_0, k_1)\, \bar{k}_0 - \delta^{\bar{t}} \frac{\partial V}{\partial y_2}(k_{\bar{t}}, k_{\bar{t}+1})\, \bar{k}_{\bar{t}+1} \\ &= \delta^{\bar{t}} \frac{\partial V}{\partial y_2}(k_{\bar{t}}, k_{\bar{t}+1})[k_{\bar{t}+1} - \bar{k}_{\bar{t}+1}]. \end{aligned}$$

Hence,

$$\epsilon < |\delta^{\bar{t}}| \left\| \frac{\partial V}{\partial y_2}(k_{\bar{t}}, k_{\bar{t}+1}) \right\| \| k_{\bar{t}+1} - \bar{k}_{\bar{t}+1} \| \tag{2.4}$$

By assumption A.3, we have $\| \bar{k}_{\bar{t}+1} \| < A$ for some $A > 0$, and since $\lim_{t\to\infty} k_t = k^\delta$ and $\partial V/\partial y_2$ is continuous, the right-hand side of (2.4) will become less than ϵ for any $\bar{t} > T'$ for some T'. Hence,

$$\lim_{T\to\infty} \sup \sum_{t=0}^{T} \delta^t V(\bar{k}_t, \bar{k}_{t+1}) \leq \sum_{t=0}^{\infty} \delta^t V(k_t, k_{t+1}). \qquad \text{Q.E.D.}$$

III. The Insensitivity of the Global Turnpike Result

We can now combine the "visit lemma" of Part I with the continuity results of Part II to establish that for δ close to one, the "global turnpike" theorem still holds.

Theorem 3. *Given an optimal growth problem* $\operatorname{Max} \sum_{t=1}^{\infty} \delta^t V(k_t, k_{t+1})$ *where* $V: A \subset R_+^{2n} \to R$ *satisfying* A.1 *to* A.9 *above and an expansible* $\underline{k}$, *there exists* $\bar{\delta} < 1$ *such that if* $1 \geq \delta \geq \bar{\delta}$, *and* $k_0 \geq \underline{k}$, *there exists an optimal path* $k(t, k_0, \delta)$ *such that* $\lim_{t\to\infty} k(t, k_0, \delta) = k^\delta$ *where* k^δ *is the unique* OSS *associated with* δ.

Proof. Note that the "visit lemma" is true for any of the equivalent norms of R^{2n}. Hence, considering the constant $\mu > 0$ given by Lemma 16, there exists δ^1 such that for $1 \geq \delta \geq \delta^1$ and $k_0 \geq \underline{k}$, there exists $\bar{t}$ such that

$\| k(\bar{t}, k_0, \delta) - k^\delta \|' < \mu$, for any optimal path $k(t, k_0, \delta)$. Let $\bar{\delta} = \max\{\delta^1, \delta^*\}$ where δ^* is given by Lemma 16. Then for $1 \geqslant \delta \geqslant \bar{\delta}$, there exists a path $k(t, k(t, k_0, \delta), \delta)$ such that

$$\lim_{t\to\infty} k(t, k(\bar{t}, k_0, \delta), \delta) = k$$

and $k(t, k(\bar{t}, k_0, \delta), \delta)$ is optimal. Hence, $k(t + \bar{t}, k_0, \delta) = k(t, k(\bar{t}, k_0, \delta), \delta)$ is optimal, and we have

$$\lim_{t\to\infty} k(t, k_0, \delta) = \lim_{t\to\infty} k(t + \bar{t}, k_0, \delta) = k^\delta.$$

Although the theorem requires the choice of an expansible $\underline{k}$, this can be relaxed if one can show that there exists $\underline{k}$ expansible such that given any k_0 with $k_0^j > 0$, $j = 1,..., n$, any optimal path $k(t, k_0, \delta)$ has the property that there exists t such that $k(t, k_0, \delta) > \underline{k}$. This can be assured using "Inada"-type conditions.

It is interesting to consider how the counterexamples to the turnpike theorem in the literature fail to meet our hypothesis. Sutherland's [18] well-known example uses δ far from 1. However, Samuelson's paper [15] contains an example due to Martin Weitzman where the turnpike result fails for any $\delta < 1$. In our notation, Weitzman's example has $V(k_t, k_{t+1}) = k_t^{1/2}(1 - k_{t+1})^{1/2}$. Hence,

$$D^2V = \begin{bmatrix} -\tfrac{1}{4}k_t^{-3/2}(1 - k_{t+1})^{1/2} & -\tfrac{1}{4}k_t^{-1/2}(1 - k_{t+1})^{-1/2} \\ -\tfrac{1}{4}k_t^{-1/2}(1 - k_{t+1})^{-1/2} & -\tfrac{1}{4}k_t^{1/2}(1 - k_{t+1})^{-3/2} \end{bmatrix}$$

and $\det(D^2V) = 0$, and A.7 is not met.

The reader should notice that although our insensitivity result relates to the discount parameter, the same method can be used for arbitrary perturbations. This may prove useful in other contexts.

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The Structure and Stability of Competitive Dynamical Systems

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1. INTRODUCTION AND SUMMARY

The theory of economic growth has itself developed at an enormous rate over the past twenty years. A careful reading of the major work in this extensive literature[1] leaves one quite definite impression: The analysis of the long-run behavior of idealized economies has been very uneven. On the one hand, quite general results are available concerning the theory of maximal growth, broadly defined to include the problem of production-maximal growth, as well as the closely related problem of consumption-optimal growth with zero (net) discounting. On the other hand, only a few simple examples and special cases have been examined in the theory of optimal growth with positive discounting, or of descriptive growth with linear saving–investment hypotheses.

In this paper we provide a general framework which encompasses these as well as many other problems in the theory of economic growth, or more broadly, the theory of economic dynamics. Our approach is to describe competitive dynamics as Hamiltonian dynamics, where the Hamiltonian can be written as a function of *present* output prices and current input stocks, and can be interpreted as the present value of net national product (equal, by duality, to the present value of net national income). Such a Hamiltonian dynamical system is competitive in the sense that it derives from the perfect-foresight, zero-profit, asset-market clearing equations

[1] Our references contain a large sample of the basic work concerned with global stability in economic growth, ranging from the early work on the "turnpike" theorem (e.g., [9, 18, 20, 23]), through the continuing work on optimal growth (e.g., [2, 11, 19, 24] for zero discounting, and [5, 15, 28, 35] for positive discounting) and on descriptive growth (e.g., [14, 32, 34, 36]), to the recent work on the "Hahn problem" (e.g., [4, 8, 12, 33]). We have not attempted to include systematic references to closely related topics, for example, local stability analysis.

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arising in descriptive growth theory, and is consistent with (i) efficiency pricing conditions developed in the Malinvaud [17] tradition, and (ii) Euler's conditions or, more generally, Pontryagin's maximum principle (in either their usual continuous-time or their analogous discrete-time formulations), applying to production-maximal or consumption-optimal growth problems.

The problems we are interested in naturally lead to cases in which the Hamiltonian function is convex in present prices (the costate variables) and concave in current stocks (the state variables). Indeed, every convex technology is essentially characterized by such a convex–concave Hamiltonian function, and every convex–concave Hamiltonian function essentially represents a unique convex technology (see, for instance, the companion analysis of Lau [16], and related exposition of Cass [7]). Since our analysis originates in this duality, it is not surprising that we find the geometry of the Hamiltonian function to be the fundamental determinant of the long-run behavior of a competitive dynamical system.

It is customary and convenient in studying the dynamics of optimal growth to cast the analysis in terms of *current* output prices rather than present output prices. In the case with zero discounting, a (golden rule path) stationary point is also a saddlepoint of the current value Hamiltonian function. Thus, a simple assumption which insures uniqueness of the stationary input stocks is that the Hamiltonian be strictly concave in current input stocks. This strict concavity condition is unnecessarily strong. An assumption of "real Hamiltonian steepness" at the stationary point suffices to insure uniqueness. Moreover, a uniform strengthening of this steepness condition also allows us to establish convergence of the optimal path to the stationary input stocks. This stability analysis for consumption-optimal growth with zero discounting can be easily extended to apply to the case of production-maximal growth. In the latter case, our strengthened assumption of "real Hamiltonian steepness" at the von Neumann ray insures convergence to that ray, and is equivalent to Radner's assumption of bounded value-loss.

For optimal growth with constant, positive discounting, a (modified golden rule path) stationary point is not a saddlepoint of the Hamiltonian function. Nevertheless, when our strengthened "real Hamiltonian steepness" condition is modified by a term which depends on the discount rate, we again establish uniqueness of and convergence to the stationary input stocks. Furthermore, even when the discount rate is only asymptotically constant, some additional tightening of this modified steepness condition still permits us to establish global stability.

Our steepness conditions lead very naturally to a particular choice of Lyapunov function for the stability arguments. This particular function

has been used previously by Samuelson [29] and Rockafellar [25] in analyzing the stability of consumption-optimal growth with zero discounting (and, under a liberal interpretation, by Radner [23] in analyzing the stability of production-maximal growth). Our extension of these earlier results provides both a coherent unification of the maximal growth models, as well as a powerful justification for our Hamiltonian approach to the analysis of competitive dynamical systems.

In the area of descriptive growth, the analysis of dynamical behavior has typically been limited to studying fairly specific examples. From our present perspective, the fundamental difficulty in obtaining definite results for more general models seems quite evident: Since the interest rate implicitly depends on the whole evolution of prices and quantities, there is no direct analog to our "real Hamiltonian steepness" conditions. Nonetheless, by exploiting the particular structure of descriptive growth under the Marxian saving–investment hypothesis, we are able to establish a broad class of circumstances in which similar conditions entail global stability. Thus, our Hamiltonian approach yields one of the first reasonably general results on the long-run performance of a decentralized, capital accumulation process. Extensions of this approach to cover a wider range of descriptive growth models appears to us an important and promising prospect.

We believe that our Hamiltonian approach has many further possibilities for application. In particular, even without complete convexity in the underlying model, the solutions to many other interesting optimal control problems must exhibit Hamiltonian dynamics of the sort we analyze in this paper, i.e., must obey dynamical laws derived from a convex–concave Hamiltonian function. Thus, for example, our results are immediately applicable to a fairly general version of the neoclassical investment model. On a broader tack, we also believe that our duality emphasis may be quite useful in investigating the various existence problems which are deliberately sidestepped in the present paper. For instance, the existence of solutions to the particular differential equations (or differential correspondences) we define as representing optimal growth would follow from a duality theorem for concave programming in some appropriate infinite-dimensional space. More speculatively, we think it may be possible to derive the solutions to particular descriptive growth models in a similar manner.

In either case, this casual conjecturing remains to be verified by further research, part of which we ourselves plan to undertake in the near future.

The extended Cambridge-versus-Cambridge debate has taught us that insights based on the aggregate production function are of only limited value for unifying models of heterogeneous capital accumulation. It is our hope that our Hamiltonian approach will, at least to some extent,

provide the unifying principle that Cambridge, Massachusetts has been seeking.

2. The General Model

A. *Technology and Market*

Let

consumption-goods output (or sometimes "utility output") $= (c_1, \ldots, c_\mu) = c$,
net investment-goods output $= (z_1, \ldots, z_\nu) = z$,
capital stocks $= (k_1, \ldots, k_\nu) = k$,
primary factors $= (l_1, \ldots, l_\epsilon) = l$, and
technology $= T \subset \{(c, z, -k, -l): (c, k, l) \geqslant 0\}$.

While the general tenor of our maintained assumptions about technology is quite conventional, there is some special structure which is either necessary or useful for accomplishing our specific goal of establishing global stability. Thus, while listing our maintained assumptions we also mix in comments about what purposes they serve (and what alternative assumptions would suffice).

Assumption (T1). Feasibility of nonnegative inputs. If $(k, l) \geqslant 0$, then there exists (c, z) such that $(c, z, -k, -l) \in T$.

This assumption simply states that, given a sign convention for measuring inputs, any input combination can be used. In conjunction with assumption (T8) below, it allows restriction of the Hamiltonian function to a simple domain of definition, the nonnegative orthant. This is merely convenient, and so (T1) could be replaced by a weaker regularity requirement concerning the projection of T onto the input space, for example, that this projection be closed.

Assumption (T2). Feasibility of boundary production. T is closed.

This assumption is a standard regularity requirement. Since so much of our analysis depends on it, directly or indirectly, it is difficult for us to see how it could be dispensed with.

Assumption (T3). Diminishing returns in production. T is convex.

While this assumption too is a standard regularity requirement, it is not indispensable. For instance, even with increasing returns to capital stock inputs, if production possibility sets are convex, then the technology is representable by a Hamiltonian function, and furthermore, an optimal

growth path is describable by a Hamiltonian dynamical system. (The last requires some additional regularity to replace convexity; such regularity is implicitly assumed, for example, when the technology is specified in terms of industry or sectoral production functions together with input stock adding-up constraints.) Though we will not put much stress on this aspect of our analysis, the potential for handling such phenomena as increasing returns is a substantial reason for devoting serious attention to Hamiltonian dynamics, as well as an important area in the future development of economic dynamics.

ASSUMPTION (T4). Productivity of the technology. There exists $(c, z, -k, -l) \in T$ such that $(c, z) > 0$.

This assumption introduces the potential for balanced growth, since it means that, with appropriate endowments of initial capital stocks and primary factors, capital stocks can be maintained forever. While such an assumption is a common feature of most growth models, it is not at all innocuous, because it completely rules out the phenomenon of depletable resources. However, developing a general theory of the behavior of prescriptive or descriptive growth models when there are depletable resources is a major task in itself, and not one we even attempt here.

ASSUMPTION (T5). Constant returns to scale. T is a cone.

This assumption is primarily a convenience for expositing our Hamiltonian approach, since, when actual technological processes exhibit diminishing returns to scale, it is tantamount to introducing a fictitious input which "earns" residual income or accounting profit. In other words, because any set can be conceived as the projection of a cross-section of a cone in one higher dimension, there is absolutely no loss of generality in conducting analysis, when it is useful, in terms of such a cone plus its restriction. Of course some care must be taken in providing economic interpretation of this maneuver (partly amplified by the discussion in [6, pp. 273–76]). Thus, in particular, the reader should beware of a literal reading of our treatment of the optimal growth model; in our specification of that model, the "primary factor" typically summarizes both the influence of exogenous factor availability, as well as the influence of diminishing marginal utility.

ASSUMPTION (T6). Necessity of primary factors: If $(c, z, -k, -l) \in T$ and $l = 0$, then $c = 0$ and $z_i \{\leqq\} 0$ according as $k_i \{\geqq\} 0$ for $i = 1, 2, \ldots, \nu$.

This assumption parallels the no-free-lunch postulate of general equilibrium analysis, but is stronger, since it requires that gross outputs be zero when primary factor inputs are zero, even when some capital stock

inputs are positive. The importance of the assumption is that, in conjunction with other of our assumptions, notably (T2), (T5) and (T7), it implies that too large capital stocks simply cannot be maintained over time. Because we also assume a bounded rate of depreciation (see (T7) immediately below), this upper bound on feasible capital stocks also provides an absolute bound on feasible produced outputs.[2] We will have need to refer to these implied restrictions later on in our analysis, and so at this point we somewhat prematurely formalize them in the following proposition (which implicitly appeals to both our maintained assumptions about technology and our special assumptions about exogenous factor availability; see (A1) and (A3) further below).

BOUNDEDNESS LEMMA. There is a finite bound $B < \infty$ such that every production point $(c, z, -k, -l) \in T$ which is replicable or feasible satisfies the boundedness restriction

$$\|(c, z, k)\| < B. \tag{B}$$

By way of further interpretation, it is interesting to observe that assumptions (T4) and (T6) play a role in our model analogous to that of the Inada conditions in the standard one-good model, and are crucial, for instance, in establishing the existence of stationary points which exhibit intertemporal consumption efficiency (modified golden rule paths).

ASSUMPTION (T7). Bounded rates of depreciation. There exists

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_\nu \end{bmatrix}$$

[2] These assertions obviously presuppose that primary factors are scarce, that is, are available in limited amounts. Their proof is relatively straightforward if we restrict attention to production points $(c, z, -k, -l) \in T$ which (i) employ a unit level of primary factors $\|l\| = 1$, and (ii) yield nonnegative investment goods output $z \geq 0$, or which are *replicable*. In this special case, the standard argument based on assumptions (T2), (T5), and (T6) establishes that the set

$$\{(c, z, k): (c, z, -k, -l) \in T, z \geq 0 \text{ and } \|l\| = 1\}$$

is bounded. When we turn attention to production points which might be observed along a path achievable from given, finite initial capital stocks and primary factors, or, which are *feasible*, their proof is of explicitly dynamical character, as well as of significantly more complexity. The useful paper by Peleg and Ryder [22] contains a complete discussion for a particular discrete-time model. The related argument applying to our continuous- and discrete-time models is similar, though not identical.

with $\lambda_i \in (0, \infty)$ (continuous time) or $\lambda_i \in (0, 1]$ (discrete time) for $i = 1, 2, ..., \nu$ such that if $(c, z, -k, -l) \in T$, then $-\Lambda k \leqslant z$, and

ASSUMPTION (T8). Free disposal in allocation. If $(c, z, -k, -l) \in T$, $(c', z' + \Lambda(k' - k), -k', -l') \leqslant (c, z, -k, -l)$ and $(c', z') \geqslant (0, -\Lambda k')$, then $(c', z', -k', -l') \in T$.

Assumptions (T7) and (T8), taken together, are nothing more than an alternative statement of the standard assumptions of depreciation at constant (relative) rates plus free disposal of (gross) output and input. While there are several implications of these assumptions which are imporant for our analysis, here we will only emphasize one, that having to do with characterizing intertemporal consumption (or, more generally, net output) efficiency. Namely, (T8) (which only makes sense in light of (T7)) entails that output, both consumption-goods *and* investment-goods, must have associated nonnegative efficiency prices. (The reason for this is basically that (T8) means that having more capital stocks today does not require having less capital stocks, and, hence, less consumption-goods output, tomorrow.) This implication provides the rationale underlying our introduction of unlimited free disposal of output in the definition of the market M below, and is especially important in justifying our particular description of optimal growth as necessarily involving asymptotically zero or finite present value of capital stocks. While it seems likely that our results can be generalized to require somewhat less special structure than is imposed by (T7) and (T8) (for example, this is suggested by Rockafellar's [26] local analysis), the complete details of such extension remain to be seen.

Corresponding to the various commodities and the technology, define

$$\begin{aligned}&\text{output prices} = (p_1, ..., p_\mu, q_1, ..., q_\nu) = (p, q),\\ &\text{input rents} = (r_1, ..., r_\nu, w_1, ..., w_\xi) = (r, w), \text{ and}\\ &\text{market} = M \subset \{(p, q, r, w): (p, q, w) \geqslant 0\}.\end{aligned}$$

M is dually related to T by

$$M = \{(p, q, r, w): pc' + qz' - rk - wl \leqslant 0 \quad \text{for all} \quad (c', z') \leqslant (c, z), (c, z, -k, -l) \in T\}, \quad (1)$$

and, as noted above, contains all the price configurations which might appear in any snapshot of a competitive dynamical system. Even without assumptions (T1)–(T8), M is a closed convex cone having the additional properties of nonnegativity of output prices, if $(p, q, r, w) \in M$, then $(p, q) \geqslant 0$, and free disposal in valuation, if $(p, q, r, w) \in M$, $(p', -r', -w') \leqslant (p, -r, -w)$ and $p' \geqslant 0$, then $(p', q', -r', -w') \in M$.

B. *Accumulation and Growth*

Initial capital stocks and primary factors are given exogenously. Hence, letting bars denote this fact, all feasible paths of real variables must satisfy, in continuous time, $t \in [0, \infty)$,

$$(c(t), z(t), -\dot{k}(t), -l(t)) \in T, \dot{k}(t) = z(t), \quad k(0) = \bar{k} \text{ and } l(t) = \bar{l}(t), \quad (2)$$

or, in discrete time, $t = 0, 1, \ldots$,

$$(c_t, z_t, -k_t, -l_t) \in T, k_{t+1} = k_t + z_t, \quad k_0 = \bar{k} \text{ and } l_t = \bar{l}_t. \quad (2')$$

In (2) it is understood that, for example, $\{z_t(t): 0 \leqslant t < \infty\}$ is summable, or $\{k_t(t): k_t(t) = k_t(0) + \int_0^t z_t(s)\,ds \text{ and } 0 \leqslant t < \infty\}$ is absolutely continuous on any finite interval $0 \leqslant t \leqslant t' < \infty$; a similar remark will apply to the corresponding price variables.

C. *Intertemporal Profit Maximization*

Parallel with (2) or (2'), all "feasible" paths of price variables must satisfy

$$(p(t), q(t), r(t), w(t)) \in M, \quad \dot{q}(t) = -r(t) \text{ and } q(0) \geqslant 0, \quad (3)$$

or

$$(p_t, q_t, r_t, w_t) \in M, \quad q_{t+1} = q_t - r_{t+1} \text{ and } q_{-1} \geqslant 0.^3 \quad (3')$$

We define paths satisfying, or solutions to (2) and (3) or (2') and (3'), to be a *competitive dynamical system*, provided, in addition, prices are non-trivial, for some t $(p(t), q(t), r(t), w(t)) \neq 0$ or $(p_t, q_t, r_t, w_t) \neq 0$,[4] and there is static profit maximization,

$$p(t)\,c(t) + q(t)\,z(t) - r(t)\,k(t) - w(t)\,l(t) = 0, \quad (4)$$

[3] We note in passing that the notion of dual instability, prominent in the early literature concerning dynamic Leontief models, seems nothing more than a special case of the observation that (3) or (3') can be essentially converted into (2) or (2') by reversing the flow of time (e.g., considering $t \in [0, -\infty)$) and (ii) adding appropriate exogenous constraints (e.g., fixing $q(0) = \bar{q}$, $p(t) = \bar{p}(t)$).

It is worth stressing explicitly that, unlike feasible paths of real variables, "feasible" paths of price variables, that is, all paths affording at best zero profit opportunities, are not subject to exogenous constraints other than nonnegativity of initial investment goods prices (and this too is actually superfluous in the case of continuous time, given our particular definition of M).

[4] Paths of price variables which are identically zero are of little interest in either prescriptive or descriptive growth theory. In fact, the same is true of paths which are identically zero after some point in time; competitive dynamical systems with such prices correspond to finite-lived economics.

or

$$p_t c_t + q_t z_t - r_t k_t - w_t l_t = 0. \tag{4'}$$

Given the definition of M in (1), Eqs. (3)–(4) or (3')–(4') are nothing but the well-known perfect foresight, intertemporal profit maximization conditions.

D. *The Hamiltonian Representation*

Static profit maximization (at nonnegative output prices $(p, q) \geqslant 0$),

$$(c, z, -k, -l) \in T,$$

and

$$pc' + qz' - rk' - wl' \leqslant pc + qz - rk - wl = 0 \quad \text{for all} \quad (c', z', -k', -l') \in T, \tag{5}$$

is equivalent to the conditions that $(c, z, -k, -l)$ is an optimal solution to the concave programming program,

$$\text{maximize } pc' + qz' \quad \text{subject to} \quad (c', z', -k, -l) \in T. \tag{NNP}$$

(p, q, r, w) is an optimal solution to the (dual) convex programming problem,

$$\text{minimize} \quad r'k + w'l \quad \text{subject to} \quad (p, q, r', w') \in M, \tag{NNI}$$

and the values of NNP ($\equiv$ Net National Product) and NNI ($\equiv$ Net National Income) are equal,

$$\begin{aligned} \text{value of NNP} &= pc + qz \\ &= \text{value of NNI} \\ &= rk + wl. \end{aligned} \tag{6}$$

This equivalence suggests the potential usefulness of defining the *Hamiltonian function*

$$H(p, q, k, l) = \text{value of NNP}. \tag{7}$$

It is fairly straightforward to establish that H exhibits the following fundamental properties.

Property (H1). H is defined on the nonnegative orthant $\{(p, q, k, l): (p, q, k, l) \geqslant 0\}$.[5]

Property (H2). H is continuous in (p, q, k, l).

[5] Note that, though T is not compact, from assumptions (T2), (T6), and (T7), a cross-section like $\{(c', z', -k', -l'): (c', z', -k', -l') \in T \text{ and } (k', l') = (k, l)\}$ is compact. The parallel definition of H in terms of the value of NNI may run into difficulty when (k, l) is on the boundary of the projection of the technology set on the input space, i.e., when $(k, l) \not> 0$; see Footnote 7 below.

Property (H3). H is nondecreasing in p and l.

Property (H4). H is convex and linear homogeneous in (p, q), and concave and linear homogeneous in (k, l).

And, most important,

Property (H5). $\partial H(p, q, k, l)/\partial(p, q) = (c, z)$,[6] and, at least for $(k, l) > 0$,[7] $\partial H(p, q, k, l)/\partial(k, l) = (r, w)$.

Thus, the generalized gradients of H with respect to (p, q) are static profit-maximizing outputs (and conversely), while, for all practical purposes, the generalized gradients of H with respect to (k, l) are competitive input rents (and conversely).

Properties (H1)–(H5) follow directly from assumptions (T1)–(T7): Roughly speaking, for the purpose of analyzing static competitive or efficient allocation, every well-behaved technology set can be fully represented by a well-behaved Hamiltonian function.[8] Furthermore, given H, one can reconstruct T (if there is unlimited free disposal of output), or at least the critical "northeast" boundary of T (if there is not), completely.

[6] Here and after we use $\partial f(x, y)/\partial x$ to denote some (perhaps particular) or all generalized gradients of $f(x, y)$ with respect to x; e.g., when, for given y, the function $f(x, y)$ and its domain $D_x(y)$ are convex in x, some or all vectors $\partial f(x, y)/\partial x$ such that

$$f(x', y) - f(x, y) \geq (x' - x)\frac{\partial f(x, y)}{\partial x} \qquad \text{for all} \quad x' \in D_x(y).$$

[7] Some version of "constraint qualification"—here we use Slater's condition, the obvious choice for our problem—is required to insure that, in addition to (7),

$$H(p, q, k, l) = \text{value of NNI}.$$

(See the extended discussion of this point in [6, pp. 277–79]. We should emphasize that for the applications in this paper, whenever (5) obtains, (6) does also, so that by presumption boundary points present no difficulty (i.e., hypothesizing a solution to (2)–(4) or (2′)–(4′) already entails $\partial H(p, q, k, l)/\partial(k, l) = (r, w)$ without qualification).

[8] It is worth at least remarking on the relative merits of our Hamiltonian function representation vis-a-vis the more common production transformation representation, say, $F(c, z, k, l) = 0$, where F is defined implicitly by the "northeast" boundary of T. In general, the former has a distinct advantage over the latter: The domain of H is the fairly simple Cartesian product $\{(p, q): (p, q) \geq 0\} \times \{(k, l): (c, z, -k, -l) \in T\}$, while the domain of F is the more complicated boundary set $\{(c, z, k, l): (c, z, -k, -l) \in T$ and (c, z) is maximal in $\{(c', z'): (c', z', -k, -l) \in T\}\}$. In other words, in order to use H, one only needs to know all feasible input combinations, while in order to use F, one also needs to know all maximal output combinations, in effect, T itself. To our minds, this is a dominating superiority in most contexts where some functional representation of T is required.

This near equivalence between the T and H representations of convex[9] technologies is, as mentioned earlier, precisely stated and fully investigated in [7, 16].

For our objective here, it is more natural to represent the technology in terms of H rather than T. In particular, the equivalence between (5) and (6) together with the properties (H1)–(H5) mean that we can very succinctly rewrite the competitive dynamical system (2)–(4) or (2′)–(4′) as the *Hamiltonian dynamical system*

$$\begin{cases} \dot{k}(t) = \dfrac{\partial H(p(t), q(t), k(t), \bar{l}(t))}{\partial q}, & k(0) = \bar{k}, \\ \dot{q}(t) = -\dfrac{\partial H(p(t), q(t), k(t), \bar{l}(t))}{\partial k}, & q(0) \geqslant 0, \end{cases} \tag{8}$$

or

$$\begin{cases} k_{t+1} = k_t + \dfrac{\partial H(p_t, q_t, k_t, \bar{l}_t)}{\partial q}, & k_0 = \bar{k}, \\ q_{t+1} = q_t - \dfrac{\partial H(p_{t+1}, q_{t+1}, k_{t+1}, \bar{l}_{t+1})}{\partial k}, & q_{-1} \geqslant 0. \end{cases} \tag{8′}$$

The most striking feature of this formulation is its suggestion that *it is properties of the technology set as reflected in properties of the Hamiltonian function which are basic to the stability analysis of competitive dynamical systems*. This suggestion is amply borne out in the sequel. The representation (8) or (8′) just as obviously provides *a very convenient framework for constructing examples of competitive dynamical systems*, as we shall also clearly illustrate in the succeeding analysis.

E. *Simplifying Assumptions*

We now make some special assumptions in order to simplify the analysis that follows.

ASSUMPTION (A1). There is a single primary factor, $\xi = 1$, and it is unchanging in amount, $l(t) = \bar{l}(t) = 1$ or $l_t = \bar{l}_t = 1$, say. This assumption is made so that we can sensibly talk about stationary points. An essentially equivalent assumption would be that, however many, all primary factors grow asymptotically at the same constant rate (the "natural" growth rate).

[9] Since an output price system "convexifies" just the production possibility sets derived from a nonconvex technology, in that case one can only recover the union of the convex hulls of cross sections of the representation T, or its "northeast" boundary, from the representation H.

ASSUMPTION (A2). There is a single consumption good, $\mu = 1$. Since we can interpret output of this good as instantaneous utility, we are able to analyze most consumption-optimal growth problems treated in the literature. Here the single final utility output might be "produced" from a multi-dimensional vector of "intermediate" consumption-goods output.[10]

For descriptive growth theory, the assumption of a single consumption good is more restrictive, but we feel that inclusion of the general case would needlessly complicate out present analysis. Alternatively, we can simply duck this issue by postulating the existence of a sufficiently well-behaved consumption aggregate.

ASSUMPTION (A3). The initial endowment of each capital good is positive and finite $0 < \bar{k} < \infty$.

This assumption together with (T6) guarantees the existence of positive capital stocks for all time, $k(t) > 0$ or $k_t > 0$, a fact we use indirectly in analyzing optimal growth, directly in analyzing descriptive growth.

F. *Saving–Investment Behavior*

To close the Hamiltonian dynamical system, (8) or (8′), requires at a minimum, specification of initial prices of capital stocks, $q(0) = \bar{q}$ or $q_{-1} = \bar{q}$, and of dated prices of consumption-goods output, $p(t) = \bar{p}(t)$ or $p_t = \bar{p}_t$.

This may be accomplished directly or indirectly, depending on the particular problem under scrutiny. Consider, for example, the problem of optimal growth with constant, positive discounting. The maximand is

$$\int_0^\infty e^{-\rho t} c(t)\, dt, \tag{9}$$

or

$$\sum_{t=0}^{\infty} (1 + \rho)^{-t} c_t, \tag{9′}$$

where $c(t)$ or c_t is interpreted as utility output and ρ its associated time

[10] Even though T is assumed to be a cone, we could allow for diminishing returns in transforming "intermediate" consumption-goods output into the final utility output. Again, this follows from the fact that any convex set can be described as the projection of a cross-section of a convex cone of one higher dimension. In this situation, however, the interpretation of the fictitious input required for marginal productivity factor payment exhaustion of product is somewhat fanciful.

discount rate, the utility or "real" interest rate. Maximization of (9) or (9′) is constrained by technology and endowments (2) or (2′). For a feasible path to be optimal, it must be efficient. Thus, it is required that there exist nonnegative imputed prices, $(p(t), q(t)) \geqslant 0$ for all t or $(p_t, q_{t-1}) \geqslant 0$ for all t, such that the laws of motion, (8) or (8′), hold along an optimal path. But since, on an optimal path, the utility interest rate is assumed equal to ρ, it is also required that the price of utility output be declining at this rate, that is, that

$$-\frac{\dot{p}(t)}{p(t)} = \rho, \tag{10}$$

or

$$-\frac{(p_t - p_{t-1})}{p_t} = \rho. \tag{10′}$$

If we now choose initial utility as the numeraire commodity, $p(0) = 1$ or $p_0 = 1$, then (10) or (10′) becomes, in closed form,

$$p(t) = e^{-\rho t}, \tag{11}$$

or

$$p_t = (1 + \rho)^{-t}. \tag{11′}$$

Hence, for this problem, on the one hand, the dated prices of utility output are directly specified, by the hypothesis of a constant utility rate of interest, and the choice of initial utility as the numeraire commodity. On the other hand, the initial prices of capital stocks (and thereby saving–investment behavior) are only indirectly specified, by means of a so-called transversality condition,

$$\lim_{t\to\infty} q(t)\, k(t) = 0 \qquad \text{for} \quad \rho > 0, \tag{12}$$

or

$$\lim_{t\to\infty} q_t k_{t+1} = 0 \qquad \text{for} \quad \rho > 0. \tag{12′}$$

In other words, it is finally required that on an optimal path, the present value of the capital stocks tends to zero. Such a transversality condition has been shown to be necessary for optimality in the discrete-time case; see [21, 22, 37]. Under what circumstances it is also necessary for optimality in the continuous-time case is at present an open question. In either case it is clear that, in conjunction with perfect foresight, intertemporal profit maximization (given dated prices of utility output), the transversality condition amounts to a duality theorem for concave programming in some

appropriate infinite-dimensional space.[11] We plan in future research to investigate the necessity of the transversality condition in the continuous-time case by exploiting this duality characterization.

Formulation of the problem of optimal growth with zero discounting is much subtler, because for $\rho = 0$ no constrained maximum to (9) or (9') will exist. If the maximand is changed to

$$\int_0^\infty [c(t) - c^*]\, dt, \tag{13}$$

or

$$\sum_{t=0}^{\infty} (c_t - c^*), \tag{13'}$$

where c^* is golden rule utility output, then a maximum may exist. As before, the laws of motion for the optimal path are given by (8) or (8') and (11) or (11'). But now, the transversality condition, (12) or (12'), is no longer suitable. The condition appropriate to the maximand (13) or (13') (again based on duality considerations) is instead

$$\lim_{t\to\infty} q(t)\, k(t) = Q^*k^*, \tag{14}$$

or

$$\lim_{t\to\infty} q_t k_{t+1} = Q^*k^*, \tag{14'}$$

[11] Namely, that maximizing (9) or (9') subject to (2) or (2') yields the same value as minimizing

$$\int_0^\infty w(t)\dot{l}(t)dt + q(0)\dot{k} \qquad \text{or} \qquad \sum_{t=0}^{\infty} w_t l_t + q_{-1}\dot{k}$$

subject to (3) or (3') plus the constraint (11) or (11'). That the two statements are equivalent is easily seen from analyzing the "feasible" behavior of

$$\int_0^T e^{-\rho t} c(t)dt + q(T)\dot{k}(T) - \int_0^T w(t)\dot{l}(t)dt,$$

or

$$\sum_{t=0}^{T-1} (1+\rho)^{-t} c_t + q_{T-1}k_T - \sum_{t=0}^{T-1} w_t l_t, \qquad \text{for} \quad T \geq 0.$$

Obviously, this particular duality theorem is only a special case of a general duality theorem involving arbitrary, exogenous specification of the weights attached to final output (e.g., terminal capital stocks as well as intermediate utility output).

In this connection, it should also be mentioned that our simplifying assumptions (A1) and (A3) play a role in this particular duality theorem similar to that of Slater's condition in the more familiar duality theorem for concave programming in Euclidean n-dimensional space, and can be weakened accordingly.

where k^* are the golden rule capital stocks, and Q^* the current investment goods prices (in terms of utility output) which support their maintenance.

As has been recognized for some time, the consumption-optimal growth with zero discounting and production-maximal growth models are very closely related, the differences being mainly in interpretation.[18] From our viewpoint, what is crucial is that both models yield canonical Hamiltonian dynamical systems, i.e., dynamical systems which are completely specified in terms of a time-autonomous Hamiltonian function. We have already seen that this is the case for consumption-optimal growth with zero discounting, since when $\rho = 0$, (11) or (11′) yields $p(t) = 1$ or $p_t = 1$, while by assumption $\bar{l}(t) = 1$ or $\bar{l}_t = 1$. For production-maximal growth such time-autonomy derives from the assumptions that there is no consumption goods output and that there is no primary factor constraint, which in our technological framework can be explicitly characterized by $c(t) = 0$ or $c_t = 0$, or better yet, what amounts to the same thing (because of free disposal in allocation (T8) or property (H3)),

$$p(t) = 0, \tag{15}$$

or

$$p_t = 0, \tag{15′}$$

and $\bar{l}(t) = \infty$ or $\bar{l}_t = \infty$. It is interesting to observe that (15) or (15′) already entails something like a transversality condition, since from (8) or (8′) we now have (here appealing to constant returns to scale (T5) or property (H3))

$$q(t)\, k(t) = q(0)\, \bar{k}, \tag{16}$$

or

$$q_t k_t = q_{-1} \bar{k}. \tag{16′}$$

And, indeed, it does turn out that, in effect, the production-maximal growth model is closed merely by postulating (15) or (15′), at least from the perspective of the type of stability analysis we will present (see our Sections 3C and 3D, below).

An alternative method of relating these two models is simply to convert consumption-optimal growth into production-maximal growth. This can be accomplished by treating both cumulated utility output and primary factor input (the latter conceived as a stock yielding service flows) as produced stocks, an approach nicely elaborated by several authors,

[18] We have in mind here the infinite horizon version of the production-maximal growth model, as analyzed, for instance, by Furuya and Inada [10]. Also, our assumptions about technology would have to be modified somewhat in order to be consistent with the discussion below. In particular, capital stock inputs must be limitational for the production-maximal growth model to make any sense.

notably Atsumi [1] and McKenzie [19]. Of course, the resulting model has some characteristics peculiar to itself, that is, that are not commonly found in the standard production-maximal growth models, and thus requires some analysis peculiar to itself.

We will not pursue either line of inquiry involving production-maximal growth models in the present paper, mainly because these models have been so extensively treated in the literature already, but also because the requisite modification of our global stability analysis for optimal growth models is fairly obvious.

Saving-investment constraints like (11) and (12), (11) and (14), and (15) or (11′) and (12′), (11′) and (14′), and (15′) do in fact close the Hamiltonian dynamical system, (8) or (8′), in the following sense: The set of initial prices of capital stocks for which there exists a path satisfying both the Hamiltonian dynamical system and the saving-investment constraints is severely restricted.[13] While this restriction is only indirect, it turns out, as we shall show in Section 3, sufficiently well-specified to render the stability analysis of the optimal (and maximal) growth models in terms of our Hamiltonian formalism almost trivial, basically because of the simple form taken by the utility output prices, (11) or (11′) (and (15) or (15′)).[14]

In descriptive growth theory the Hamiltonian dynamical system,[15] (8) or (8′), is typically closed by appending an instantaneous saving–investment relation of the general form

$$S\left(p(t), -\frac{\dot{p}(t)}{p(t)}, q(t), k(t), l(t)\right) = 0, \tag{17}$$

or

$$S\left(p_t, -\frac{p_t - p_{t-1}}{p_t}, q_t, k_t, l_t\right) = 0. \tag{17′}$$

[13] Regarding optimal growth, the severest restriction occurs when, for instance, T is endowed with sufficient additional structure so that M exhibits the following properties: (i) if $(1, q, r, w) \in M$, then $w \geq 0$, and (ii) if $(1, q^i, r^i, w^i) \in M$ for $i = 1, 2$ are distinct, then $c + [\lambda q^1 + (1-\lambda)q^2]z - [\lambda r^1 + (1-\lambda)r^2]k - [\lambda w^1 + (1-\lambda)w^2] < 0$ for all $0 < \lambda < 1$, $(c, z, -k, -1) \in T$. Then the initial prices of capital stocks are unique. Less onerous additional structure will also suffice for this result.

[14] What renders the analysis so simple is not so much the fact that the interest rate, $-\dot{p}(t)/p(t)$ or $-(p_t - p_{t-1})/p_t$, is constant, as the fact that it is exogenous, that is, is determined independently of output prices and input stocks, $(p(t), q(t), k(t), l(t))$ or (p_t, q_t, k_t, l_t). Indeed, at the end of Section 3 we describe how our results are easily modified to cover the case where the interest rate is only asymptotically constant, though still exogenous.

[15] That competitive dynamical systems can be explicitly represented as Hamiltonian dynamical systems has been more or less commonly recognized among capital theorists for a number of years, especially in relation to prescriptive growth theory. Hahn's fairly specialized version of this fact [13] is the only published use of such representation in descriptive growth theory of which we are aware, though there may well be others.

For global stability analysis, the most critical feature of (17) or (17′) is that the interest rate, $-(\dot{p}(t)/p(t))$ or $-(p_t - p_{t-1})/p_t$, is endogenous, and, hence, varies with the evolution of the economy. A familiar example of such a saving–investment relation is the "Keynesian" hypothesis,

$$q(t)\,z(t) = s[r(t)\,k(t) + w(t)\,\bar{l}(t)], \tag{18}$$

or

$$q_t z_t = s[r_t k_t + w_t \bar{l}_t], \tag{18′}$$

where $0 < s < 1$ is the constant saving rate from NNI.[16] Another well-known example which can be cast in the form (17) or (17′) is the "Marxian" saving–investment hypothesis that wages are consumed, profits are reinvested,

$$q(t)\,z(t) = r(t)\,k(t), \tag{19}$$

or

$$q_t z_t = r_t k_t\,. \tag{19′}$$

In Section 4, we closely examine the stability properties of the descriptive growth model which incorporates the behavior described by (19) or (19′).[17]

Both (18) or (18′) and (19) or (19′) are particular variants of the general linear saving–investment hypothesis

$$q(t)\,z(t) = s_r r(t)\,k(t) + s_w w(t)\,\bar{l}(t) - (1 - s_g)\,p(t)\,\frac{d[q(t)/p(t)]}{dt}\,k(t), \tag{20}$$

or

$$q_t z_t = s_r r_t k_t + s_w w_t \bar{l}_t - (1 - s_g)\,p_t \left[\frac{q_t}{p_t} - \frac{q_{t-1}}{p_{t-1}}\right] k_t\,, \tag{20′}$$

where $0 \leqslant s_r\,, s_w\,, s_g \leqslant 1$ are the constant saving rates from alternative functional income streams, including capital gains relative to consumption,

$$p(t)\,\frac{d[q(t)/p(t)]}{dt}\,k(t) = \left[\dot{q}(t) - \frac{\dot{p}(t)}{p(t)}\,q(t)\right] k(t),$$

[16] Some versions of (18) or (18′) are in terms of gross rather than net quantities; see, for example, [32, 34] Of course, such a specification requires an explicit formulation of the way depreciation occurs, as well as of the technology for producing gross outputs. The standard formulation postulates constant depreciation rates (parallel with the linear form of the saving–investment hypothesis itself).

[17] Notice that Marxian saving–investment behavior has the special property that it can be *entirely* described in terms of the Hamiltonian function. Thus, stability for this model can *only* involve properties of the Hamiltonian function, a special circumstance, which, together with the particular form of Marxian saving–investment behavior, greatly simplifies its analysis.

or

$$p_t\left[\frac{q_t}{p_t}-\frac{q_{t-1}}{p_{t-1}}\right]k_t=\left[(q_t-q_{t-1})-\frac{p_t-p_{t-1}}{p_t}\frac{p_t}{p_{t-1}}q_t\right]k_t\,.$$

Shell, Sidrauski, and Stiglitz [32] give an extended discussion of the rationale behind including capital gains in perceived income, and its implications for capital accumulation.

Clearly, there are many other interesting specifications of (17) or (17'). In any case, the general role of such a saving–investment relation, like that of the saving–investment constraints for the optimal and maximal growth models, is to restrict the permissible solutions to the general Hamiltonian dynamical system, hopefully, enough so that definite qualitative properties can be ascertained. For the purpose of analyzing stability, we have found it useful to conceive of (17) or (17') as possibly restricting (8) or (8') in several different ways: (i) by providing a direct additional restriction on the domain of the Hamiltonian function (and, hence, Hamiltonian dynamical system),

$$(p, q, k, l)\in D=\{(p, q, k, l): S(p, i, q, k, l)=0\}; \tag{21}$$

(ii) by providing a direct or indirect endogenous determination of the interest rate, i (or, because without loss of generality $p(0)=1$ or $p_0=1$, of the price of consumption goods output),

$$i\in I(p, q, k, l) \qquad \text{for} \quad (p, q, k, l)\in D; \tag{22}$$

and (iii) by providing an additional indirect restriction on the initial prices of capital stocks,

$$q(0)\subset \bar{Q}(0) \qquad \text{or} \qquad q_{-1}\in \bar{Q}_{-1}\,. \tag{23}$$

As compared to the optimal or maximal growth models, stability of the descriptive growth model is potentially easier to analyze on account of (21), harder to analyze on account of (22). In our experience with a few leading cases, we have usually found the complexities stemming from (22) to outweigh the simplifications stemming from (21).

3. Stability of the Optimal Growth Model

A. *Preliminary Comments*

Our analysis of stability in the optimal growth model is conducted mainly in terms of continuous time; at the end of the section we briefly outline the parallel analysis in terms of discrete time. A substantial part of

the underlying results, those concerning existence and uniqueness of stationary points, is common to both. When doing so will cause no confusion, we will simply suppress the time variable.

For the optimal growth model, it turns out to be most convenient to conduct the analysis in terms of *current* rather than present values. We denote current prices and rentals by upper case letters,

$$(Q(t), R(t), W(t)) \equiv \frac{(q(t), r(t), w(t))}{p(t)}, \tag{24}$$

or

$$(Q_t, R_t, W_t) \equiv \frac{(q_t, r_t, w_t)}{p_t}. \tag{24'}$$

Then bringing together (8), (11), and (12) or (14), or (8'), (11') and (12'), or (14'), and utilizing our other normalizing assumption (A1) along with this normalizing definition (24) or (24'), the competitive dynamical system for optimal growth can be written compactly as

$$\begin{cases} \dot{k} = \dfrac{\partial H(Q, k)}{\partial Q}, & k(0) = \bar{k}, \\ \dot{Q} = -\dfrac{\partial H(Q, k)}{\partial k} + \rho Q, & Q(0) \geqslant 0, \\ \lim\limits_{t \to \infty} Qe^{-\rho t}k = 0 & \text{for } \rho > 0, \\ \qquad = Q^*k^* & \text{for } \rho = 0, \end{cases} \tag{25}$$

or

$$\begin{cases} k_{t+1} = k_t + \dfrac{\partial H(Q_t, k_t)}{\partial H}, & k_0 = \bar{k}, \\ Q_{t+1} = Q_t - \dfrac{\partial H(Q_{t+1}, k_{t+1})}{\partial k} + \rho Q_t, & Q_{-1} \geqslant 0, \\ \lim\limits_{t \to \infty} Q_t(1 + \rho)^{-t} k_{t+1} = 0 & \text{for } \rho > 0, \\ \qquad = Q^*k^* & \text{for } \rho = 0, \end{cases} \tag{25'}$$

where we define the current value Hamiltonian function $H(Q, k) \equiv H(1, Q, k, 1) \equiv H(1, q/p, k, 1)$, the current value of imputed NNP (= the current value of imputed NNI).

B. *Stationary Points*

As done earlier in our discussion of transversality conditions, we denote the variables corresponding to stationary points for the system (25) or (25') with asterisks. Thus, in particular,

$$z^* = \frac{\partial H(Q^*, k^*)}{\partial Q} = 0,$$
$$-R^* + \rho Q^* = -\frac{\partial H(Q^*, k^*)}{\partial k} + \rho Q^* = 0. \tag{26}$$

Also, as is conventional, we will refer to stationary points, (Q^*, k^*), (and, occasionally and loosely, other of the variables corresponding to them, R^*, W^*, c^*, and z^*) as modified golden rule paths.

If the discount rate, ρ, is too high, then there may be no (nontrivial) modified golden rule path. However, utilizing our various assumptions about the technology it is fairly straightforward to establish the

EXISTENCE OF A MODIFIED GOLDEN RULE PATH. *Let*

$$\bar{\rho} = \sup\{\rho: (c, z, -k, -l) \in T, (c, z - \rho k) > 0\},$$

and suppose $\rho \in [0, \bar{\rho})$. *Then there exists* $(1, Q^*, R^*, W^*) \in M$ *and* $(c^*, z^*, -k^*, -1) \in T$ *such that* (i) $R^* = \rho Q^*$, (ii) $c^* > 0$, $z^* = 0$, *and* (iii) $c^* + Q^*z^* - R^*k^* - W^* = 0$.

Proof. The proof consists simply in carrying out the program outlined in [6, pp. 289–90]. In particular, we assert (without going into details) that (i) the mapping from the nonempty, compact, convex set $K = \{k: (c, z, -k, -l) \in T, \|(c, z, k)\| \leq B \text{ and } l \leq 1\}$ into itself defined by

$$\Phi(k) = \{k': (c', z', -k', -l') \text{ is an optimal solution to maximize } c' \text{ subject to } (c', z', -k', -l') \in T, \|(c', z', k')\| \leq B, z' \geq \rho k', l' \leq 1\} \quad \text{for } k \in K \tag{27}$$

is a nonempty, convex-valued, upper semicontinuous correspondence (so that Kakutani's fixed point theorem can be appealed to), and (ii) a fixed point of the mapping (27), say k^*, yields a (nontrivial) modified golden rule path as optimal solutions to the dual concave and convex programming problems

$$\text{maximize } c \quad \text{subject to } (c, z, -k, -l) \in T, z - \rho k \geq -\rho k^*, l \leq 1,$$

and

$$\text{minimize } W + \rho Q k^* \quad \text{subject to } (p, Q, R, W) \in M, R - \rho Q = 0, p = 1.^{18}$$

[18] It is worth noting, referring to the property (B), that only the bound associated with replicable production points is actually relevant to this argument. And this particular bound enters in two ways, first, to provide the compactness required for the application of Kakutani's theorem, and second, to permit dropping the constraint $\|(c, z, k)\| \leq B$, once the existence of a fixed point has been established.

Since, just to be able to talk about stability, we will require the existence of some (nontrivial) modified golden rule path, hereafter we simply hypothesize:

EXISTENCE ASSUMPTION. $\rho \in [0, \bar{\rho})$. (E)

Notice that, as in the one-good model satisfying Inada's conditions, $\bar{\rho} = \infty$ is possible.

We next take up the question of the uniqueness of *the* modified golden rule capital stocks, k^*, which in our approach turns out closely related to the question of the convergence of the optimal paths defined by (25) to *some* modified golden rule capital stocks, $k^*, k^{**}, \ldots$. Our strategy is to find conditions in terms of the geometry of the current value Hamiltonian, $H(Q, k)$, which insure both uniqueness and convergence, the two properties which together define global stability, or simply, stability.

Though the zero discount rate case has been quite extensively treated in the literature, we begin our discussion with a full account of that special case. We do so partly to provide a self-contained, complete analysis of stability for optimal growth, but mostly to provide an intuitive, heuristic motivation for our later discussion of the general case. We should expressly warn the reader at the outset that, aside from the statement and verification of various uniqueness and stability conditions, the balance of this subsection is mostly devoted to motivating, explaining and interpreting, in other words, to exposition, not analysis.

If $\rho = 0$, then a stationary point, (Q^*, k^*) (now referred to as a golden rule path), must be a saddle point of the current value Hamiltonian, $H(Q, k)$,

$$H(Q^*, k) \leqslant H(Q^*, k^*) \leqslant H(Q, k^*) \qquad \text{for all} \quad (Q, k) \geqslant 0,$$

or

$$H(Q, k^*) - H(Q^*, k) \geqslant 0 \qquad \text{for all} \quad (Q, k) \geqslant 0, \quad (28)$$

since it satisfies (26), and $H(Q, k)$ is convex in Q and concave in k. Now suppose k^* were not unique, i.e., that (28) were also to hold with (Q^{**}, k^{**}) in place of (Q^*, k^*), where $k^{**} \neq k^*$. Then (28) would also achieve its minimum value of zero at (Q^{**}, k^{**}) as well as (Q^*, k^*). This suggests that any reasonably general condition which guarantees that k^* is unique must also, in effect, guarantee that (28) is a strict inequality for $k \neq k^*$.[19] Thus, one obvious sufficient condition for uniqueness is simply

[19] As Lionel McKenzie has emphasized, there are many specific technologies for which the last is not true, and yet k^* is unique. Uzawa's two-sector model is perhaps the best known of these (see Footnote 31 below). The point is that what we are concerned with here are conditions which don't require more detailed specification of the technology, for example, that it has an industry or sectoral structure.

that $H(Q, k)$ be strictly concave in k (symmetrically, that $H(Q, k)$ be strictly convex in Q is a sufficient condition for uniqueness of Q^*; see, for example, [25]). However, this is clearly stronger than needed, since the equally obvious direct condition, possibly valid without strict concavity, works just as well.

UNIQUENESS ASSUMPTION FOR $\rho = 0$.

$$k \neq k^* \Rightarrow H(Q, k^*) - H(Q^*, k) > 0, \tag{29}$$

or equivalently,[20]

$$k \neq k^* \Rightarrow (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) > 0. \quad (U^0)$$

(U^0) can be aptly called the *real Hamiltonian steepness condition*, since (29) means that looking from (Q^*, k^*), the Hamiltonian function, $H(Q, k)$, has negative steepness in all directions $(0, k - k^*) \neq 0$, or alternatively, that when $k \neq k^*$, the convex function $H(Q, k^*) - H(Q^*, k)$ lies strictly above its horizontal support at $(Q, k) = (Q^*, k^*)$.

Convergence to the golden rule capital stocks, k^*, can be established if our real Hamiltonian steepness condition holds uniformly in k.

STABILITY ASSUMPTION FOR $\rho = 0$. For every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|k - k^*\| \geq \epsilon \Rightarrow (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) \geq \delta. \quad (S^0)$$

The proof of this assertion will be a particular application of the proof for the general model when $\rho \geq 0$, presented in the next subsection.

[20] This equivalence rests heavily on the convexity–concavity of $H(Q, k)$. On the one hand, (29) implies (U^0), since

$$\begin{aligned}(Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) &= \left[H(Q, k) + \frac{\partial H(Q, k)}{\partial k}(k^* - k)\right] \\ &\quad - \left[H(Q, k) + (Q^* - Q)\frac{\partial H(Q, k)}{\partial Q}\right] \\ &\geq H(Q, k^*) - H(Q^*, k).\end{aligned}$$

On the other hand, denial of (29) implies the denial of (U^0), since if $k' \neq k^*$ and $H(Q', k^*) - H(Q^*, k') = 0$, then $H(Q^*, k') = H(Q^*, k^*)$, $\partial H(Q^*, k')/\partial k = 0$, and $(Q^* - Q^*)(\partial H(Q^*, k')/\partial Q) - (\partial H(Q^*, k')/\partial k)(k - k^*) = 0$.

(U^0) and (S^0) can be interpreted in straightforward economic terms, and thereby related to the previous literature. Expanding the basic expression in both conditions yields

$$
\begin{aligned}
&(Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) \\
&\quad = (Q - Q^*)\, z - R(k - k^*) \\
&\quad = (Q - Q^*)(z - z^*) - (R - R^*)(k - k^*) \\
&\qquad \text{(since by definition } z^* = R^* = 0\text{)} \\
&\quad = (1 - 1)(c - c^*) + (Q - Q^*)(z - z^*) \\
&\qquad - (R - R^*)(k - k^*) - (W - W^*)(1 - 1) \\
&\quad = -(c + Q^*z - R^*k - W^*) - (c^* + Qz^* - Rk^* - W) \\
&\qquad \text{(since by hypothesis} \\
&\qquad c + Qz - Rk - W = c^* + Q^*z^* - R^*k^* - W^* = 0\text{).}
\end{aligned}
\tag{30}
$$

Now let $x = (c, z, -k, -1)$ denote any efficient production point and $\pi = (1, Q, R, W)$ its associated current competitive prices. Then the last line in (30) becomes

$$-\pi x^* - \pi^* x, \tag{31}$$

the sum of the potential losses from adopting the golden rule production point x^* at current competitive prices π plus those from adopting the efficient production point x at golden rule prices π^*. Thus, the conditions (U^0) or (S^0) are nothing more than the requirements that these potential losses be positive or uniformly positive, respectively, when an efficient production point does not utilize golden rule capital stocks. In particular, (S^0) is therefore seen to be nothing but a symmetric adaptation of Radner's bounded value-loss condition (introduced in [23, pp. 101–2]) to the present model.[21]

We now turn to consideration of the general optimal growth model with constant, nonnegative discounting, $\rho \geqslant 0$. As we have just seen, the

[21] For the same sort of reason that (U^0) and (29) are equivalent, (S^0) is also easily shown to be equivalent to a direct analog of Radner's bounded value-loss condition, namely

$$\| k - k^* \| > \epsilon \Rightarrow \pi^* x > \delta.$$

Indeed, if we convert the consumption-optimal growth model with $\rho = 0$ into a production-maximal growth model in the manner suggested earlier, then the latter condition is precisely the appropriate bounded value-loss condition (that is, after taking account of how this artificial variant differs from the standard formulation of the production-maximal growth model).

analysis of uniqueness and convergence when $\rho = 0$ hinges on the fact that then (Q^*, k^*) is a saddlepoint of $H(Q, k)$. When $\rho > 0$, this saddlepoint property no longer obtains. Notice, however, that for the general model, a modified golden rule path, (Q^*, k^*), is *something like* a saddlepoint for the modified current value Hamiltonian function, $H(Q, k) - \rho Qk$. That is, from (26) and convexity–concavity, we have

$$H(Q^*, k) - \rho Q^* k \leqslant H(Q^*, k^*) - \rho Q^* k^* \leqslant H(Q, k^*) - \rho Q^* k^* \quad \text{for all} \quad (Q, k) \geqslant 0,$$

or

$$H(Q, k^*) - \rho Q^* k^* - [H(Q^*, k) - \rho Q^* k] \geqslant 0 \quad \text{for all} \quad (Q, k) \geqslant 0,$$

or

$$H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) \geqslant 0 \quad \text{for all} \quad (Q, k) \geqslant 0. \tag{32}$$

Obviously, the reason we say "something like" has to do with the asymmetry of terms involving ρ. Applying reasoning similar to that leading to (29) and (U^0) now results in both strong and weak sufficient conditions for uniqueness of k^* (again because of asymmetry in terms involving ρ, which in effect makes the comparison involved in (U^s) below quantitative, rather than qualitative, as in that involved in (29) above.)

UNIQUENESS ASSUMPTION FOR $\rho \geqslant 0$ (STRONG VERSION).

$$k \neq k^* \Rightarrow H(Q, k^*) - H(Q^*, k) + \rho Q(k - k^*) > 0,$$

or

$$k \neq k^* \Rightarrow H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*) \tag{U^s}$$

and

UNIQUENESS ASSUMPTION FOR $\rho \geqslant 0$.

$$k \neq k^* \Rightarrow (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*). \tag{U}$$

We repeat for emphasis: By convexity-concavity, (U^s) implies (U), but the converse implication is generally false unless $\rho = 0$, in which case (U) simply reduces to (U^0).

The geometric and economic interpretations of (U) are not nearly as transparent for $\rho > 0$ as they were for $\rho = 0$. Some idea of what (U)

means geometrically can be gleaned from the observation that its strengthening (U^s) amounts to a modified real Hamiltonian steepness condition: When $k \neq k^*$, the convex function $H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*)$ lies strictly above the indefinite quadratic $-\rho(Q - Q^*)(k - k^*)$. This interpretation also reemphasizes the self-evident importance of variation in Q (as well as k) in the statement (and verification) of either (U^s) or (U). Notice especially that both conditions are automatically satisfied whenever $-\rho(Q - Q^*)(k - k^*) < 0$ because of convexity–concavity and (32).

Again reasoning from the case $\rho = 0$, some idea of the economic content of (U) can be gotten from repeating the exercise which yielded (30), and which here leads to exactly the same expression, since on a modified golden rule path $z^* = 0$ but $R^* = \rho Q^*$. Thus, the fundamental inequality in (U) can be expressed as

$$-\pi x^* - \pi^* x > -\rho(Q - Q^*)(k - k^*)$$

or

$$-\pi x^* - \pi^* x > \rho Q^*(k - k^*) + \rho Q(k^* - k), \tag{33}$$

"the sum of potential losses exceeds the sum of potential interest." The idea embodied in (33) seems to revolve around what it denies; figuratively speaking, if (33) were not true, then there would appear to be potential gains in trade away from the modified golden rule path. Or putting it another way, and getting a little bit ahead of our story (see (S) and the subsequent argument below), society may not countenance growth toward the modified golden rule capital stocks if, when once achieved, it would then appear that society could actually have done better elsewhere.[22]

We close this subsection by again remarking that all the foregoing discussion applies equally well to both continuous time and discrete time.

C. *Stability Analysis*

We suppose that a solution to (25) exists. Then, consider the behavior of the (Lyapunov) current valuation function

$$V = (Q - Q^*)(k - k^*). \tag{34}$$

[22] A more concrete interpretation may be possible. When $\rho = 0$, (S^0) can be used to show that *any* path converging to the golden rule capital stocks is superior to *all* paths which do not (see, for example, [11]). When $\rho > 0$, no such very crude comparison can be coaxed from (S), since, so to speak, the initial evolution of a path weighs much more heavily than its terminal evolution. It is an interesting, open question whether, nevertheless, some much more delicate but just as easily interpretable comparison can be developed on the basis of (S).

Direct calculation shows that

$$\begin{aligned}\frac{d(Ve^{-\rho t})}{dt} &\sim \dot{V} - \rho V \\ &= (Q - Q^*)\, k + \dot{Q}(k - k^*) - \rho(Q - Q^*)(k - k^*) \\ &= (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \left(\frac{\partial H(Q, k)}{\partial k} - \rho Q\right)(k - k^*) \\ &\quad - \rho(Q - Q^*)(k - k^*) \\ &= (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \left(\frac{\partial H(Q, k)}{\partial k} - \rho Q^*\right)(k - k^*) \\ &\geqslant H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) \\ &\geqslant 0, \end{aligned} \tag{35}$$

the last two inequalities again following from convexity–concavity of $H(Q, k)$ and the "saddlepoint" property of (Q^*, k^*), (32). Furthermore, the transversality condition in (25) requires that

$$\begin{aligned}\lim_{t\to\infty} Ve^{-\rho t} &= \lim_{t\to\infty} (Q - Q^*)(k - k^*)\, e^{-\rho t} \\ &\leqslant \lim_{t\to\infty} (Qe^{-\rho t}k + Q^*e^{-\rho t}k^*) \\ &= 0 \qquad \text{for} \quad \rho > 0 \\ &(= 2Q^*k^* \qquad \text{for} \quad \rho = 0). \end{aligned} \tag{36}$$

Hence, since $Ve^{-\rho t}$ is increasing and nonpositive for $\rho > 0$ (bounded for $\rho = 0$), V must have an upper bound, say, $V \leqslant V^* < \infty$. If, in addition, we knew that $k \neq k^*$ entails $\dot{V} > 0$ (with some uniformity in the relation between the size of $(k - k^*)$ and the size of $\dot{V}$), then a routine argument (the details of which are provided below) would establish that

$$\lim_{t\to\infty} k = k^*.^{23}$$

[23] The reader may have begun to wonder why we focus on stability in terms of k rather than in terms of (Q, k). One reason is the fact that we don't even know for sure that Q is bounded (though $Qe^{-\rho t}$ likely is). Thus, the sort of analysis we present below concerning k cannot be carried out concerning Q. Perhaps a better (economic) reason is that we don't really care about the asymptotic behavior of prices in this context. The interesting aspect of the modified golden rule path is its real, not its price side, and we may well get convergence to the former even when Q^* is not unique. For example, the special model where $n = 1$ and $H(Q, k) = (1 + Q)f(k) - \lambda Qk$ can be employed to construct a solution to (25) on which, because k^* is, but Q^* is not unique, for given (Q^*, k^*), $\lim_{t\to\infty} k = k^*$, but $\lim_{t\to\infty} Q \neq Q^*$.

How do we insure the last? Casual inspection of the expressions in (35) suggests simply requiring that (U^g) or (U) hold uniformly in k.

STABILITY ASSUMPTION FOR $\rho \geqslant 0$ (STRONG VERSION). For every $\epsilon > 0$, there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*) + \delta. \quad (S^g)$$

and

STABILITY ASSUMPTION FOR $\rho \geqslant 0$. For every $\epsilon > 0$, there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$(Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*) + \delta. \quad (S)$$

The reader may want to check for himself whether (S) holds in particular examples. Consider, for instance, the one-good model with a strictly concave intensive production function, $f(k) = F(k, 1)$, a constant depreciation rate, $f'(0) > \lambda > f'(\infty) \geqslant 0$, and utility output linear in consumption output, utility output $= c$ (without loss of generality). In this model $H(Q, k) = \max[1, Q]f(k) - \lambda Qk$, so that (S) holds when the discount rate, ρ, is sufficiently small. Notice that for this model, or its standard variants,[24] even our modified real Hamiltonian steepness condition (U) is far stronger than required for convergence. This is because, with a single capital good, the (Q, k) space is two-dimensional, while with both a single capital good and an intensive production technology which is strictly convex in capital intensity, the real side of an optimal path is unique. Thus, orbiting is not possible. Moreover, since the intensive production technology does not permit an unbounded capital intensity, the real side of an optimal path is also bounded. Hence, for such models, an optimal path must converge to *some* modified golden rule capital stock.

However, one should not be misled by these simple examples. In general, on optimal paths which are convergent, the valuation function, V, is also convergent, but not necessarily monotonically. This strongly suggests to us that any general argument for stability is going to require that V (or something very like it) be monotonically increasing, which in turn strongly

[24] See for example, [5, 15], or, for a pedagogic survey, [31]. A critical feature of all these models is that there is only a single capital good.

suggests that (S) (or something very like it) is the weakest general stability condition possible. Putting the matter another way, it seems very unlikely to us that, in more than two dimensions, orbiting on an optimal path can be ruled out unless some form of real Hamiltonian steepness obtains.[25]

Returning to the main point of this subsection, we now explicitly demonstrate the

STABILITY OF OPTIMAL GROWTH WITH CONSTANT, NONNEGATIVE DISCOUNTING. *Assumptions* (E) *and* (S) *imply that the real side of the solutions to* (25) *converges to unique (nontrivial) modified golden rule capital stocks.*

Proof. To begin with, we have from the definition of V and the stability condition (S) that

$$\begin{aligned}
\dot{V} &= (Q - Q^*)\dot{k} + \dot{Q}(k - k^*) \\
&= (Q - Q^*)\frac{\partial H(Q, k)}{\partial k} - \left(\frac{\partial H(Q, k)}{\partial k} - \rho Q\right)(k - k^*) \\
&= \left[(Q - Q^*)\frac{\partial H(Q, k)}{\partial k} - \left(\frac{\partial H(Q, k)}{\partial k} + \rho Q^*\right)(k - k^*)\right] \\
&\quad + \rho(Q - Q^*)(k - k^*) \\
&\geq 0,
\end{aligned}$$

and from (35) and (36) that

$$V \leq V^* < \infty.$$

Hence, putting these together, we have that

$$\lim_{t \to \infty} V = V^\infty \leq V^* < \infty.$$

Suppose that $\lim_{t\to\infty} k = k^*$ were not true, i.e., that for some $\epsilon > 0$, there were a sequence of points $\{t_j\}$ such that $\| k(t_j) - k^* \| > 2\epsilon$. Then, since (B) and (T7) entail uniform continuity of $k(t)$ on the half line $[0, \infty)$, there would also have to be a sequence of intervals $\{[\underline{t}_j, \bar{t}_j]\}$ such $\bar{t}_j - \underline{t}_j >$

[25] Perhaps this is an appropriate place to point out that, when stripped of our particular interpretation, our stability argument applies to any sufficiently regular dynamical system of the form (25), with or without convexity, provided the right-hand side of the inequality in (S) is further strengthened to read $\max[0, -\rho(Q - Q^*)(k - k^*) + \delta]$. (This strengthening incorporates the one direct implication of convexity–concavity actually used in the stability argument, namely, the final inequality in (35).) Of course, without convexity, neither the transversality condition (more generally, boundedness) nor the steepness condition (more generally, definiteness) typically makes much sense, which for us significantly reduces possible interest in such generalization.

$\Delta t > 0$ and $\| k(t) - k^* \| > \epsilon$ for $t \in [\underline{t}_j, \bar{t}_j]$. But this would imply, now utilizing condition (S) to bound uniformly the derivative

$$\dot{V}(t) = \Big[(Q(t) - Q^*) \frac{\partial H(Q(t), k(t))}{\partial Q} - \Big(\frac{\partial [H(Q(t), k(t))}{\partial k} + \rho Q^*\Big)(k(t) - k^*)\Big] + \rho(Q(t) - Q^*)(k(t) - k^*),$$

that there is a $\delta > 0$ such that

$$\dot{V}(t) > \delta \qquad \text{for} \quad t \in [\underline{t}_j, \bar{t}_j].$$

Hence, for sufficiently large t' we would have both $V^\infty - \delta \leqslant V(t) \leqslant V^\infty$ and

$$V(t) \geqslant V(t') + \sum_{t' \leqslant \underline{t}_j < \bar{t}_j < t} \delta(\bar{t}_j - \underline{t}_j)$$

$$\geqslant V(t') + [\max\{j: t' \leqslant \underline{t}_j \leqslant \bar{t}_j \leqslant t\} - \min\{j: t' \leqslant \underline{t}_j \leqslant \bar{t}_j \leqslant t\}]\, \delta \Delta t$$

for $t \geqslant t'$, which are inconsistent, and the theorem is proved.

It should be clear that stronger "$\epsilon - \delta$" conditions would suffice for our argument, for instance (S^a), or the conditions used by Rockafellar [26], or the conditions used by Brock and Scheinkman [3].

We close this discussion by commenting briefly on the optimal growth model with a discount rate, $\rho(t) = -\dot{p}(t)/p(t)$, which is only asymptotically constant and nonnegative, $\lim_{t \to \infty} \rho(t) = \rho \geqslant 0$.[26] For this model, we have

$$\dot{V} = (Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho(t)\, Q(k - k^*)$$

$$= (Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) + \rho(Q - Q^*)(k - k^*) + (\rho(t) - \rho)(Q - Q^*)(k - k^*) + (\rho(t) - \rho)\, Q^*(k - k^*).$$

Since the last term in this expansion may be negative (the second to the last term may be negative too, but can eventually be safely neglected), our

[26] Symmetrically, we now assume that the valuation function, $V(t)$, is only uniformly bounded from above, $V(t) < V^* < \infty$ for all t. It is perhaps worth noting that this discussion applies equally well to any descriptive growth model for which these same properties, $\lim_{t \to \infty} \rho(t) = \rho > 0$ and $V(t) < V^* < \infty$ for all t, can be established.

argument above will not go through without some further strengthening of our modified real Hamiltonian steepness condition (U) beyond (S), basically to negate the effect of Q potentially being "too far" from Q^*. In particular, if we substitute the quantity $\max[\delta, -\delta(Q - Q^*)(k - k^*)]$ for the constant δ in the right-hand side of the inequality in (S), then the proof of stability involves only relatively minor modification, the details of which we omit.[27]

D. *Discrete Time*

Recall that the analysis of existence and uniqueness of stationary points is the same in either continuous time or discrete time. The reason for this fortunate circumstance is that, for an economy in balanced growth, currently compounding interest on this period's capital value (continuous time) is indistinguishable from accrued simple interest on last period's capital value (discrete time), provided both interest rates are the same. For the analysis of behavior away from stationary points, however, the intrinsic difference in the timing of interest calculations between continuous time and discrete time becomes of paramount importance.

In terms of modifying our analysis in the preceding subsection to accomodate discrete time, this shows up almost immediately once we write down the appropriate analogs to (34) and (35).

$$V_t = (Q_{t-1} - Q^*)(k_t - k^*) \tag{34'}$$

and

$$\begin{aligned}
&V_{t+1}(1+\rho)^{-(t+1)} - V_t(1+\rho)^{-t} \\
&\quad \sim V_{t+1} - (1+\rho)V_t \\
&\quad = (Q_t - Q^*)(k_{t+1} - k^*) - (1+\rho)(Q_{t-1} - Q^*)(k_t - k^*) \\
&\quad = (Q_t - Q^*)\left[\frac{\partial H(Q_t, k_t)}{\partial Q} + (k_t - k^*)\right] \\
&\qquad - (1+\rho)\left[\frac{1}{1+\rho}\left(\frac{\partial H(Q_t, k_t)}{\partial k} + Q_t\right) - Q^*\right](k_t - k^*) \\
&\quad = (Q_t - Q^*)\frac{\partial H(Q_t, k_t)}{\partial Q} - \frac{\partial H(Q_t, k_t)}{\partial k}(k_t - k^*) - \rho Q^*(k_t - k^*) \\
&\quad \geqslant H(Q_t, k^*) - H(Q^*, k_t) - \rho Q^*(k_t - k^*) \\
&\quad \geqslant 0.
\end{aligned} \tag{35'}$$

[27] Except to remark, for those interested in the details, that the essential difference comes in restricting attention to t sufficiently large so that

$$\max[|\rho(t) - \rho|, |\rho(t) - \rho|V^*, |(\rho(t) - \rho)Q^*(k - k^*)|] < \delta/3.$$

Equations (34′) and (35′) again lay a solid foundation for a routine argument establishing stability, provided only that the following inequality holds uniformly in $k_t \neq k^*$:

$$V_{t+1} - V_t = (Q_t - Q^*)\frac{\partial H(Q_t, k_t)}{\partial Q} - \frac{\partial H(Q_t, k_t)}{\partial k}(k_t - k^*) + \rho Q_{t+1}(k_t - k^*) > 0. \quad (37)$$

But notice the slippage of the time subscript in the last term of the middle expression! While this natural slippage highlights the fundamental character of the difference between continuous time and discrete time, the obvious method for getting around it also reemphasizes the central importance of both the price and quantity equations in our analysis. For, once more utilizing the second equation in (25′), we see that

$$\rho Q_{t+1}(k_t - k^*) = \frac{\rho}{1+\rho}\left(\frac{\partial H(Q_t, k_t)}{\partial Q} + Q_t\right)(k_t - k^*). \quad (38)$$

Hence, substituting from (38) into (37), and carrying out some algebraic rearrangement yields the direct analog of (S) for discrete time.

STABILITY ASSUMPTION FOR $\rho \geq 0$ (DISCRETE TIME). For every $\epsilon > 0$, there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$(1+\rho)(Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*) + \delta. \quad (S')$$

But now notice the addition of the factor $1 + \rho$ in the first term of the lefthand side!

Some brief reflection on the distinction between the stability conditions (S′) and (S) is revealing. Evidently, they will generally coincide only with a zero discount rate, $\rho = 0$. For a given current Hamiltonian function, $H(Q, k)$, and positive discount rate, $\rho > 0$, both, just one or the other, or neither may hold. Hence, in particular, (S′) may hold when (U), and, therefore, (S) as well, does not; in other words, for the system (25′), establishing convergence may be possible when establishing stability is not.[28] For example, in the one-good model mentioned in the last sub-

[28] Thus, it would be more accurate to refer to (S′) as a convergence, rather than a stability condition. However, we have yet to find a simple example in which (S′) holds, (U) does not, and there are actually several stationary capital stocks.

The reader should beware the fact that the comparison we are making here is somewhat artificial, since we are considering both continuous- and discrete-time conditions in terms of the *same* Hamiltonian function.

section, (S') is in fact slightly weaker than (S). For another example, consider the (mathematically simplest) quadratic Hamiltonian model where $\nu = 1$ and $H(Q, k) = (\alpha/2)(Q - Q^*)^2 + \gamma(Q - Q^*)(k - k^*) - (\beta/2)(k - k^*)^2 + \rho Q^*(k - k^*)$ with $\alpha, \beta > 0$. Here (S') amounts to the requirement that $4\alpha\beta > [(1 + \gamma)^2/(1 + \rho)]\rho^2$, but (U) or (S) to the requirement that $4\alpha\beta > \rho^2$. Clearly, since $-1 \leqslant \gamma \leqslant \infty$ entails $0 < (1 + \gamma)^2/(1 + \rho) < \infty$, any one of the four combinations of validity for these two parametric conditions is possible.

But then we really should not be surprised that the alternative treatments of time are just not the same. On the one hand, as we demonstrated earlier in the paper, the standard single capital good models must eventually converge to some modified golden rule capital stock in continuous time. On the other hand, Sutherland's illuminating example [35, example on p. 588] taught us some time ago that these same models may forever orbit in discrete time. (Incidentally, Sutherland's particular model essentially reduces to the quadratic Hamiltonian model with parameter values such that $4\alpha\beta = 23/16 < [(1 + \gamma)^2/(1 + \rho)]\rho^2 = 121/48 < \rho^2 = 4$.)

The general problem of relating results for continuous time with those for discrete time seems to us worth further, detailed investigation.

4. Stability of the Descriptive Growth Model with Marxian Saving-Investment Behavior

A. *Preliminary Comments*

It was our original hope that our Hamiltonian approach would provide a basis for unifying the stability analysis of maximal, optimal, and descriptive growth models. While we have not yet given up that hope, we are also not yet prepared to handle anything like the descriptive growth model with a general saving-investment relation (17) or (17'). Instead, the discussion of this section deals only with the Marxian saving-investment hypothesis, and that only in continuous time.

To the experienced capital theorist, it comes, of course, as no surprise that stability for descriptive growth models is much more fragile, that is, requires much more structure, than stability for optimal growth models. Even with a single capital good, perpetual orbiting in a descriptive growth setting is not an uncommon phenomenon. Just consider, for instance, the special saving-investment hypothesis (19) or (19') we shall later concentrate attention on. Right away we have to contend with the well-known example of Inada [14], based on Uzawa's [36] famous two-sector model,

which shows that orbiting may arise whenever the investment-goods sector is more capital intensive than is the consumption-goods sector. Moreover, we must also contend with the fact that, in discrete time, even the one-good model with Marxian saving–investment behavior is likely to exhibit orbiting.[29]

In purely technical terms, there are major obstacles to simply extending the analysis of the previous section to descriptive growth models. For example, when the interest rate is endogenous and depends on the state of the economy, what particular value of the interest rate should form the basis for our stability condition (S)? We have seen that even the seemingly slight change from an exogenous, interest rate which is constant to one which is only asymptotically constant required some additional tightening of that condition. Or, for another example, what economic forces will insure boundedness of our valuation function in a decentralized, competitive dynamical system? There seems to be no intrinsic reason why every such system should mimic even a sufficiently constrained optimization problem, and therefore exhibit this more complete form of duality.

The worked-out examples of descriptive growth models all seem to have two distinctive characteristics: The underlying technology is simple enough so that it can be completely parameterized, while the saving–investment relation (typically it too can be completely parameterized) is special enough so that it severely restricts the domain of solutions to the Hamiltonian dynamical system. These special properties are then heavily exploited to show directly that paths which do not converge must eventually leave the permissible domain of solutions. Can these arguments even be systematically extended to a general model using our Hamiltonian approach? At the end of the section we will briefly comment on the extent of our success relative to the worked-out examples with Marxian saving–investment behavior (expecially [8, 14, 33, 36]), and on the possible future directions for applying our Hamiltonian approach to descriptive growth models with more general saving–investment behavior.

B. *The Formal Model*

In this section the choice of numeraire is fairly complicated. In general, the prices will be present values, but with the numeraire implicitly chosen so that the initial value of capital stocks equals the (constant) current value of capital stocks along the golden rule path. The express motivation

[29] In fact, given any three capital intensities, $0 < k_0 < k^* < k_1 < \infty$, it is easy to construct a strictly increasing, strictly concave intensive production function such that the golden rule capital stock is k^*, but the only solution for $k = k_0$ is

$$k_{2t} = k_0, k_{2t+1} = k_1 \quad \text{for} \quad t = 0, 1, \ldots.$$

for this particular choice, like that for the subsequent analysis, is that our results generalize particular properties intrinsic to the worked-out examples mentioned above. However, at several points in the discussion (notably, in the statement of uniqueness and stability conditions) it will be convenient to revert to current values (recalling that $(Q, R, W) = (q, r, w)/p$). Also, we will find it convenient to alternate freely between the Hamiltonian and conventional representations of the model. Finally, because we are only analyzing the model in continuous time, we will suppress the time variable.

The complete competitive dynamical system incorporating all of our maintained assumptions about technology, resources, and behavior can be written concisely as

$$\begin{cases} \dot{k} = \dfrac{\partial H(p, q, k, 1)}{\partial q} = \dfrac{\partial H(Q, k)}{\partial Q} = z, & k(0) = \bar{k} > 0, \\ \dot{q} = -\dfrac{\partial H(p, q, k, 1)}{\partial k} = -p\dfrac{\partial H(Q, k)}{\partial k} = -r, & q(0) \geqslant 0, \\ p\dfrac{\partial H(p, q, k, 1)}{\partial p} = pc = \dfrac{\partial H(p, q, k, 1)}{\partial l} l = w, \end{cases} \tag{39}$$

where again we use $H(Q, k) = H(1, q/p, k, 1)$. Since this model would not be a very interesting description of an economy otherwise, we append one further restriction on its solutions, that

$$w > 0, \tag{40}$$

the present wage rate, and, hence, current consumption output and its present value are all positive. The basis of our whole analysis is the simple observation that, because of linear homogeneity of H in (p, q) and the last equation in (39),

$$(\dot{qk}) = q\dot{k} + \dot{q}k = q\frac{\partial H(p, q, k, 1)}{\partial q} - \frac{\partial H(p, q, k, 1)}{\partial k}k = qz - rk =$$

$$(pc + qz - rk - w) - (pc - w) = -(pc - w) = 0$$

or

$$qk = q(0)\,\bar{k}, \text{ a constant.} \tag{41}$$

C. *Stationary Points*

As in our earlier discussion of optimal growth, we are interested in stationary points defined in terms of current values (even though our competitive

dynamical system is defined in terms of present values). The only stationary points of this sort which are consistent with (39)–(40) are those at which either the present value of investment-goods output is zero, i.e., the interest rate is not zero, or at which all present values are constant, i.e., the interest rate is zero. Existence of the latter, that is, of a golden rule path, is guaranteed by our earlier argument. Thus, for our purposes now, we need a somewhat different uniqueness assumption, namely the

POSITIVITY AND UNIQUENESS ASSUMPTION. (i) $Q^*k^* > 0$, and (ii)

$$k \neq k^* \Rightarrow (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) = (Q - Q^*)z - R(k - k^*) > 0 \quad \text{(P)}$$

when $Q(\partial H(Q, k)/\partial Q) = Qz = (\partial H(Q, k)/\partial k)\, k = Rk$.

Various additional restrictions on the technology (having to do with the necessity and productivity of capital) would entail positivity $Q^*k^* > 0$. The second part of (P) is essentially a weakening of (U^0) which focuses on just those perturbations satisfying the saving–investment hypothesis; it would obviously be satisfied whenever (U^0) is.

Given the first part of (P), if $q(0)\,\bar{k} > 0$, then the price side of the solutions to (39) can be normalized so that $qk = q(0)\,\bar{k} = Q^*k^*$.[30] Thus, we have either

$$qk = 0 \qquad \text{or} \qquad qk = Q^*k^* > 0. \tag{42}$$

D. *Stability Analysis*

Here, we focus our attention on a current valuation function, $v = (q - Q^*)(k - k^*)$. For this system, from (42), v is bounded by

$$v = (q - Q^*)(k - k^*) = qk - qk^* - Q^*k + Q^*k^* \leqslant 2Q^*k^* < \infty, \tag{43}$$

while, from (39) and (41), v behaves according to

$$\begin{aligned} \dot{v} &= \dot{q}(k - k^*) + (q - Q^*)\,\dot{k} \\ &= (\dot{q}k) - \dot{q}k^* - Q^*\dot{k} \\ &= \frac{\partial H(p, q, k, 1)}{\partial k}\,k^* - Q^*\frac{\partial H(p, q, k, 1)}{\partial q} \\ &= rk^* - Q^*z. \end{aligned} \tag{44}$$

[30] Otherwise any normalization, e.g., $p(0) = 1$, will do.

In order for v to be sufficiently positive to guarantee convergence of k to k^*, we also need a somewhat different stability assumption than before, namely, the

INTENSITY AND STABILITY ASSUMPTION. (i) $(p, q, r, w) \in M$, $(c, z, -k, -l) \in T$, $pc + qz - rk - w = 0$, $pc = w > 0$, and $qk = Q^*k^* > 0 \Rightarrow rk^* \{\gtrless\} 0$ according as $p \{\gtrless\} 1$, or $rk^* \geqslant Rk^*$, and (ii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$(Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k} (k - k^*) = (Q - Q^*) z - R(k - k^*) > \delta \quad (I)$$

when $Q(\partial H(Q, k)/\partial Q) = Qz - (\partial H(Q, k)/\partial k) k = Rk$.

We should emphasize that it is the first part of (I) which is really fundamental, since it entails that (44) can be bounded below by

$$rk^* - Q^*z \geqslant Rk^* - Q^*z, \tag{45}$$

and in that condition, the normalization $qk = Q^*k^*$ which is really crucial, since, without it, finding fairly general stability conditions seems an almost impossible task. Again, this particular normalization was suggested by careful scrutiny of the worked-out examples, especially the one- and two-sector models. Notice, by way of interpretation, that the conclusion of the first part of (I) can be restated as (because $Qk \{\gtrless\} Q^*k^*$ according as $p \{\gtrless\} 1$ when $qk = Q^*k^*$)

$$Rk^* \{\gtrless\} 0 \qquad \text{according as} \quad Qk \{\gtrless\} Q^*k^*, \tag{46}$$

"aggregate" net rentals are positive or negative depending on whether "aggregate" capital intensity is less or greater than its golden rule path value.

A reader steeped in the tradition of neoclassical growth theory might plausibly conjecture that (46) is closely related to the condition that

$$c \{\gtrless\} c^* \qquad \text{according as} \quad Qk \{\gtrless\} Q^*k^*, \tag{47}$$

aggregate consumption is less or greater than its golden rule path value depending on whether "aggregate" capital intensity is. Unfortunately, neither (46) nor (47) implies the other. It is interesting to notice, however, that (47) can be strengthened to provide an alternative stability condition,

since (44) can also be bounded below by

$$rk^* - Q^*z = -(pc^* + qz^* - rk^* - w) - (c + Q^*z - R^*k - W^*) - (p - 1)(c - c^*)$$
$$\geqslant -(p - 1)(c - c^*). \tag{48}$$

By virtue of (48), (47) entails that (again because $Qk \{\gtreqless\} Q^*k^*$ according as $p \{\gtreqless\} 1$ when $qk = Q^*k^*$)

$$rk^* - Q^*z \{\geqq 0\} \qquad \text{according as} \quad Qk \{\gtreqless\} Q^*k^*, \tag{49}$$

a condition which can obviously be elaborated to yield $\dot{v}$ sufficiently positive to guarantee convergence of k to k^*. In either case, (46) or (47), the parallel with intrinsic properties of the one-good model is self-evident.

We are now in a position to demonstrate the

STABILITY OF DESCRIPTIVE GROWTH WITH MARXIAN SAVING–INVESTMENT BEHAVIOR. *Assumptions* (P) *and* (I) *imply that the real side of the solutions to* (39) *converges to unique (nontrivial) golden rule capital stocks.*

Proof. According to (42), there are two distinct cases to consider. (i) $qk = 0$: Here $k > 0$ implies $q = 0$ implies $r = 0$ (as $\dot{q} = -r$) implies $rk^* = 0$. Hence, from (44) $\dot{v} = -Q^*z$, while from just the second part of (I)

for every $\epsilon > 0$ there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow -Q^*z > \delta$.

Thus, by the sort of reasoning used in our earlier stability argument, (43) entails $\lim_{t\to\infty} k = k^*$. (ii) $qk = Q^*k^* > 0$: Here, for any point along a solution to (39)–(40), the first part of (I) allows us to strengthen the second part to

$$\text{for every } \epsilon > 0 \text{ there is a } \delta > 0 \text{ such that } \| k - k^* \| > \epsilon \Rightarrow \dot{v} = rk^* - Q^*z \geqslant Rk^* - Q^*z > \delta.$$

Thus, as before, (43) entails $\lim_{t\to\infty} k = k^*$.

E. *Worked-out Examples and Extensions*

It is fairly easy to show that (P) and (I) are satisfied in Uzawa's two-sector model, provided that the consumption-goods sector is more capital intensive than the investment-goods sector so that Inada's type of counter-example is excluded.[81] It is much less obvious whether our conditions

[81] We note in passing that this model does not satisfy a stronger uniqueness or stability assumption, like (U^0) or (S^0) of the last section. Many feasible allocations with $k \neq k^*$ can be static profit maximizing at (modified) golden rule prices.

apply to the other worked-out examples. Thus far we have been able to show, by an extremely tedious argument, that the Shell–Stiglitz example [33] satisfies (P) and a weaker version of (I), which may imply stability as well, but at present it appears that it does not satisfy (I). We have not yet tackled the other worked-out examples (in particular, the Caton–Shell example [8]). Even for the descriptive growth model with Marxian saving–investment behavior, much more work needs to be done!

Analysis of this model is clearly simplified because the saving–investment hypothesis can be described entirely in terms of H, and because the only consistent, meaningful stationary points are golden rule paths. The next obvious candidate for analysis is a model with the Keynesian saving–investment hypothesis, which has neither of these special features. Our failure thus far to find general results for such a model seems to be related to our inability to see how to utilize the restrictions imposed by (18), if indeed they can be. In any case, the investigation of this, and further similar extensions, seem a fascinating and important task for future research.

Of course, it may well turn out that the stability notion used in our present analysis is simply far too rigid for any such contemplated extensions. For example, it may be more appropriate to ask whether there is some bounded set of capital stocks, perhaps exhibiting other special properties, which serves as a global attractor for the real side of the solutions to a Hamiltonian dynamical system. This broadened stability notion would then obviously accommodate such phenomena as arise in Sutherland's or Inada's examples. It might also be sufficient to completely dispel Hahn-like worries [12] about the "gross" inefficiency of capitalist development. However, on the other hand, it would definitely entail a significant loss in our ability to do meaningful comparative dynamics.

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Saddle Points of Hamiltonian Systems in Convex Lagrange Problems Having a Nonzero Discount Rate

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Problems are studied in which an integral of the form $\int_0^{+\infty} L(k(t), \dot{k}(t))e^{-\rho t}\,dt$ is minimized over a class of arcs k: $[0, +\infty) \to R^n$. It is assumed that L is a convex function on $R^n \times R^n$ and that the discount rate ρ is positive. Optimality conditions are expressed in terms of a perturbed Hamiltonian differential system involving a Hamiltonian function $H(k, q)$ which is concave in k and convex in q, but not necessarily differentiable. Conditions are given ensuring that, for ρ sufficiently small, the system has a stationary point, in a neighborhood of which one has classical "saddle point" behavior. The optimal arcs of interest then correspond to the solutions of the system which tend to the stationary point as $t \to +\infty$. These results are motivated by questions in theoretical economics and extend previous work of the author for the case $\rho = 0$. The case $\rho < 0$ is also covered in part.

1. INTRODUCTION

Let L. $R^n \times R^n \to (-\infty, +\infty]$ be convex, lower, semicontinuous, and not identically $+\infty$, and let $\rho \geqslant 0$. For each $c \in R^n$, let

$$\phi(c) = \inf \left\{ \int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt \;\middle|\; k(0) = c \right\}, \tag{1.1}$$

where the infimum is over the class of all *arcs* (taken here to mean *absolutely continuous functions*) k: $[0, +\infty) \to R^n$ such that $e^{-\rho t}k(t)$ remains bounded as $t \to +\infty$. The integral in (1.1) has a classical value, possibly infinite, unless neither the positive nor the negative part of the integrand $L(k(t), \dot{k}(t))\, e^{-\rho t}$ (a measurable function of t) is summable over $[0, +\infty)$; in the latter case, we consider the integral to have the value $+\infty$ by convention. The convexity of L implies the convexity of ϕ as an extended-real-valued function on R^n.

Our interest lies in the existence and characterization of the arcs k, if any, for which the infimum in (1.1) is attained. An important aid in this

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regard is the study of a generalized Hamiltonian subdifferential "equation"

$$(-e^{\rho t}\dot{w}(t), \dot{k}(t)) \in \partial H(k(t), e^{\rho t}w(t)) \qquad \text{a.e.,} \tag{1.2}$$

which after the change of variables,

$$q(t) = e^{\rho t}w(t), \tag{1.3}$$

can be rewritten in the autonomous form

$$(-\dot{q}(t) + \rho q(t), \dot{k}(t)) \in \partial H(k(t), q(t)) \qquad \text{a.e.} \tag{1.4}$$

The Hamiltonian H is defined here by the conjugacy formula

$$H(k, q) = \sup\{z \cdot q - L(k, z) \mid z \in R^n\}. \tag{1.5}$$

By virtue of the assumptions on L, $H(k, q)$ is concave in k, convex in q, and the inverse formula

$$L(k, z) = \sup\{z \cdot q - H(k, q) \mid q \in R^n\} \tag{1.6}$$

is valid (cf. [3]). The set $\partial H(k, q)$ consists of the *subgradients* of H at (k, q), i.e., the pairs $(r, z) \in R^n \times R^n$ such that

$$H(k, q') \geq H(k, q) + (q' - q) \cdot z \qquad \text{for all} \quad q' \in R^n, \tag{1.7}$$

$$H(k', q) \leq H(k, q) + (k' - k) \cdot r \qquad \text{for all} \quad k' \in R^n. \tag{1.8}$$

Established theory (cf. [3]) tells us that if $k(t)$ and $q(t)$ satisfy the (perturbed) Hamiltonian system (1.4) over a real interval J, then for every bounded subinterval $[t_0, t_1]$ of J, the integral

$$\int_{t_0}^{t_1} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt \tag{1.9}$$

is minimized with respect to the class of all arcs over $[t_0, t_1]$ having the same endpoints as k at $t = t_0$ and $t = t_1$. At the same time, there is a dual property: for a certain function M, the integral

$$\int_{t_0}^{t_1} M(q(t), \dot{q}(t) - \rho q(t))\, e^{-\rho t}\, dt \tag{1.10}$$

is minimized with respect to the class of all arcs over $[t_0, t_1]$ having the same endpoints as q at $t = t_0$ and $t = t_1$. The function M is defined by

$$\begin{aligned} M(q, s) &= \sup\{k \cdot s + z \cdot q - L(k, z) \mid (k, z) \in R^n \times R^n\} \\ &= \sup\{k \cdot s + H(k, q) \mid k \in R^n\}, \end{aligned} \tag{1.11}$$

and one has, reciprocally,

$$L(k, z) = \sup\{k \cdot s + z \cdot q - M(q, s) | (q, s) \in R^n \times R^n\}. \quad (1.12)$$

Indeed, in the terminology of [3] the Lagrangians

$$\tilde{L}(t, k, z) = L(k, z)\, e^{-\rho t}, \quad (1.13)$$

$$\tilde{M}(t, w, v) = M(e^{\rho t} w, e^{\rho t} v)\, e^{-\rho t} \quad (1.14)$$

are dual to each other, whence the result [3, p. 213].

Henceforth, let $(\bar{k}, \bar{q})$ denote a pair in $R^n \times R^n$ such that

$$(\rho \bar{q}, 0) \in \partial H(\bar{k}, \bar{q}). \quad (1.15)$$

(We shall comment in Section 6 on the existence of such a pair $(\bar{k}, \bar{q})$.) Relation (1.15) means that $(\bar{k}, \bar{q})$ is a *stationary point* of the system (1.4), in the sense that the constant functions $k(t) \equiv \bar{k}$ and $q(t) \equiv \bar{q}$ satisfy the system over $J = (-\infty, +\infty)$. In a previous paper [2], we investigated for the case $\rho = 0$ the behavior of the system near such a stationary point, particularly the existence of solutions $(k(t), q(t))$ tending to $(\bar{k}, \bar{q})$ as $t \to +\infty$ or as $t \to -\infty$. This was shown to be closely related to question of optimality for the minimization problems in (1.1), as well as for a class of dual problems involving M. The analysis was carried out under the assumption that H was *strictly* concave–convex in a neighborhood of $(\bar{k}, \bar{q})$.

The purpose of the present paper is to extend some of the results to the case of sufficiently small $\rho > 0$, making use of a strengthened strict concavity–convexity assumption on H. Economic motivation may be found in the interesting paper of Case and Shell [1], which contains certain related results based on a somewhat different set of technical assumptions.

It will be convenient to make a translation of variables,

$$x = k - \bar{k}, \qquad p = q - \bar{q}, \quad (1.16)$$

so that the stationary point of the Hamiltonian system appears at the origin and the finiteness of certain integrals is more apparent. Specifically, let

$$H_0(x, p) = H(\bar{k} + x, \bar{q} + p) - H(\bar{k}, \bar{q}) - \rho x \bar{q}. \quad (1.17)$$

Then H_0 is a concave–convex function which, according to (1.15), satisfies

$$(0, 0) \in \partial H_0(0, 0), \quad (1.18)$$

or in other words, the minimax saddle point condition

$$H_0(x, 0) \leqslant H_0(0, 0) \leqslant H_0(0, p) \qquad \text{for all} \quad x \in R^n, p \in R^n. \tag{1.19}$$

Moreover, one has

$$H_0(0, 0) = 0. \tag{1.20}$$

The (perturbed) Hamiltonian system

$$(-\dot{p}(t) + \rho p(t), \dot{x}(t)) \in \partial H_0(x(t), p(t)) \tag{1.21}$$

is clearly equivalent to the previous system (1.4) under (1.16).

Let us also define

$$L_0(x, z) = L(\bar{k} + x, z) - L(\bar{k}, 0) - \bar{q} \cdot (z - \rho x), \tag{1.22}$$

$$M_0(p, u) = M(\bar{q} + p, -\rho\bar{q} + u) - M(\bar{q}, -\rho\bar{q}) - \bar{k} \cdot u. \tag{1.23}$$

These formulas yield (by the theory of conjugate convex functions, cf. [3]) the relations

$$H_0(x, p) = \sup\{z \cdot p - L_0(x, z) \mid z \in R^n\}, \tag{1.24}$$

$$L_0(x, z) = \sup\{z \cdot p - H_0(x, p) \mid p \in R^n\}, \tag{1.25}$$

$$\begin{aligned} M_0(p, u) &= \sup\{x \cdot u + z \cdot p - L_0(x, z) \mid (x, z) \in R^n \times R^n\} \\ &= \sup\{x \cdot u + H_0(x, p) \mid x \in R^n\} \end{aligned} \tag{1.26}$$

$$L_0(x, z) = \sup\{x \cdot u + z \cdot p - M_0(p, u) \mid (p, u) \subset R^n \times R^n\}, \tag{1.27}$$

inasmuch as (1.3) implies

$$-L(\bar{k}, 0) = H(\bar{k}, \bar{q}) = M(\bar{q}, -\rho\bar{q}) + \rho\bar{k} \cdot \bar{p}. \tag{1.28}$$

Observe that

$$L_0(x, z) \geqslant L_0(0, 0) = 0 \qquad \text{for all} \quad (x, z) \in R^n \times R^n, \tag{1.29}$$

$$M_0(p, u) \geqslant M_0(0, 0) = 0 \qquad \text{for all} \quad (p, u) \in R^n \times R^n. \tag{1.30}$$

Let us say that a finite function h on a convex set $C \subset R^n$ is α-*convex*, where $\alpha \in R$, if for all $x \in R^n$, $x' \in R^n$ and $\lambda \in [0, 1]$, it is true that

$$h((1 - \lambda)x + \lambda x') \leqslant (1 - \lambda)h(x) + \lambda h(x') - \tfrac{1}{2}\alpha\lambda(1 - \lambda)|x - x'|^2, \tag{1.31}$$

where $|\cdot|$ denotes the Euclidean norm. Obviously, this is ordinary convexity if $\alpha = 0$ and a form of strict convexity (*strong* convexity) if

$\alpha > 0$. One can verify that h is α-convex if and only if the function $h(x) - \frac{1}{2}\alpha \mid x \mid^2$ is convex. Thus, if C is open and h is twice differentiable, α-convexity is equivalent to the condition that

$$v \cdot Q(x)\, v \geqslant \alpha \mid v \mid^2 \qquad \text{for all} \quad x \in C,\ v \in R^n,$$

where $Q(x)$ is the matrix of second partial derivatives of h at x. Another easily derived characterization, for C open, is that h is α-convex if and only if for each $z \in C$, there exists $y \in R^n$ such that

$$h(x') \geqslant h(x) + (x' - x) \cdot y + \tfrac{1}{2}\alpha \mid x' - x \mid^2 \qquad \text{for all} \quad x' \in C. \quad (1.32)$$

At all events, if h is α-convex on C (not necessarily open) and $y \in \partial h(x)$, then (1.32) holds.

We shall say h is *α-concave* if, in place of (1.31), we have

$$h((1-\lambda)\, x + \lambda x') \geqslant (1-\lambda)\, h(x) + \lambda h(x') + \tfrac{1}{2}\alpha\lambda(1-\lambda) \mid x - x' \mid^2. \quad (1.33)$$

CURVATURE ASSUMPTION. *We suppose throughout this paper that, for certain values $\alpha > 0$ and $\beta > 0$, the Hamiltonian H is locally α-concave-β-convex near the stationary point $(\bar{k}, \bar{q})$, or in other words, that there exists a convex neighborhood $U \times V$ of $(0, 0)$ in $R^n \times R^n$ such that $H_0(x, p)$ is (finite and) α-concave in $x \in U$ for each $p \in V$ and β-convex in $p \in V$ for each $x \in U$. Moreover, the discount rate $\rho > 0$ is small enough so that*

$$\rho^2 < 4\alpha\beta. \quad (1.34)$$

Let K_+ denote the set of all pairs $(a, b) \in R^n \times R^n$ such that the Hamiltonian system (1.21) has a solution $(x(t), p(t))$ over $[0, +\infty)$ satisfying

$$(x(0), p(0)) = (a, b), \quad (1.35)$$

$$e^{-\rho t} x(t) \cdot p(t) \to 0 \text{ as } t \to +\infty. \quad (1.36)$$

The first of our main results is the following.

THEOREM 1. *There is an open neighborhood $U_+ \times V_+$ of $(0, 0)$ (arbitrarily small, with $U_+ \subset U$ and $V_+ \subset V$) such that $K_+ \cap (U_+ \times V_+)$ is the graph of a homeomorphism of U_+ onto V_+, and for each $(a, b) \in K_+ \cap (U_+ \times V_+)$ the solution to the system (1.21) over $[0, +\infty)$ satisfying (1.35) and (1.36) is unique, remains in $K_+ \cap (U_+ \times V_+)$ and converges to $(0, 0)$ as $t \to +\infty$.*

To give an interpretation to the set K_+, we introduce the functions

$$f_+(a) = \inf\left\{\int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \;\middle|\; x(0) = a\right\}, \tag{1.37}$$

$$g_+(b) = \inf\left\{\int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \;\middle|\; p(0) = b\right\}, \tag{1.38}$$

where each infimum is taken over the class of *all* arcs (absolutely continuous R^n-valued functions) defined on $[0, +\infty)$ and satisfying the endpoint condition in question. Note that the integrals are well-defined (possibly $+\infty$), due to the lower semicontinuity and nonnegativity of the functions L_0 and M_0. In fact, (1.29) and (1.30) imply that the functions f_+ and g_+ on R^n are nonnegative and vanish at 0. Of course, these functions are all convex; this follows immediately from the convexity of L_0 and M_0.

For $a = c - \bar{k}$, the minimization problem in (1.37) is equivalent to the one defining the value $\phi(c)$ at the beginning of our introduction, as will be demonstrated in the next section (Proposition 2). This equivalence could fail if one were to drop from the definition of $\phi(c)$ the restriction to arcs k such that $e^{-\rho t}k(t)$ remains bounded as $t \to +\infty$; see Example 2 in Section 6.

THEOREM 2. *Let U_+ and V_+ be neighborhoods of 0 with the properties in Theorem 1. Then f_+ is finite and continuously differentiable on U_+, g_+ is finite and continuously differentiable on V_+, and for $(a, b) \in U_+ \times V_+$, one has*

$$f_+(a) + g_+(b) \geqslant -a \cdot b, \tag{1.39}$$

with

$$\begin{aligned}(a, b) \in K_+ &\Leftrightarrow f_+(a) + g_+(b) = -a \cdot b \\ &\Leftrightarrow b = -\nabla f_+(a) \Leftrightarrow a = -\nabla g_+(b).\end{aligned} \tag{1.40}$$

Moreover, if $(x(t), p(t))$ is the solution to the Hamiltonian system (1.21) over $[0, +\infty)$ corresponding to $(a, b) \in K_+ \cap (U_+ \times V_+)$ as in Theorem 1, then the arc x uniquely furnishes the minimum in the definition (1.37) of $f_+(a)$, while the arc p uniquely furnishes the minimum in the definition (1.38) of $g_+(b)$.

Complementary results are obtainable for behavior over the interval $(-\infty, 0]$. These are of less interest for economic applications, but they do shed further light on the qualitative nature of the Hamiltonian dynamical system. They can also be interpreted equivalently, under a reversal of time, as results over $[0, +\infty)$ for a *negative discount rate* ρ.

Let K_- denote the set of all pairs $(a, b) \in R^n \times R^n$ such that the system (1.21) has a solution $(x(t), p(t))$ over $(-\infty, 0]$ satisfying (1.35), and

$$e^{-\rho t}x(t) \cdot p(t) \to 0 \text{ as } t \to -\infty. \tag{1.41}$$

Define

$$f_-(a) = \inf \left\{ \int_{-\infty}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \;\middle|\; x(0) = a \right\}, \tag{1.42}$$

$$g_-(b) = \inf \left\{ \int_{-\infty}^{0} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \;\middle|\; p(0) = b \right\}. \tag{1.43}$$

The functions f_- and g_- are nonnegative, convex, and they vanish at 0.

THEOREM 1′. *There is an open neighborhood $U_- \times V_-$ of (0, 0) (arbitrarily small, with $U_- \subset U$ and $V_- \subset V$) such that $K_- \cap (U_- \times V_-)$ is the graph of a homeomorphism of U_- onto V_-, and for each $(a, b) \in K_- \cap (U_- \times V_-)$ the solution to the system (1.21) over $(-\infty, 0]$ satisfying (1.35) and (1.41) is unique, remains in $K_- \cap (U_- \times V_-)$ and converges to (0, 0) as $t \to -\infty$. Moreover,*

$$K_+ \cap K_- = \{(0, 0)\}. \tag{1.44}$$

THEOREM 2′. *Let U_- and V_- be neighborhoods of 0 with the properties in Theorem 1′. Then f_- is finite and continuously differentiable on U_-, g_- is finite and continuously differentiable on V_-, and for $(a, b) \in U_- \times V_-$, one has*

$$f_-(a) + g_-(b) \geqslant a \cdot b, \tag{1.45}$$

with

$$\begin{aligned} (a, b) \in K_- &\Leftrightarrow f_-(a) + g_-(b) = a \cdot b \\ &\Leftrightarrow b = \nabla f_-(a) \Leftrightarrow a = \nabla g_-(b). \end{aligned} \tag{1.46}$$

Moreover, if $(x(t), p(t))$ is the solution to the Hamiltonian system (1.21) over $(-\infty, 0]$ corresponding to $(a, b) \in K_- \cap (U_- \times V_-)$ as in Theorem 1′, then the arc x uniquely furnishes the minimum in the definition (1.42) of $f_-(a)$, while the arc p uniquely furnishes the minimum in the definition (1.43) of $g_-(b)$.

Theorems 1 and 1′ say that the behavior of the system (1.21) near the rest point (0, 0) resembles that of a classical saddlepoint in the theory of differential equations. At least locally, K_+ and K_- are n-dimensional manifolds intersecting only at (0, 0), and comprised, respectively, of the trajectories that tend to (0, 0) as $t \to +\infty$ and as $t \to -\infty$.

Theorem 2 will be derived from Theorem 1 in Section 2, while Theorem 1 itself will be established at the end of Section 4. The proofs of Theorem 1′ and 2′ are parallel, but in certain respects simpler; they will be treated in Section 5. Various examples and a result about the existence of points $(\bar{x}, \bar{q})$ satisfying the stationary point condition (1.15) will be treated in Section 6. In Section 3, we develop some facts which enable us, in Section 4, to deduce the local results from more special theorems based on a *global* assumption of α-concavity–β-convexity.

2. Local Behavior and Optimality

We start by establishing some bounds that, as a by-product, make clear the equivalence of the original class of variational problems defining the values $\phi(c)$ and the notationally more convenient class of "translated" problems defining the values $f_+(a)$.

Proposition 1. *The convex function L_0 is finite on a neighborhood of* $(0, 0)$, *and there exist real numbers $\mu_0 > 0$ and μ_1 such that*

$$L_0(x, z) \geq \mu_0[|x| + |z - \rho x|] - \mu_1 \qquad \text{for all} \quad (x, z) \in R^n \times R^n. \quad (2.1)$$

Similarly, the convex function M_0 is finite on a neighborhood of $(0, 0)$, *and there exist real numbers $\nu_0 > 0$ and ν_1 such that*

$$M_0(p, s - \rho p) \geq \nu_0[|p| + |s - \rho p|] - \nu_1 \qquad \text{for all} \quad (p, x) \in R^n \times R^n. \quad (2.2)$$

Proof. According to our curvature assumption, the convex function $H_0(0, \cdot)$ is finite and strictly convex in a neighborhood of $p = 0$. It follows from this and the minimax property (1.19) that the supremum in formula (1.25) is uniquely attained at $p = 0$. But $H_0(0, \cdot)$ is conjugate to $L_0(0, \cdot)$ by (1.24), so this implies 0 is the unique subgradient of $L_0(0, \cdot)$ at $z = 0$ [5, Theorem 23.5]. Hence, $L_0(0, \cdot)$ is finite on a neighborhood of 0 (since otherwise the subgradient set would have to be unbounded or empty [5, Theorem 23.4]). Thus, the convex set

$$\operatorname{dom} L_0 = \{(x, z) \in R^n \times R^n \mid L_0(x, z) < +\infty\} \qquad (2.3)$$

contains $(0, z)$ for all z sufficiently near 0. On the other hand, the image of $\operatorname{dom} L_0$ under the projection $(x, z) \to x$ consists of all x such that the convex function $L_0(x, \cdot)$ is not identically $+\infty$, or what is equivalent in view of the conjugacy relation (1.24), such that the convex function $H_0(x, \cdot)$ nowhere takes the value $-\infty$. This image therefore contains the neigh-

borhood U in our curvature assumption, since a convex function cannot ake on the value $-\infty$ anywhere if it is finite on some nonempty open set ɔ, Theorem 7.2]. Thus 0 lies in the interior of the x-projection of dom L_0, while 0 is also in the interior of the z-cross-section of dom L_0 corresponding to $x = 0$. These properties imply that (0, 0) is an interior point of dom L_0 [5, Theorem 7.8], or in other words that L_0 is finite on a neighborhood of (0, 0). Of course, L_0 is then continuous near (0, 0) by convexity. Thus there exist $\nu_0 > 0$ and ν_1 such that

$$L_0(x, z) \leqslant \nu_1 \qquad \text{if} \quad |x| \leqslant \nu_0 \text{ and } |z| \leqslant \nu_0 .$$

Applying this to formula (1.26) (for M_0 in terms of L_0), we obtain

$$\begin{aligned} M_0(p, u) &\geqslant \sup\{x \cdot p + z \cdot u - \nu_1 \,|\, |x| \leqslant \nu_0 , |z| \leqslant \nu_0\} \\ &= \nu_0 |p| + \nu_0 |u| - \nu_1 , \end{aligned}$$

which is the desired inequality (2.2).

By a parallel argument, the concave function $H_0(\cdot, 0)$ is finite and strictly concave on a neighborhood of $x = 0$, so that by (1.19), the supremum in formula (1.26) (for M_0 in terms of H_0) is attained uniquely for $x = 0$. Since this formula expresses the convex function $M_0(0, \cdot)$ as the conjugate of $-H(\cdot, 0)$, we are able to conclude, just as above, that 0 is the unique subgradient of $M_0(0, \cdot)$ at 0, and, hence, that $M_0(0, \cdot)$ is finite on a neighborhood of 0. The convex set

$$\text{dom } M_0 = \{(p, u) |\ M_0(p, u) < +\infty\} \tag{2.4}$$

therefore contains $(0, u)$ for all u sufficiently near 0. The p-projection of dom M_0 also contains the neighborhood V of 0 in our curvature assumption, because of (1.26). We deduce from this that (0, 0) is an interior point of dom M_0 and consequently a point in a neighborhood of which M_0 is finite and continuous. Let the numbers $\mu > 0$, $\mu' > 0$, and μ_1 be such that

$$M_0(p, u) \leqslant \mu_1 \qquad \text{if} \quad |p| \leqslant \mu \text{ and } |u| \leqslant \mu'.$$

We then have from (1.27) that

$$\begin{aligned} L_0(x, z) &\geqslant \sup\{x \cdot u + p \cdot z - \mu_1 \,|\, |p| \leqslant \mu, |u| \leqslant \mu'\} \\ &= \mu |x| + \mu' |z| - \mu_1 . \end{aligned}$$

Using the fact that

$$|z| \geqslant |z - \rho x| - \rho |x| ,$$

we get

$$L_0(x, z) \geqslant (\mu - \rho\mu')|\, x \,| + \mu' \,|\, z - \rho x \,| - \mu_1 .$$

The desired inequality (2.1) therefore holds for

$$\mu_0 = \min\{\mu - \rho\mu', \mu'\},$$

provided μ' is taken small enough so that this value is positive.

The next proposition is the one specifically establishing the asserted equivalence between the minimization problems defining $\phi(c)$ in (1.1) (where $e^{-\rho t}k(t)$ is bounded) and $f_+(a)$ in (1.37) for $x = k - \bar{k}$, $a = c - \bar{k}$. (In the problem for $f_+(a)$, it is not stipulated in advance that $e^{-\rho t}x(t)$ be bounded, but this turns out to be a consequence of the finiteness of the integral.)

PROPOSITION 2. (a) *If the arc* $x : [0, +\infty) \to R^n$ *is such that* $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ *is summable in* t *over* $[0, +\infty)$, *then*

$$\lim_{t \to +\infty} e^{-\rho t}x(t) = 0, \tag{2.5}$$

and for $k(t) = \bar{k} + x(t)$, *we have* $L(k(t), \dot{k}(t))\, e^{-\rho t}$ *summable in* t, *with*

$$\int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt = \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt - [H(\bar{k}, \bar{q})/\rho] - x(0) \cdot \bar{q}. \tag{2.6}$$

On the other hand, if $L(k(t), \dot{k}(t))\, e^{-\rho t}$ *is majorized by a summable function of* t *over* $[0, +\infty)$, *and also*

$$\limsup_{t \to +\infty} e^{-\rho t}k(t) \cdot \bar{q} > -\infty, \tag{2.7}$$

then $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ *is indeed summable over* $[0, +\infty)$. (b) *If the arc* $p : [0, +\infty) \to R^n$ *is such that* $M_0(p(t), \dot{p}(t) - p\rho(t))\, e^{-\rho t}$ *is summable in* t *over* $[0, +\infty)$, *then*

$$\lim_{t \to +\infty} e^{-\rho t}p(t) = 0, \tag{2.8}$$

and for $q(t) = \bar{q} + p(t)$, *we have* $M(q(t), \dot{q}(t) - \rho q(t))\, e^{-\rho t}$ *summable in* t, *with*

$$\int_0^{+\infty} M(q(t), \dot{q}(t) - \rho q(t))\, e^{-\rho t}\, dt = \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt - [H(\bar{k}, \bar{q})/\rho] + k \cdot q(0). \tag{2.9}$$

On the other hand, if $M(q(t), \dot{q}(t) - pq(t))\, e^{-\rho t}$ *is majorized by a summable function of* t *over* $[0, +\infty)$, *and also*

$$\limsup_{t \to +\infty} e^{-\rho t} q(t) \cdot \bar{k} > -\infty, \tag{2.10}$$

then $M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}$ *is summable over* $[0, +\infty)$.

Proof. If $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ is summable, then by virtue of inequality (2.1) in Proposition 1, we have

$$\int_0^{+\infty} |x(t)|\, e^{-\rho t}\, dt < +\infty \qquad \text{and} \qquad \int_0^{+\infty} |\dot{x}(t) - x(t)|\, e^{-\rho t}\, dt < +\infty. \tag{2.11}$$

Setting $v(t) = e^{-\rho t} x(t)$, so that

$$\dot{v}(t) = [\dot{x}(t) - \rho x(t)]\, e^{-\rho t},$$

we see from (2.11) that

$$\int_0^{+\infty} |v(t)|\, dt < +\infty \qquad \text{and} \qquad \int_0^{+\infty} |\dot{v}(t)|\, dt < +\infty.$$

The finiteness of the second integral shows that $v(t)$ tends to a limit as $t \to +\infty$, while the finiteness of the first integral shows that the limit is 0. Thus (2.5) is true. Since

$$\begin{aligned} L(k(t), \dot{k}(t))\, e^{-\rho t} = {} & L_0(x(t), \dot{x}(t))\, e^{-\rho t} \\ & - H(\bar{k}, \bar{q})\, e^{-\rho t} + (d/dt)\, e^{-\rho t} x(t) \cdot \bar{q} \end{aligned}$$

by (1.22) and (1.28), we then have (2.6) and the summability of $L(k(t), \dot{k}(t))\, e^{-\rho t}$. Conversely, if the latter expression is majorized by a summable function of t, then so is $(d/dt)\, e^{-\rho t} x(t) \cdot \bar{q}$ by (2.12), since $L_0 \geqslant 0$. This implies that $e^{-\rho t} x(t) \cdot \bar{q}$ tends to a certain limit other than $+\infty$ as $t \to +\infty$. The limit cannot be $-\infty$ by assumption (2.7), and, therefore, it is finite. In other words, $(d/dt)\, e^{-\rho t} x(t)$ is actually summable, which leads via (2.12) to the conclusion that $L(k(t), \dot{k}(t))\, e^{-\rho t}$ also majorizes a summable function (the right side of (2.12) with the L_0 term deleted) and, hence, is summable. But then by (2.12), $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ must likewise be summable.

The proof of part (b) of Proposition 2 is much the same.

COROLLARY. $\phi(\bar{k} + a) = f_+(a) - [H(\bar{k}, \bar{q})/\rho] - a \cdot \bar{q}$.

A fundamental fact about "truncated" variational problems over the finite interval $[0, T]$ will now be stated. Much of our analysis of the problems over $[0, +\infty)$ is dependent on limit arguments concerning what happens to this case as $T \to +\infty$.

PROPOSITION 3. *For arbitrary arcs* $x : [0, T] \to R^n$ *and* $p : [0, T] \to R^n$ *with* $0 < T < +\infty$, *one has*

$$\int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_0^T M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \geq e^{-\rho T} x(T) \cdot p(T) - x(0) \cdot p(0). \tag{2.13}$$

Furthermore, equality holds in (2.13) *if and only if* x *and* p *satisfy the Hamiltonian system* (1.21) *over* $[0, T]$.

Proof. Formula (1.26) tells us that

$$L_0(x, z) + M_0(p, u) \geq x \cdot u + z \cdot p, \tag{2.14}$$

with equality if and only if

$$(u, p) \in \partial L_0(x, z), \tag{2.15}$$

or equivalently (cf. [5, Theorem 37.5])

$$(-u, z) \in \partial H_0(x, p). \tag{2.16}$$

Therefore,

$$\begin{aligned} &L_0(x(t), \dot{x}(t))\, e^{-\rho t} + M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t} \\ &\quad \geq e^{-\rho t}[x(t) \cdot \dot{p}(t) + \dot{x}(t) \cdot p(t) - \rho x(t) \cdot p(t)] \\ &\quad = (d/dt)\, e^{-\rho t} x(t) \cdot p(t), \end{aligned}$$

with equality if and only if (1.21) holds. The result is then immediate.

COROLLARY. *For arbitrary arcs* $x : [0, +\infty) \to R^n$ *and* $p : [0, +\infty) \to R^n$ *such that* $e^{-\rho t} x(t) \cdot p(t) \to 0$ *as* $t \to +\infty$, *one has*

$$\int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \geq -x(0) \cdot p(0), \tag{2.17}$$

with equality if and only if x *and* p *satisfy the Hamiltonian system* (1.21) *over* $[0, +\infty)$. *In particular,*

$$K_+ \subset \{(a, b) \mid f_+(a) + g_+(b) + a \cdot b \leq 0\}. \tag{2.18}$$

Proof of Theorem 2 *using Theorem* 1. Fix any $(a, b) \in K_+$. Let x and

denote the corresponding unique solution to the system (1.21) over $[0, +\infty)$ satisfying (in line with Theorem 1)

$$(x(0), p(0)) = (a, b), \tag{2.19}$$

$$(x(t), p(t)) \to (0, 0) \text{ as } t \to +\infty. \tag{2.20}$$

Then equality holds in (2.17) by the corollary immediately above, so that

$$\begin{aligned} +\infty &> \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \\ &= -a \cdot b - \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt. \end{aligned} \tag{2.21}$$

If $x' : [0, +\infty)$ is any other arc with $x'(0) = a'$, say, and

$$+\infty > \int_0^{\infty} L_0(x(t), \dot{x}'(t))\, e^{-\rho t}\, dt,$$

we have

$$\lim_{t \to +\infty} e^{-\rho t} x'(t) \cdot p(t) = 0$$

by (2.20) and property (2.5) of Proposition 2. Therefore, again by the corollary to Proposition 3,

$$\begin{aligned} &\int_0^{+\infty} L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt \\ &\quad \geqslant -a' \cdot b - \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt. \end{aligned} \tag{2.22}$$

Focusing attention on the case where $a' = a$, we see from (2.21) and (2.22) that

$$\int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt = f_+(a). \tag{2.23}$$

Similar reasoning establishes that

$$\int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt = g_+(b), \tag{2.24}$$

and in consequence, by way of (2.21),

$$f_+(a) = -a \cdot b - g_+(b). \tag{2.25}$$

We conclude further from (2.22) and the corollary to Proposition 3 that if x' were any arc with $x'(0) = a'$ and

$$\int_0^{+\infty} L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt = -a' \cdot b - g_+(b),$$

then (x', p) would have to satisfy the Hamiltonian system over $[0, +\infty)$, which by the uniqueness assertion in Theorem 1 would necessitate $x'(t) \equiv x(t)$. In particular, taking $a' = a$, we observe that the arc x uniquely gives the minimum in the definition of $f_+(a)$; in fact, the relation

$$f_+(a') \geqslant -a' \cdot b - g_+(b) \tag{2.26}$$

holds, with equality uniquely when $a' = a$. Analogously, the arc p uniquely gives the minimum in the definition of $g_+(b)$, and one has

$$g_+(b') \geqslant -a \cdot b' - f_+(a)$$

for all $b' \in R^n$. Combining (2.25) and (2.26), we get the subgradient relation

$$f_+(a') \geqslant f_+(a) - (a' - a) \cdot b \qquad \text{for all} \quad a' \in R^n, \tag{2.27}$$

or symbolically, $-b \in \partial f_+(a)$. By the same token, we have

$$g_+(b') \geqslant g_+(b) - (b' - b) \cdot a \qquad \text{for all} \quad b' \in R^n, \tag{2.28}$$

or in other words $-a \in \partial g_+(b)$. This establishes all of Theorem 2 except for the differentiability assertion. For the latter, let θ denote the homeomorphism whose graph is $K_+ \cap (U_+ \times V_+)$; thus, $(a, \theta(a)) \in K_+$ for all $a \in U_+$. Then $-\theta(a) \in \partial f_+(a)$, and since θ is continuous, we must actually have $-\theta(a) = \nabla f_+(a)$ [5, Theorems 25.1, 25.5, and 25.6]. Thus, f_+ is continuously differentiable on U_+; similarly, g_+ is continuously differentiable on V_+. This completes the proof.

In the next two sections, Theorem 1 itself will be proved, but for this purpose a further consequence of Proposition 3 will eventually be required. We state it now for convenience. For $0 < T < +\infty$, let

$$f_T(a, a') = \inf \left\{ \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \;\middle|\; x(0) = a,\ x(T) = a' \right\}, \tag{2.29}$$

$$g_T(b, b') = \inf \left\{ \int_0^T M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \;\middle|\; p(0) = b,\ p(T) = b' \right\}, \tag{2.30}$$

where again the infima are over all arcs (absolutely continuous R^n-valued functions on $[0, T]$) satisfying the given terminal constraints. It is evident

from (1.29) and (1.30) that the functions f_T and g_T on $R^n \times R^n$ are convex, nonnegative, and vanish at (0, 0).

PROPOSITION 4. *One has*

$$f_T(a, a') + g_T(b, b') \geq e^{-\rho T} a' \cdot b' - a \cdot b \qquad \text{for all} \quad (a, a') \text{ and } (b, b'). \tag{2.31}$$

If $(x(t), p(t))$ *satisfies the Hamiltonian system* (1.21) *over* $[0, T]$ *with* $(x(0), p(0)) = (a, b)$ *and* $(x(T), p(T)) = (a', b')$, *then equality holds in* (2.31), x *yields the minimum in the definition of* $f_T(a, a')$, *and* p *yields the minimum in the definition of* $g_T(b, b')$. *The converse implication is also true.*

Proof. This is obvious from Proposition 3.

3. Reduction from the Local to the Global Case

The next results will be used ultimately to show that, for the purpose of proving Theorem 1, our basic curvature assumption can just as well be cast in a global form. Certain facts about uniqueness of solutions to the Hamiltonian system are also implied by these results.

PROPOSITION 5. *If* $(x_1(t), p_1(t))$ *and* $(y_2(t), p_2(t))$ *are solutions to the Hamiltonian system* (1.21) *over an interval* J, *then the inequality*

$$(d/dt)\, e^{-\rho t}(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \geq 0 \qquad \text{a.e.} \tag{3.1}$$

holds on J, *with strict inequality over portions of* J *where* $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$ *and at least one of the two solutions lies in the neighborhood* $U \times V$ *in the curvature assumption.*

Over portions of J *where* $(x_1(t), p_1(t)) \neq (x_2(t), p_2(t))$ *and both of the solutions lie in* $U \times V$, *one actually has*

$$\begin{aligned}(d/dt)\, e^{-\rho t}(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \\ \geq \sigma_0 \,|\, e^{-\rho t}(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t))| \qquad \text{a.e.,}\end{aligned} \tag{3.2}$$

where

$$\sigma_0 = 2 \min\{\alpha, \beta\} > 0, \tag{3.3}$$

as well as

$$\begin{aligned}(d/dt)(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \\ \geq \sigma_1[|\, x_1(t) - x_2(t)| + |\, p_1(t) - p_2(t)|]^2 \qquad \text{a.e.}\end{aligned} \tag{3.4}$$

where

$$\sigma_1 = (4\alpha\beta - \rho^2)/4(\alpha + \beta + \rho) > 0. \tag{3.5}$$

Proof. We have by definition of ∂H_0 that

$$H_0(x_1(t), p_2(t)) \geqslant H_0(x_1(t), p_1(t)) + \dot{x}_1(t) \cdot (p_2(t) - p_1(t)), \tag{3.6}$$

$$H_0(x_2(t), p_1(t)) \leqslant H_0(x_1(t), p_1(t)) + (-\dot{p}_1(t) + \rho p_1(t)) \cdot (x_1(t) - x_2(t)), \tag{3.7}$$

$$H_0(x_2(t), p_1(t)) \geqslant H_0(x_2(t), p_2(t)) + \dot{x}_2(t) \cdot (p_1(t) - p_2(t)), \tag{3.8}$$

$$H_0(x_1(t), p_2(t)) \leqslant H_0(x_2(t), p_2(t)) + (-\dot{p}_2(t) + \rho p_2(t)) \cdot (x_2(t) - x_1(t)). \tag{3.9}$$

These inequalities yield

$$\begin{aligned}(\dot{x}_1(t) - \dot{x}_2(t)) \cdot (p_1(t) - p_2(t)) + (x_1(t) - x_2(t)) \cdot (\dot{p}_1(t) - \dot{p}_2(t)) \\ -\rho(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \geqslant 0 \qquad \text{a.e.}\end{aligned} \tag{3.10}$$

Multiplying the latter by $e^{-\rho t}$, we get (3.1). If $(x_1(t), p_1(t))$, say, lies in $U \times V$, where H_0 is in particular strictly concave–convex, we have strict inequality in (3.6) unless $p_1(t) = p_2(t)$, as well as strict inequality in (3.7) unless $x_1(t) = x_2(t)$. In this context, therefore, strict inequality holds in (3.10), and hence, in (3.1), unless $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$.

Over subintervals where both solutions lie in $U \times V$, we can improve the argument by adding the term $\frac{1}{2}\beta \mid p_1(t) - p_2(t)\mid^2$ to the right sides of (3.6) and (3.8), while subtracting $\frac{1}{2}\alpha \mid x_1(t) - x_2(t)\mid^2$ from the right sides of (3.7) and (3.9). In this way, (3.10) is strengthened to

$$\begin{aligned}(\dot{x}_1(t) - \dot{x}_2(t)) \cdot (p_1(t) - p_2(t)) + (x_1(t) - x_2(t)) \cdot (\dot{p}_1(t) - \dot{p}_2(t)) \\ - \rho(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \geqslant \alpha \mid x_1(t) - x_2(t)\mid^2 \\ + \beta \mid p_1(t) - p_2(t)\mid^2 \qquad \text{a.e.}\end{aligned} \tag{3.11}$$

Using the fact that

$$\mid u\mid^2 + \mid v\mid^2 \geqslant 2 \mid u \cdot v\mid,$$

we see that

$$\begin{aligned}\alpha \mid x_1(t) - x_2(t)\mid^2 + \beta \mid p_1(t) - p_2(t)\mid^2 \\ \geqslant \sigma_0 \mid (x_1(t) - x_2(t)) \cdot (p_1(t) + p_2(t))\mid.\end{aligned} \tag{3.12}$$

When (3.12) is juxtaposed with (3.11) and both sides are multiplied by

$e^{-\sigma t}$, we obtain (3.2). In establishing (3.4), the first thing to record is that 3.11) also implies

$$\begin{aligned}(d/dt)(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \\ \geqslant \alpha \mid x_1(t) - x_2(t)\mid^2 + \rho(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \\ + \beta \mid p_1(t) - p_2(t)\mid^2 \\ \geqslant \alpha \mid x_1(t) - x_2(t)\mid^2 - \rho \mid x_1(t) - x_2(t)\mid \mid p_1(t) - p_2(t)\mid \\ + \beta \mid p_1(t) - p_2(t)\mid^2 \qquad \text{a.e.}\end{aligned} \tag{3.13}$$

The proof of (3.4) can be completed by showing that for all real numbers $\lambda \geqslant 0$ and $\mu \geqslant 0$, one has

$$\alpha\lambda^2 - \rho\lambda\mu + \beta\mu^2 \geqslant \sigma_1(\lambda + \mu)^2. \tag{3.14}$$

This inequality is trivial of course if $\lambda = 0 = \mu$, so we can suppose that $\lambda + \mu > 0$ and rewrite (3.14) as

$$\alpha\theta^2 - \rho\theta(1 - \theta) + \beta(1 - \theta)^2 \geqslant \sigma_1 , \tag{3.15}$$

where $\theta = \lambda/(\lambda + \mu)$. The validity of (3.15) for all $\theta \in [0, 1]$ is seen by calculating the minimum value of the left side of (3.15) as a quadratic (convex) function of $\theta \in (-\infty, +\infty)$ and showing that it in fact equals σ_1, which is positive by assumption (1.34).

COROLLARY 1. *If* $(x_1(t), p_1(t))$ *and* $(x_2(t), p_2(t))$ *are solutions to the Hamiltonian system over an interval J, then the expression*

$$\theta(t) = e^{-\sigma t}(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \tag{3.16}$$

is nondecreasing over J, in fact strictly increasing over those portions of J where $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$ *and at least one of the solutions* $U \times V$.

Over portions of J where $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$ *and both of the solutions lie in* $U \times V$, *one has* $e^{-\sigma_0 t}\theta(t)$ *nondecreasing where* $\theta(t) > 0$, *and* $e^{\sigma_0 t}\theta(t)$ *nondecreasing where* $\theta(t) < 0$.

Proof. The function θ is absolutely continuous, so it is nondecreasing over intervals where $\dot\theta(t) \geqslant 0$ almost everywhere, and it is strictly increasing over intervals where $\dot\theta(t) > 0$ almost everywhere. The justification of the final assertion of the corollary is seen by rewriting (3.2) as

$$\dot\theta(t) - \sigma_0 \mid \theta(t)\mid \geqslant 0 \qquad \text{a.e.}$$

Multiplying both sides of this by $e^{-\sigma_0 t}$, one finds that

$$(d/dt)\, e^{-\sigma_0 t}\theta(t) \geq 0 \qquad \text{where} \quad \theta(t) \geq 0,$$

$$(d/dt)\, e^{\sigma_0 t}\theta(t) \geq 0 \qquad \text{where} \quad \theta(t) \leq 0.$$

COROLLARY 2. *If among the solutions to the Hamiltonian system* (1.21) *over* $[0, +\infty)$ *satisfying*

$$(x(0), p(0)) = (a, b) \in U \times V, \tag{3.17}$$

$$\lim_{t \to +\infty} e^{-\rho t} x(t) \cdot p(t) = 0, \tag{3.18}$$

there is one such that $(x(t), p(t))$ *remains bounded and in* $U \times V$ *as* $t \to +\infty$, *then it is the unique solution to the system over* $[0, +\infty)$ *having either* $x(0) = a$ *or* $p(0) = b$.

Proof. Let $(x'(t), p'(t))$ also satisfy (3.18) with either $x'(0) = a$ or $p'(0) = b$. The corollary to Proposition 3 gives us

$$\int_0^{+\infty} L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt + \int_0^{+\infty} M_0(p'(t), \dot{p}'(t) - \rho p'(t))\, e^{-\rho t}\, dt$$
$$= -x'(0) \cdot p'(0) < +\infty, \tag{3.19}$$

and, hence, by Proposition 2, we have

$$\lim_{t \to +\infty} e^{-\rho t} x'(t) = \lim_{t \to +\infty} e^{-\rho t} p'(t) = 0. \tag{3.20}$$

By virtue of the boundedness of $(x(t), p(t))$, the expression

$$\theta(t) = e^{-\rho t}(x'(t) - x(t)) \cdot (p'(t) - p(t)) \tag{3.21}$$

therefore satisfies

$$\lim_{t \to +\infty} \theta(t) = 0 = \theta(0). \tag{3.22}$$

This implies via Corollary 1 that $x'(t) = x(t)$ and $p'(t) = p(t)$ for all $t \in [0, +\infty)$.

Corollary 2 is the basis for the uniqueness assertion in Theorem 1. The rest of Theorem 1 relates only to the local behavior of H_0 and the corresponding Hamiltonian system near (0, 0). In the proof, therefore, there is no harm in replacing H_0 by any more convenient function that agrees with it on a neighborhood of (0, 0). The proposition below will allow us in this manner to derive Theorem 1 by way of results that are more global in nature.

PROPOSITION 6. *Let $C \times D$ be any compact convex neighborhood of $(0, 0)$ contained in the neighborhood $U \times V$ in the curvature assumption on H_0. Then there exists a function H_1 on $R^n \times R^n$ agreeing with H_0 on $C \times D$, such that $H_1(x, p)$ is everywhere finite, α-concave in x, and β-concave in p.*

Moreover, H_1 can be constructed in such a manner that the corresponding convex Lagrangians

$$L_1(x, z) = \sup\{p \cdot z - H_1(x, p) | \, p \in R^n\}, \tag{3.23}$$

$$M_1(p, u) = \sup\{u \cdot x + H_1(x, p) | \, x \in R^n\}, \tag{3.24}$$

are finite throughout $R^n \times R^n$.

Proof. Let

$$\bar{H}_0(x, p) = H_0(x, p) + \tfrac{1}{2}\alpha \, | \, x \, |^2 - \tfrac{1}{2}\beta \, | \, p \, |^2. \tag{3.25}$$

Then H_0 is concave–convex on $U \times V$. The construction given in [2, Proof of Proposition 3.1] furnishes a finite, concave–convex function $\bar{H}_1$ on $R^n \times R^n$ agreeing with H_0 on $C \times D$. Let

$$H_1(x, p) = \bar{H}_1(x, p) - \tfrac{1}{2}\alpha \, | \, x \, |^2 + \tfrac{1}{2}\beta \, | \, p \, |^2. \tag{3.26}$$

Then $H_1(x, p)$ is everywhere finite, α-concave in x, β-convex in p, and $H_1(x, p) = H_0(x, p)$ for (x, p) in $C \times D$. Moreover

$$\lim_{\lambda \to +\infty} H_1(x, \lambda p)/\lambda = +\infty, \tag{3.27}$$

$$\lim_{\lambda \to +\infty} H_1(\lambda x, p)/\lambda = -\infty, \tag{3.28}$$

so that the functions L_1 and M_1 in (3.23) and (3.24) must be finite everywhere [5, Corollary 13.3.1].

4. GLOBAL RESULTS

In view of Proposition 6, there is no loss of generality if in the rest of the development of the proof of Theorem 1 we invoke the following.

GLOBAL CURVATURE ASSUMPTION. *The function H_0 is actually finite and α-concave-β-convex throughout $R^n \times R^n$, i.e., the earlier curvature assumption is valid with $U \times V = R^n \times R^n$. Furthermore, the functions L_0 and M_0 are finite throughout $R^n \times R^n$.*

A crucial consequence of this assumption is the following property.

PROPOSITION 7. *Under the global curvature assumption, if* $(x(t), p(t))$ *satisfies the Hamiltonian system* (1.21) *over* $[0, +\infty)$, *then either*

$$\lim_{t \to +\infty} e^{-\rho t} x(t) \cdot p(t) = +\infty, \tag{4.1}$$

or one has

$$\lim_{t \to +\infty} (x(t), p(t)) = (0, 0). \tag{4.2}$$

Proof. We first apply Corollary 1 of Proposition 5 to $(x_1(t), p_1(t)) \equiv (x(t), p(t))$ and $(x_2(t), p_2(t)) \equiv (0, 0)$ to see that the function

$$\theta(t) = e^{-\rho t} x(t) \cdot p(t)$$

is nondecreasing, and in fact $e^{-\rho_0 t}\theta(t)$ is nondecreasing on subintervals where $\theta(t) > 0$. Thus (4.1) holds unless $\theta(t) \leq 0$ for all $t \geq 0$. Suppose now that the latter is true, so that also

$$x(t) \cdot p(t) \leq 0 \qquad \text{for all} \quad t \geq 0. \tag{4.3}$$

From Proposition 5 we have at the same time

$$(d/dt)\, x(t) \cdot p(t) \geq \sigma_1[|x(t)| + |p(t)|]^2 \geq \sigma_1 |(x(t), p(t))|^2 \qquad \text{a.e.} \tag{4.4}$$

Hence, for all $T > 0$,

$$\sigma_1 \int_0^T |(x(t), p(t))|^2\, dt \leq x(T) \cdot p(T) - x(0) \cdot p(0) \leq -x(0) \cdot p(0). \tag{4.5}$$

This yields

$$\int_0^{+\infty} \zeta(t)\, dt < +\infty, \qquad \text{where} \quad \zeta(t) = |(x(t), p(t))|^2. \tag{4.6}$$

Since the concave–convex function H_0 is everywhere finite, its subdifferential multifunction ∂H_0 is bounded on bounded sets [6, Lemma 4], so that in particular there is a number λ such that the elements of the set $\partial H_0(a, b)$ are bounded in norm by λ when $|(a, b)| \leq 1$. Then, since $(x(t), p(t))$ satisfies the system (1.21), we have

$$|(x(t), p(t))| \leq \lambda + \rho \qquad \text{whenever} \quad |(x(t), p(t))| \leq 1. \tag{4.7}$$

For $\zeta(t)$ as in (4.6), this means that

$$\dot{\zeta}(t) \leq 2(\lambda + \rho) \qquad \text{whenever} \quad \zeta(t) \leq 1. \tag{4.8}$$

We proceed now to show that (4.6) and (4.8) imply

$$\lim_{t \to +\infty} \zeta(t) = 0, \tag{4.9}$$

a property equivalent, of course, to the desired conclusion (4.2). Fix any $\epsilon \in (0, 1)$ and let

$$S = \{t \in [0, +\infty) | \ \zeta(t) \leqslant \epsilon/2\}.$$

The set S is closed by the continuity of $\zeta(t)$, so its complement in $[0, +\infty)$ is the union of a sequence of intervals. The finiteness of the integral in (4.6) ensures that the intervals among these having length $\epsilon/2(\lambda + \rho)$ or greater are all contained in $[0, T]$ for some T sufficiently large. Then for every $t > T$, there exists $t_0 \in S$ such that $| t - t_0 | < \epsilon/4(\lambda + \rho)$. But (4.8) implies

$$\zeta(t) < \zeta(t_0) + 2(\lambda + \rho)| t - t_0 | \qquad \text{if} \quad t_0 \in S \text{ and } 2(\lambda + \rho)| t - t_0 | \leqslant \epsilon/2. \tag{4.10}$$

Thus $\zeta(t) \leqslant \epsilon$ if $t > 0$. Since ϵ can be taken arbitrarily small, (4.9) is indeed correct.

We next state a result for the functions f_T and g_T in (2.29) and (2.30) that does not make fullest use of the global curvature assumption, although the latter will enter via Proposition 7 when we argue later by taking the limit as $T \to +\infty$.

THEOREM 3. *Under the global curvature assumption, the function f_T for $0 < T < +\infty$ is everywhere continuously differentiable and strictly convex on $R^n \times R^n$, and the infimum in its definition is always attained by a unique arc. The same properties hold for g_T . Furthermore, one has the conjugacy relations*

$$g_T(b, b') = \max_{(a,a')} \{e^{-\rho T}a' \cdot b' - a \cdot b - f_T(a, a')\} = f_T^*(-b, e^{-\rho T}b'), \tag{4.11}$$

$$f_T(a, a') = \max_{(b,b')} \{e^{-\rho T}a' \cdot b' - a \cdot b - g_T(b, b')\} = g_T^*(-a, e^{-\rho T}a'), \tag{4.12}$$

and the gradient relation

$$(-b, e^{-\rho T}b') = \nabla f_T(a, a') \Leftrightarrow (-a, e^{-\rho T}a') = \nabla g_T(b, b'). \tag{4.13}$$

The conditions in (4.13) *are satisfied if and only if*

$$f_T(a, a') + g_T(b, b') = e^{-\rho T}a' \cdot b' - a \cdot b. \tag{4.14}$$

Proof. Relations (4.11) and (4.12) in "sup" form, and the existence of minimizing arcs, follow from [4, Corollary 2 of Theorem 1] and the global finiteness of the functions L_0 and M_0. This finiteness also implies from the definitions (2.29) and (2.30) that f_T and g_T are finite everywhere, and hence that "sup" can be strengthened to "max" in passing to the conjugate functions f_T^* and g_T^* [5, pp. 217–218]. In view of (4.11) and (4.12), we have the subdifferential relation

$$(-b, e^{-\rho T}b') \in \partial f_T(a, a') \Leftrightarrow (-a, e^{-\rho T}a') \in \partial g_T(b, b'), \tag{4.15}$$

these conditions being equivalent to (4.14). Suppose now for $i = 1, 2$ that (a_i, a_i') and (b_i, b_i') are such that these conditions hold, and let $x_i(t)$ and $p_i(t)$ be corresponding arcs over $[0, T]$ furnishing the minima in the definition of $f_T(a_i, a_i')$ and $g_T(b_i, b_i')$. Then, according to the converse part of Proposition 4, $(x_i(t), p_i(t))$ satisfies the system (1.21) with

$$(x_i(0), p_i(0)) = (a_i, b_i) \qquad \text{and} \qquad (x_i(T), p_i(T)) = (a_i', b_i'). \tag{4.16}$$

Invoking Corollary 1 of Proposition 5, we see that the function θ in (3.16) satisfies $\theta(T) > \theta(0)$, unless $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$ for all $t \in [0, T]$. But

$$\theta(0) = (a_1 - a_2) \cdot (b_1 - b_2), \qquad \theta(T) = e^{-\rho T}(a_1' - a_2') \cdot (b_1' - b_2'). \tag{4.17}$$

Therefore, the equation $(a_1, a_1') = (a_2, a_2')$ implies $(b_1, b_1') = (b_2, b_2')$, and conversely. This shows that the subdifferential multifunctions ∂f_T and ∂g_T are actually one-to-one functions, so that f_T and g_T must be differentiable and strictly convex [5, Corollary 26.3.1]. The argument also shows the uniqueness of the minimizing arcs over $[0, T]$, and the proof of Theorem 3 is, therefore, complete.

The first consequence of Theorem 3 which we derive concerns the existence of minimizing arcs in the definitions of the functions f_+ and g_+.

PROPOSITION 8. *Under the global curvature assumption, the convex function f_+ is everywhere finite on R^n, and for each $a \in R^n$, there is a unique arc p over $[0, +\infty)$ furnishing the minimum in the definition of $f_+(a)$.*

Proof. Since L_0 is finite and has the properties (1.29), it is evident that

$$0 \leqslant f_+(a) \leqslant f_T(a, 0) < +\infty \qquad \text{for all} \quad T > 0. \tag{4.18}$$

Hence, f_+ is finite. Now fix any $a \in R^n$. The definitions of f_+ and f_T yield the identity

$$f_+(a) = \inf_{a' \in R^n} \{f_T(a, a') + e^{-\rho T} f_+(a')\} \qquad \text{for all} \quad T > 0. \tag{4.19}$$

We know from Theorem 3 that f_T is finite, strictly convex, and cofinite (i.e., has an everywhere-finite conjugate function f_T^*). In particular, this implies that, as a function of a', $f_T(a, a')$ is strictly convex and satisfies the growth condition

$$\lim_{\lambda \to +\infty} [f_T(a, a' + \lambda a'') - f_T(a, a')]/\lambda = +\infty \quad \text{whenever} \quad a'' \neq 0 \tag{4.20}$$

[5, Corollary 13.3.1]. The convexity of f_+ ensures, of course, that the difference quotient $[f_+(a' + \lambda a'') - f_+(a')]/\lambda$ is always nondecreasing in λ, so it follows that for each $T > 0$, the function

$$h_T(a') = f_T(a, a') + e^{-\rho T} f_+(a') \tag{4.21}$$

is everywhere finite on R^n, strictly convex and satisfies

$$\lim_{\lambda \to +\infty} [h_T(a' + \lambda a'') - h_T(a')]/\lambda = +\infty \quad \text{whenever} \quad a'' \neq 0. \tag{4.22}$$

Therefore, h_T attains its minimum over R^n at a unique point [5, Theorem 27.2]. Let us denote this point by $x(T)$, defining also $x(0) = a$. We then have the identiy

$$f_+(a) = f_T(a, x(T)) + e^{-\rho T} f_+(x(T)) \quad \text{for all} \quad T > 0. \tag{4.23}$$

Note that this identity would also have to be satisfied by any arc giving the minimum in the definition of $f_+(a)$, so the function $x : [0, +\infty) \to R^n$ that we have constructed is the unique candidate for such an arc. To verify that x is absolutely continuous, we temporarily fix T and let y denote the unique arc over $[0, T]$ giving the minimum in the definition of $f_T(a, x(T))$ (cf. Theorem 3). For all $S \in (0, T)$, it is true that

$$f_T(a, x(T)) = f_S(a, y(S)) + e^{-\rho S} f_{T-S}(y(S), x(T)), \tag{4.24}$$

$$f_+(y(S)) \leqslant f_{T-S}(y(S), x(T)) + e^{-\rho(T-S)} f_+(x(T)). \tag{4.25}$$

combining (4.24) and (4.25) with (4.23), we obtain

$$f_+(a) \geqslant f_S(a, y(S)) + e^{-\rho S} f_+(y(S)). \tag{4.26}$$

On the other hand, the formula

$$f_+(a) = \min_{a' \in R^n} \{f_S(a, a') + e^{-\rho S} f_+(a')\} \tag{4.27}$$

holds, with the minimum attained uniquely at the point $x(S)$. Therefore (4.26) implies $y(S) = x(S)$. This is true for all $S \in (0, T)$, so the arcs x and

y coincide over $[0, T]$. Thus x is absolutely continuous, and for every $T > 0$ we have

$$f_T(a, x(T)) = \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt. \tag{4.28}$$

Plugging the latter into (4.23), we get

$$f_+(a) \geqslant \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \qquad \text{for all} \quad T > 0,$$

and, therefore,

$$f_+(a) \geqslant \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt. \tag{4.29}$$

Thus x must give the minimum in the definition of $f_+(a)$ and is the unique arc to do so.

The argument establishing the assertions of Proposition 8 about g_+ is entirely parallel.

PROPOSITION 9. *Under the global curvature assumption, we have*

$$f_+(a) = \lim_{T \to +\infty} [\min_{a' \in R^n} f_T(a, a')], \tag{4.30}$$

$$g_+(b) = \lim_{T \to +\infty} [\min_{b' \in R^n} g_T(b, b')]. \tag{4.31}$$

Proof. Let

$$\psi_T(a) = \min_{a' \in R^n} f_T(a, a'). \tag{4.32}$$

(The "min", in place of "inf", is appropriate because of the growth property of f_T displayed in (4.20).) The identity

$$f_T(a, a') = \inf_{c \in R^n} \{f_S(a, c) + e^{-\rho S} f_{T-S}(c, a')\} \qquad \text{for} \quad 0 < S < T \tag{4.33}$$

shows that

$$f_T(a, a') \geqslant \inf_{c \in R^n} f_S(a, c) = \psi_S(a), \tag{4.34}$$

and it is true, therefore, that

$$\psi_T(a) \geqslant \psi_S(a) \qquad \text{for} \quad 0 < S < T < +\infty. \tag{4.35}$$

At the same time we have

$$f_+(a) \leqslant \psi_T(a) \qquad \text{for all } T > 0 \tag{4.36}$$

by (4.21) and the nonnegativity of f_+. Hence, the limit

$$\psi_+(a) \triangleq \lim_{t \to +\infty} \psi_T(a) \tag{4.37}$$

exists and satisfies

$$0 \leqslant \psi_+(a) \leqslant f_+(a) < +\infty. \tag{4.38}$$

We must show that also $\psi_+(a) \geqslant f_+(a)$. The convexity of f_T implies that of the functions ψ_T and, hence, that of ψ_+. The next step consists of demonstrating that ψ_+, like f_+, satisfies

$$\psi_+(a) = \min_{a' \in R^n} \{f_T(a, a') + e^{-\rho T}\psi_+(a')\} \qquad \text{for all} \quad T > 0. \tag{4.39}$$

Certainly the definitions imply

$$\psi_{T+S}(a) = \inf_{a' \in R^n} \{f_T(a, a') + e^{-\rho T}\psi_S(a')\} \tag{4.40}$$

for all $T > 0, S > 0$. In particular, then, we have

$$\psi_{T+S}(a) \leqslant f_T(a, a') + e^{-\rho T}\psi_S(a') \qquad \text{for all} \quad a' \in R^n, \tag{4.41}$$

and passing to the limit as $S \to +\infty$, we get

$$\psi_+(a) \leqslant f_T(a, a') + e^{-\rho T}\psi_+(a') \qquad \text{for all} \quad a' \in R^n. \tag{4.42}$$

On the other hand, let us fix any $a \in R^n$ and $T > 0$, and consider the function

$$k_S(a') = f_T(a, a') + e^{-\rho T}\psi_S(a'). \tag{4.43}$$

Since ψ_S is finite and convex, while f_T is strictly convex with the growth property (4.20), we have k_S strictly convex with

$$\lim_{\lambda \to +\infty} [k_S(a' + \lambda a'') - k_S(a')]/\lambda = +\infty \qquad \text{if} \quad a'' \neq 0. \tag{4.44}$$

It follows that k_S attains its minimum over R^n at a unique point a_S'. Then

$$\psi_{T+S}(a) = f_T(a, a_S') + e^{-\rho T}\psi_S(a_S'). \tag{4.45}$$

Observe that this relation entails

$$f_T(a, a_S') \leqslant \psi_{T+S}(a) \leqslant \psi_+(a),$$

and, hence,

$$a_S' \in B = \{a' \in R^n \mid f_T(a, a') \leqslant \psi_+(a)\}. \tag{4.46}$$

The set B is bounded, by virtue of the growth property (4.20) [5, Theorems 8.4 and 27.1(f)], so that (4.46) implies the existence of a cluster point of a_S' as $S \to +\infty$. Since the functions ψ_S are convex and converge pointwise to ψ_+ on R^n, they actually converge uniformly on all bounded sets [5, Theorem 10.8]. Therefore, in passing to the limit as $S \to +\infty$ in (4.45), we have

$$\psi_+(a) = f_T(a, a') + e^{-\rho T}\psi_+(a'), \tag{4.47}$$

where a' is any cluster point of a_s' as $s \to +\infty$. Thus, equality does hold in (4.42) for some a', and (4.39) is correct.

We next apply to (4.39) the same argument we applied in the proof of Proposition 8 to the parallel formula for f_+. This yields the existence for each $a \in R^n$ of an arc x over $[0, +\infty)$ satisfying $x(0) = a$,

$$\psi_+(a) = f_T(a, x(T)) + e^{-\rho T}\psi_+(x(T)), \tag{4.48}$$

$$f_T(a, x(T)) = \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt, \tag{4.49}$$

for all $T > 0$. But then

$$\psi_+(a) \geq \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \geq f_+(a). \tag{4.50}$$

This completes the proof of Proposition 9, the argument for g_+ being parallel.

PROPOSITION 10. *Let F_+ and G_+ be the functions defined like f_+ and g_+ in* (1.37) *and* (1.38), *but with the infima taken only over arcs which are bounded over* $[0, +\infty)$. *Then*

$$\lim_{T \to +\infty} f_T(a, a') = F_+(a) \qquad \text{for all} \quad a' \in R^n, \tag{4.51}$$

$$\lim_{T \to +\infty} g_T(b, b') = G_+(b) \qquad \text{for all} \quad b' \in R^n. \tag{4.52}$$

Proof. The properties (1.29) of L_0 imply that $f_T(a, 0)$ is nonincreasing as a function of $T > 0$, and $f_T(a, 0) \geq F_+(a)$. (Any arc x over $[0, T]$ with $x(0) = a$ and $x(T) = 0$ can be continued over $[0, +\infty)$ by defining $x(t) = 0$ for all $t > T$.) Thus, the function

$$\tilde{F}_+(a) \triangleq \lim_{T \to +\infty} f_T(a, 0) \geq F_+(a) \tag{4.53}$$

is well defined. For $T > S > 0$, we have

$$f_{T+S}(a, 0) \leq f_T(a, a') + e^{-\rho T} f_S(a', 0), \tag{4.54}$$

$$f_T(a, a') \leq f_{T-S}(a, 0) + e^{-\rho(T-S)} f_S(0, a'). \tag{4.55}$$

Taking the limit in these inequalities as $T \to +\infty$ for fixed S and a', we obtain, respectively,

$$\tilde{F}_+(a) \leqslant \liminf_{T\to+\infty} f_T(a, a'), \tag{4.56}$$

$$\limsup_{T\to+\infty} f_T(a, a') \leqslant \tilde{F}_+(a), \tag{4.57}$$

and, hence,

$$\lim_{T\to+\infty} f_T(a, a') = \tilde{F}_+(a). \tag{4.58}$$

We finish the proof of (4.51) by showing that $\tilde{F}_+(a) \leqslant F_+(a)$. Fix $a \in R^n$ and $\epsilon > 0$, and let x be a bounded arc over $[0, +\infty)$ such that $x(0) = a$ and

$$\int_0^\infty L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < F_+(a) + \epsilon. \tag{4.59}$$

Then for all $T > 0$, it is true that

$$F_+(a) + \epsilon > \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \geqslant f_T(a, x(T)). \tag{4.60}$$

According to (4.58), the functions $a' \to f_T(a, a')$ converge as $T \to +\infty$ to the constant function $a' \to \tilde{F}_+(a)$, and since the functions are convex, the convergence must be uniform on all bounded sets [5, Theorem 10.8], in particular on the set $\{x(T) \mid 0 \leqslant T < +\infty\}$ Therefore,

$$\lim_{T\to+\infty} f_T(a, x(T)) = \tilde{F}_+(a), \tag{4.61}$$

and (4.60) thus implies $F_+(a) + \epsilon \geqslant \tilde{F}_+(a)$. Since $\epsilon > 0$ was arbitrary, we are able to conclude $F_+(a) \geqslant \tilde{F}_+(a)$ as aimed.

The argument for g_+ and G_+ is parallel.

Our main "global" result can now be treated.

THEOREM 4. *Under the global curvature assumption, the functions f_+ and g_+ are everywhere continuously differentiable and strictly convex on R^n, and they satisfy the conjugacy relations*

$$g_+(b) = \max_{a\in R^n} \{-a \cdot b - f_+(a)\} = f_+{}^*(-b), \tag{4.62}$$

$$f_+(a) = \max_{b\in R^n} \{-a \cdot b - g_+(b)\} = g_+{}^*(-a). \tag{4.63}$$

Furthermore, one has

$$\begin{aligned}(a, b) \in K_+ &\Leftrightarrow f_+(a) + g_+(b) + a \cdot b = 0 \\ &\Leftrightarrow b = -\nabla f_+(a) \Leftrightarrow a = -\nabla g_+(b).\end{aligned} \tag{4.64}$$

For each $(a, b) \in K_+$, there is a unique solution $(x(t), p(t))$ to the Hamiltonian system (1.21) *over* $[0, +\infty)$ *satisfying* (1.35) *and* (1.36), *and it tends to* $(0, 0)$ *as* $t \to +\infty$. *In fact, x is the unique arc furnishing the minimum in the definition of $f_+(a)$, while p is the unique arc furnishing the minimum in the definition of $g_+(b)$.*

Proof. Defining ψ_T as in (4.32), we have from Theorem 3 the relation

$$\begin{aligned} g_T(b, 0) &= \max_{(a,a')} \{-a \cdot b - f_T(a, a')\} \\ &= \max_{a} \{-a \cdot b - \psi_T(a)\} = \psi_T^*(-b). \end{aligned} \tag{4.65}$$

As $T \to +\infty$, the convex functions ψ_T converge pointwise to f_+ by Proposition 9, while the convex functions $b \to g_T(b, 0)$ converge pointwise to G_+ by Proposition 10. Thus ψ_T and the conjugate ψ_T^* converge to finite limit functions as $T \to +\infty$, implying that these limit functions must be conjugate to each other (cf. [6; 7; 5, Theorem 10.8]). Therefore, $G_+(b) = f_+^*(-b)$, so that (again by virtue of the finiteness of the two functions)

$$G_+(b) = \max_{a \in R^n} \{-a \cdot b - f_+(a)\} \tag{4.66}$$

and, reciprocally,

$$f_+(a) = \max_{b \in R^n} \{-a \cdot b - G_+(b)\}. \tag{4.67}$$

The next thing to note is that G_+ satisfies the identity

$$G_+(b) = \inf_{b' \in R^n} \{g_T(b, b') + e^{-\rho T} G_+(b')\} \qquad \text{for all } T > 0. \tag{4.68}$$

This is evident from the definition of G_+ in Proposition 10. The same argument used in connection with Formula (4.19) in the proof of Proposition 8 shows for each $b \in R^n$ the existence of a unique arc p over $[0, +\infty)$ satisfying $p(0) = b$ and

$$G_+(b) = g_T(b, p(T)) + e^{-\rho T} G_+(p(T)) \qquad \text{for all } T > 0, \tag{4.69}$$

$$g_T(b, p(T)) = \int_0^T M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \qquad \text{for all } T > 0. \tag{4.70}$$

Now fix any $a \in R^n$ and let x be the unique arc over $[0, +\infty)$ giving the minimum in the definition of $f_+(a)$, as exists by Proposition 8. Then

$$f_+(a) = f_T(a, x(T)) + e^{-\rho T} f_+(x(T)) \qquad \text{for all } T > 0, \tag{4.71}$$

$$f_T(a, x(T)) = \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \qquad \text{for all } T > 0. \tag{4.72}$$

Let b be such that the maximum in (4.67) is attained, and let p be a corresponding arc over $[0, +\infty)$ with $p(0) = b$ satisfying (4.69) and (4.70). We then have from (4.69) and (4.71) that

$$\begin{aligned} 0 &= f_+(a) + G_+(b) + a \cdot b \\ &= f_T(a, x(T)) + g_T(b, p(T)) + e^{-\rho T}[f_+(x(T)) + G_+(p(T))] + a \cdot b \\ &= f_T(a, x(T)) + g_T(b, p(T)) - e^{-\rho T} x(T) \cdot p(T) + a \cdot b \\ &\quad + e^{-\rho T}[f_+(x(T)) + G_+(p(T)) + x(T) \cdot p(T)], \end{aligned} \tag{4.73}$$

and, consequently, by virtue of the conjugacy relations (4.66) and (4.11),

$$f_T(a, x(T)) + g_T(b, p(T)) - e^{-\rho T} x(T) \cdot p(T) + a \cdot b = 0 \quad \text{for all} \quad T > 0, \tag{4.74}$$

$$f_+(x(T)) + G_+(p(T)) + x(T) \cdot p(T) = 0 \quad \text{for all} \quad T > 0. \tag{4.75}$$

Substituting (4.70) and (4.72) into (4.74) and applying Proposition 3, one sees that $(x(t), p(t))$ is a solution to the Hamiltonian system (1.21). Moreover, (4.74) implies in conjunction with (4.69) and (4.71) that

$$\begin{aligned} \limsup_{T \to +\infty} e^{-\rho T} x(T) \cdot p(T) &= \limsup_{T \to +\infty} [f_T(a, x(T)) + g_T(b, p(T)) + a \cdot b] \\ &\leqslant f_+(a) + G_+(b) + a \cdot b < +\infty, \end{aligned} \tag{4.76}$$

and, hence, via Proposition 7, that

$$\lim_{t \to +\infty} (x(t), p(t)) = (0, 0). \tag{4.77}$$

The minimizing arc x is thus bounded over $[0, +\infty)$, and therefore $f_+(a) = F_+(a)$. This has been verified for an arbitrary $a \in R^n$, so actually $f_+ = F_+$.

A parallel argument shows that likewise $g_+ \equiv G_+$. Thus, the formulas derived above for f_+ and G_+ are actually valid for f_+ and g_+; both (4.62) and (4.63) are valid, and for each (a, b) in the set

$$\begin{aligned} K_+' &= \{(a, b) \mid f_+(a) + g_+(b) + a \cdot b = 0\} \\ &= \{(a, b) \mid f_+(a) + g_+(b) + a \cdot b \leqslant 0\} \\ &= \{(a, b) \mid -b \in \partial f(a)\} = \{(a, b) \mid -a \in \partial g_+(b)\}, \end{aligned} \tag{4.78}$$

there exists a solution to the system (1.21) over $[0, +\infty)$ with $(x(0), p(0)) = (a, b)$, satisfying (4.77) and (by (4.75))

$$(x(T), p(T)) \in K_+' \quad \text{for all} \quad T > 0. \tag{4.79}$$

But this implies $K_+' \subset K_+$, whereas on the other hand, the inclusion $K_+ \subset K_+'$ follows from the corollary to Proposition 3. Therefore, $K_+' = K_+$, and for each $(a, b) \in K_+$, there is a solution to the system (1.21) over $[0, +\infty)$ with $(x(0), p(0)) = (a, b)$ which remains in K_+ and tends to (0, 0) as $t \to +\infty$. Corollary 2 of Proposition 5 tells us this must be the unique solution to (1.21) satisfying

$$\lim_{t \to \infty} e^{-\rho t} x(t) \cdot p(t) = 0,$$

and having either $x(0) = a$ or $p(0) = b$. Hence, for each $a \in R^n$, there is no more than one $b \in R^n$ with $(a, b) \in K_+$, and for each $b \in R^n$ there is no more than one $a \in R^n$ with $(a, b) \in K_+$. Since $K_+ = K_+'$, this says that the subgradient sets $\partial f_+(a)$ and $\partial g_+(b)$ never contain more than one element. It follows that f_+ and g_+ are continuously differentiable [5, Theorems 23.4, 25.1, and 25.5] and in view of their conjugacy relationship, also strictly convex [5, Theorem 26.3].

COROLLARY. *Under the global curvature assumption, one has* $F_+ = f_+$ *and* $G_+ = g_+$.

Proof of Theorem 1. As already has been noted, there is no loss of generality if the global curvature assumption is invoked in the proof of Theorem 1. (The uniqueness assertion is covered by Corollary 2 of Proposition 3.) Thus, we can place ourselves in the context of Theorem 4, according to which K_+ is the graph of a homeomorphism from R^n onto R^n (namely the mapping $-\nabla f_+$, whose inverse is $-\nabla g_+$). The task is to show that, given any neighborhood $U \times V$ of (0, 0) in $R^n \times R^n$, there exists an open neighborhood $U_+ \times V_+$ of (0, 0) such that $K_+ \cap (U_+ \times V_+)$ is the graph of a homeomorphism from U_+ onto V_+, and for each $(a, b) \in K_+ \cap (U_+ \times V_+)$, the solution to the system (1.21) starting at (a, b) and tending to (0, 0) stays entirely within $K_+ \cap (U_+ \times V_+)$. Since K_+ is already the graph of a global homeomorphism, the local homeomorphism property will certainly be satisfied if U_+ and V_+ are taken to be of the form

$$\begin{aligned} U_+ &= \{a \mid \exists b \quad \text{with} \quad (a, b) \in K_+ \cap W\}, \\ V_+ &= \{b \mid \exists a \quad \text{with} \quad (a, b) \in K_+ \cap W\}, \end{aligned} \tag{4.80}$$

where W is some open neighborhood of (0, 0); one then has

$$K_+ \cap (U_+ \times V_+) = K_+ \cap W. \tag{4.81}$$

Thus the proof is reduced to showing that every neighborhood $U \times V$ of

$(0, 0)$ contains an open neighborhood W of $(0, 0)$ such that, for each $(a, b) \in K_+ \cap W$, the solution to the system (1.21) starting at (a, b) and tending to $(0, 0)$ stays entirely within $K_+ \cap W$. Actually, such a solution $(x(t), p(t))$ remains in K_+ by definition and, hence, by Theorem 4, it satisfies

$$f_+(x(t)) + g_+(p(t)) + x(t) \cdot p(t) = 0 \qquad \text{for all} \quad t > 0. \tag{4.82}$$

Moreover, $x(t) \cdot p(t)$ is nondecreasing in t; this follows from Proposition 5 in the case of $(x_1(t), p_1(t)) = (x(t), p(t))$ and $(x_2(t), p_2(t)) \equiv (0, 0)$. Hence, the expression $f_+(x(t)) + g_+(p(t))$ is nonincreasing. This indicates that the desired properties can be obtained by taking W to be of the form

$$W = \{(a, b) | f_+(a) + g_+(b) < \epsilon\}, \tag{4.83}$$

provided this set is indeed, for $\epsilon > 0$ sufficiently small, an open neighborhood of $(0, 0)$ contained in whatever neighborhood $U \times V$ has been specified. But the latter properties follow from the fact that f_+ and g_+ are finite, strictly convex functions (hence, continuous) satisfying by definition

$$f_+(a) \geqslant f_+(0) = 0 \qquad \text{and} \qquad g_+(b) \geqslant g_+(0) = 0 \tag{4.84}$$

(cf. [5, Theorem 27.2]).

5. Results for the Interval $(-\infty, 0]$

Most of the results for $[0, +\infty)$ can easily be derived in a parallel form for $(-\infty, 0]$ but with some important simplifications. In building up to the proofs of Theorems 1′ and 2′, we begin with facts corresponding to those in Proposition 2.

Proposition 2′. *The inequalities*

$$\int_{-\infty}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < +\infty$$

and (5.1)

$$\int_{-\infty}^{0} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt < +\infty$$

imply, respectively, that

$$\lim_{t \to -\infty} e^{-\rho t} x(t) = 0 \qquad \textit{and} \qquad \lim_{t \to -\infty} e^{-\rho t} p(t) = 0. \tag{5.2}$$

Proof. The argument is based on Proposition 1 and follows exactly the same lines as the first part of the proof of Proposition 2.

COROLLARY. *If* (5.1) *holds, then*

$$\lim_{t \to -\infty} e^{-\rho t} x(t) \cdot p(t) = 0. \tag{5.3}$$

PROPOSITION 3'. *For* $-\infty < T < 0$, *one always has*

$$\int_T^0 L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_T^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \geqslant x(0) \cdot p(0) - e^{-\rho T} x(T) \cdot p(T). \tag{5.4}$$

Furthermore, equality holds here if and only if $(x(t), p(t))$ *satisfies the Hamiltonian system* (1.21) *over* $[T, 0]$.

Proof. Same argument as for Proposition 3.

COROLLARY 1. *For arbitrary arcs* x *and* p *over* $(-\infty, 0]$, *one has*

$$\int_{-\infty}^0 L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_{-\infty}^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \geqslant x(0) \cdot p(0), \tag{5.5}$$

with equality if and only if $(x(t), p(t))$ *is a solution to system* (1.21) *over* $(-\infty, 0]$ *which satisfies* (5.3).

COROLLARY 2. *One has*

$$f_-(a) + g_-(b) \geqslant a \cdot b \qquad \text{for all} \quad (a, b) \in R^n \times R^n, \tag{5.6}$$

$$K_- \subset \{(a, b) \mid f_-(a) + g_-(b) = a \cdot b\}. \tag{5.7}$$

In fact, K_- *consists precisely of the pairs* (a, b) *belonging to the set on the right in* (5.7) *such that the infima in the definitions of* $f_-(a)$ *and* $g_-(b)$ *are both attained by arcs* x *and* p, *respectively; such pairs of arcs give the solutions* $(x(t), p(t))$ *to the system* (1.21) *described in the definition of* K_-, *and they always satisfy* (5.2) *and consequently*

$$\lim_{t \to -\infty} (x(t), p(t)) = (0, 0). \tag{5.8}$$

Note the absence in Corollary 1 of any assumption on $e^{-\rho t} x(t) \cdot p(t)$, which is justified by the corollary to Proposition 2'. This has produced the

general inequality (5.6). We were able to derive the similar inequality for f_+ and g_+ only under the *global* curvature assumption.

Proof of Theorem 2' Using Theorem 1'. This is obvious in view of Corollary 2 above, except for the assertions about ∇f_- and ∇g_-. The relations (5.6) and (5.7) do imply that

$$K_- \subset \{(a, b) | \ b \in \partial f_-(a)\}, \tag{5.9}$$

and, hence, the homeomorphism Θ in Theorem 1' from U_- onto V_- whose graph is $K_- \cap (U_- \times V_-)$ satisfies

$$\Theta(a) \in \partial f_-(a) \qquad \text{for all} \quad a \in U_- . \tag{5.10}$$

But f_- is a convex function which is finite on U_-. Therefore, $\partial f_-(a)$ reduces almost everywhere on U_- to the gradient $\nabla f_-(a)$ [5, Theorem 25.5]. Moreover, at the remaining points $a \in U_-$, $\partial f_-(a)$ can be constructed as the convex hull of the limiting values of gradients $\nabla f_-(a')$ existing at points a' near a [5, Theorem 25.6]. The existence of a homeomorphism Θ satisfying (5.10) therefore implies that $\partial f_-(a)$ reduces everywhere on U_- to a single element, and this must then be the gradient $\nabla f_-(a)$. Thus (5.10) is equivalent to

$$\Theta(a) = \nabla f_-(a) \qquad \text{for all} \quad a \in U_- , \tag{5.11}$$

and in particular f_- is continuously differentiable on U_-. This, with a similar argument for g_-, justifies the second and third equivalences in (1.46).

The next result corresponds to Proposition 7 and Corollary 2 of Proposition 5.

PROPOSITION 11. *Let $(x(t), p(t))$ be a solution to the system* (1.21) *over* $(-\infty, 0]$ *with* $(x(t), p(t)) = (a, b)$. *Then either*

$$\lim_{t \to -\infty} e^{-\rho t} x(t) \cdot p(t) = -\infty \tag{5.12}$$

or one has (5.3), *and hence* $(a, b) \in K_+$. *If* (5.3) *holds and* $(x(t), p(t))$ *remains in* $U \times V$ (*the neighborhood of* (0, 0) *in the basic curvature assumption*) *for all* $t \in (-\infty, 0]$, *then any solution* $(x'(t), p'(t))$ *to* (1.21) *likewise with property* (5.3) *and having either* $x'(0) = a$ *or* $p'(0) = b$, *must satisfy*

$$(x'(t), p'(t)) = (x(t), p(t)) \qquad \textit{for all} \quad t \in (-\infty, 0]. \tag{5.13}$$

Proof. From Proposition 3′, we have

$$e^{-\rho T}x(T)\cdot p(T) = x(0)\cdot p(0) - \int_T^0 L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt - \int_T^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \tag{5.14}$$

for all $T < 0$. If the limit of the right side of (5.11) is not $-\infty$, then (5.3) holds by the corollary to Proposition 2′, and indeed the stronger relations (5.2) are valid. If $(x'(t), p'(t))$ also has these properties and has either $x'(0) = a$ or $p'(0) = b$, the function

$$\theta(t) = e^{-\rho t}(x'(t) - x(t))\cdot(p'(t) - p(t)) \tag{5.15}$$

satisfies

$$\lim_{t\to-\infty}\theta(t) = 0 = \theta(0). \tag{5.16}$$

Assuming that $(x(t), p(t)) \in U \times V$ for all $t \in (-\infty, 0]$, we must have (5.13) because of Corollary 1 of Proposition 5.

PROPOSITION 8′. *Under the global curvature assumption, the convex function f_- is everywhere finite on R^n, and for each $a \in R^n$, there is a unique arc x over $(-\infty, 0]$ furnishing the minimum in the definition of $f_-(a)$. Similarly, g_- is finite everywhere on R^n, and for each $b \in R^n$, there is a unique arc p over $(-\infty, 0]$ furnishing the minimum in the definition of $g_-(b)$.*

Proof. Closely parallel to Proposition 8. No new version of Theorem 3 need be stated for the intervals $[T, 0]$, since for $T < 0$, one has

$$\inf\left\{\int_T^0 L_0(x(t), x(t))\, e^{-\rho t}\, dt \;\middle|\; x(T) = a', x(0) = a\right\} = e^{-\rho T} f_{-T}(a', a), \tag{5.17}$$

$$\inf\left\{\int_T^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \;\middle|\; p(T) = b', p(0) = b\right\} = e^{-\rho T} g_{-T}(b', b). \tag{5.18}$$

We next derive facts corresponding to Propositions 9 and 10.

PROPOSITION 12. *Under the global curvature assumption, one has*

$$\lim_{T\to-\infty} e^{-\rho T} f_{-T}(a', a) = f_-(a) \quad \text{if } a' = 0, \qquad = +\infty \quad \text{if } a' \neq 0, \tag{5.19}$$

$$\lim_{T\to-\infty} e^{-\rho T} g_{-T}(b', b) = g_-(b) \quad \text{if } b' = 0, \qquad = +\infty \quad \text{if } b' \neq 0, \tag{5.20}$$

and consequently also

$$\lim_{T\to-\infty}\ [\min_{a'\in R^n} e^{-\rho T} f_{-T}(a', a)] = f_-(a), \tag{5.21}$$

$$\lim_{T\to-\infty}\ [\min_{b'\in R^n} e^{-\rho T} g_{-T}(b', b)] = g_-(b). \tag{5.22}$$

Proof. Note first from (5.14) that

$$e^{-\rho T} f_{-T}(0, a) \geqslant f_-(a) \qquad \text{for all} \quad T < 0, \tag{5.23}$$

where the left side is nonincreasing as $T \to -\infty$ and, hence, approaches a limit. To show that the limit is $f_-(a)$, select an arbitrary $\epsilon > 0$ and any arc x over $(-\infty, 0]$ such that $x(0) = a$ and

$$\int_{-\infty}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < f_-(a) + \epsilon. \tag{5.24}$$

For all T sufficiently low, one will have

$$\int_{T+1}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < f_-(a) + \epsilon. \tag{5.25}$$

Let $\bar{x}$ denote the arc corresponding to the minimum in the definition of $f_1(0, x(T+1))$ (as exists by Theorem 3), and define the arc x' over $[T, 0]$ by

$$\begin{aligned} x'(t) &= \bar{x}(t - T) \qquad \text{for} \quad T \leqslant t \leqslant T + 1, \\ &= x(t) \qquad\quad \text{for} \quad T + 1 \leqslant t \leqslant 0. \end{aligned} \tag{5.26}$$

Then, by (5.17),

$$\begin{aligned} e^{-\rho T} f_{-T}(0, a) &\leqslant \int_T^0 L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt \\ &= \int_T^{T+1} L_0(\bar{x}(t), \dot{\bar{x}}(t))\, e^{-\rho t}\, dt + \int_{T+1}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \\ &= e^{-\rho T} f_1(0, x(T+1)) + \int_{T+1}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt. \end{aligned} \tag{5.27}$$

On the other hand, since f_1 is convex with $0 = f_1(0, 0) = \min f_1$, it is true (for $T < 0$) that

$$0 \leqslant e^{-\rho T} f_1(0, x(T+1)) \leqslant f_1(0, e^{-\rho T} x(T+1)). \tag{5.28}$$

But (5.24) implies via Proposition 2′ that

$$\lim_{T\to-\infty} e^{-\rho T} x(T+1) = 0. \tag{5.29}$$

Hence,

$$\lim_{T \to -\infty} e^{-\rho T} f_1(0, x(T+1)) = 0, \tag{5.30}$$

because f_1 is continuous (Theorem 3). From this and inequalities (5.25) and (5.27), it is clear that one will have

$$e^{-\rho T} f_{-T}(0, a) < f_-(a) + \epsilon \tag{5.31}$$

for all T sufficiently low. The first limit assertion in (5.19) is thereby established. The second part of (5.19) follows from the fact that

$$\lim_{T \to -\infty} f_{-T}(a', a) = f_+(a') > 0 \qquad \text{if} \quad a' \neq 0, \tag{5.32}$$

which is valid because of Proposition 10 and the corollary to Theorem 4; since f_+ is nonnegative and strictly convex by Theorem 4, it cannot vanish except at the origin. The implication from (5.19) to (5.21) can be verified using the convexity of the expression $e^{-\rho T} f_{-T}(a', a)$ as a function of a' for fixed a.

The proofs of (5.20) and (5.22) are analogous.

THEOREM 4'. *Under the global curvature assumption, the functions f_- and g_- are everywhere continuously differentiable and strictly convex on R^n, and they satisfy the conjugacy relations*

$$g_-(b) = \max_{a \in R^n} \{a \cdot b - f_-(a)\} = f_-{}^*(b), \tag{5.33}$$

$$f_-(a) = \max_{b \in R^n} \{a \cdot b - g_-(b)\} = g_-{}^*(a). \tag{5.34}$$

Furthermore, one has

$$\begin{aligned} (a, b) \in K_- &\Leftrightarrow f_-(a) + g_-(b) = a \cdot b \\ &\Leftrightarrow b = \nabla f_-(a) \Leftrightarrow a = \nabla g_-(b). \end{aligned} \tag{5.35}$$

For each $(a, b) \in K_-$, there is a unique solution $(x(t), p(t))$ to the Hamiltonian system (1.21) *over $(-\infty, 0]$ satisfying $(x(0), p(0)) = (a, b)$ and* (5.3), *and it tends to* (0, 0) *as $t \to -\infty$. In fact, x is the unique arc furnishing the minimum in the definition of $f_-(a)$, while p is the unique arc furnishing the minimum in the definition of $g_-(b)$.*

Proof. Let

$$\phi_T(a) = \min_{a' \in R^n} e^{-\rho T} f_{-T}(a', a) \qquad \text{for} \quad T < 0. \tag{5.36}$$

Then ϕ_T is a finite convex function. Using the conjugacy relations in Theorem 3, we calculate that

$$\begin{aligned}\phi_T^*(b) &= \max_{a\in R^n}\{a\cdot b - \phi_T(a)\}\\ &= e^{-\rho T}\max_{(a,a')}\{e^{\rho T}a\cdot b - a\cdot 0 - f_{-T}(a', a)\} \qquad (5.37)\\ &= e^{-\rho T}g_{-T}(0, b).\end{aligned}$$

By Proposition 12, therefore, the functions ϕ_T converge pointwise to f_- as $T\to-\infty$, while their conjugates ϕ_T^* converge pointwise to g_-. Since f_- and g_- are finite everywhere (Proposition 8'), this implies that f_- and g_- are conjugate to each other (cf. [6, 7]), i.e., Formulas (5.33) and (5.34) are correct. (The finiteness justifies writing "max" in place of "inf" [5, Theorems 23.4 and 23.5].) Proposition 8' and Corollary 2 of Proposition 3' give us now the facts stated after (5.35), as well as the first equivalence in (5.35). Since f_- and g_- are conjugate to each other, we certainly have

$$f_-(a) + g_-(b) = a\cdot b \Leftrightarrow b\in\partial f_-(a) \Leftrightarrow a\in\partial g_-(b) \qquad (5.38)$$

[5, Theorem 23.5], and, hence,

$$K_- = \{(a, b)|\ b\in\partial f_-(a)\} = \{(a, b)|\ a\in\partial g_-(b)\}. \qquad (5.39)$$

The uniqueness properties in Proposition 11 assert that K_- is the graph of a one-to-one mapping, and it therefore follows that f_- and g_- are continuously differentiable and strictly convex [5, Corollary 26.3.1]. The proof of Theorem 4' is thereby finished.

Proof of Theorem 1'. First we prove (1.44). It is clear that

$$(0, 0)\in K_+\cap K_-,$$

since the Hamiltonian system (1.21) is satisfied by $(x(t), p(t))\equiv(0, 0)$. Suppose that also $(a, b)\in K_+\cap K_-$. Then there is a solution $(x(t), p(t))$ to the Hamiltonian system over $(-\infty, +\infty)$ satisfying $(x(0), p(0)) = (a, b)$ and

$$0 = \lim_{t\to+\infty}\theta(t) = \lim_{t\to-\infty}\theta(t), \qquad \text{where}\quad \theta(t) = e^{-\rho t}x(t)\cdot p(t). \qquad (5.40)$$

Applying Corollary 1 of Proposition 5 to $(x_1(t), p_1(t))\equiv(x(t), p(t))$ and $(x_2(t), p_2(t))\equiv(0, 0)$, we find that this implies $(x(t), p(t)) = (0, 0)$ for all $t\in(-\infty, +\infty)$. In particular, $(a, b) = (0, 0)$. Therefore (1.44) is correct.

The assertion in Theorem 1′ about uniqueness of solutions is covered by Proposition 11. The rest of the proof is just like that of Theorem 1. We invoke the global curvature assumption and apply Theorem 4. The sets U_- and V_- are defined by

$$U_- = \{a \mid \exists b \quad \text{with} \quad (a, b) \in K_- \cap W\}, \tag{5.41}$$

$$V_- = \{b \mid \exists a \quad \text{with} \quad (a, b) \in K_- \cap W\}, \tag{5.42}$$

where

$$W = \{(a, b) \mid f_-(a) + g_-(b) \leqslant \epsilon\} \tag{5.43}$$

for ϵ sufficiently small. For each $(a, b) \in K_- \cap W$, the corresponding solution $(x(t), p(t))$ to the system (1.21) over $(-\infty, 0]$, as in the definition of K_-, remains in K_- and, hence, satisfies (by Theorem 4)

$$f_-(x(t)) + g_-(p(t)) = x(t) \cdot p(t) \qquad \text{for} \quad -\infty < t \leqslant 0. \tag{5.44}$$

But the right side of (5.44) is nondecreasing as a function of t by Proposition 5, and, hence, the left side is also nondecreasing. This shows that $(x(t), p(t))$ remains in W for all $t \in (-\infty, 0]$. The details of the proof can thus be effected as in the case of Theorem 1.

6. Some Complementary Results

We present now two counterexamples, as well as a theorem on the existence of a stationary point $(\bar{k}, \bar{q})$ in certain cases where the discount rate ρ is sufficiently small. For other existence results of somewhat different import, see [1].

Example 1. This will demonstrate that strict concavity–convexity of H_0 near $(0, 0)$ is not enough to guarantee "saddle point behavior" of the dynamical system for a given $\rho > 0$, and that something like the inequality in the basic curvature assumption is necessary. Let $n = 1$ and

$$H(x, p) = -\tfrac{1}{2}x^2 + \tfrac{1}{2}p^2 + xp \qquad \text{for all} \quad (x, p) \in R \times R. \tag{6.1}$$

Then H is strictly concave in x and strictly convex in p, with $H(0, 0) = 0$ and $\nabla H(0, 0) = (0, 0)$. Thus $(\bar{k}, \bar{q}) = (0, 0)$ satisfies the rest-point condition (1.15), and we have $H_0 = H$. The perturbed Hamiltonian system (1.21) reduces to

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \rho - 1 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \tag{6.2}$$

a differential equation whose behavior around (0, 0) is easily analyzed. One finds that there is "saddle point behavior" if $0 \leqslant \rho < 2$, but if $\rho > 2$, then *all* solutions to (6.2) (other than $(x(t), p(t)) \equiv (0, 0)$) diverge from (0, 0) as $t \to +\infty$.

EXAMPLE 2. The purpose of this example is to show that the problem in the definition (1.1) of $\phi(c)$, if deprived of the boundedness restriction on $e^{-\rho t}k(t)$, may fail to be reducible to the problem in the definition (1.37) of $f_+(a)$ as in Proposition 2 and its corollary. Taking $n = 1$, we define

$$L(k, z) = r(k) + r(z) + 4(z - \rho k) \qquad \text{for} \quad (k, z) \in R \times R, \tag{6.3}$$

where

$$\begin{aligned} r(z) &= \tfrac{1}{2}z^2 \qquad \text{if} \quad |z| \leqslant 1, \\ &= |z| \qquad \text{if} \quad |z| \geqslant 1. \end{aligned} \tag{6.4}$$

Note that r is a differentiable convex function on the real line whose conjugate is

$$\begin{aligned} r^*(p) &= \tfrac{1}{2}|p|^2 \qquad \text{if} \quad |p| \leqslant 1, \\ &= +\infty \qquad \text{if} \quad |p| > 1. \end{aligned} \tag{6.5}$$

From the definition (1.5), we have

$$H(k, q) = -r(k) + r^*(q - 4) + 4\rho k. \tag{6.6}$$

Thus, the rest point condition (1.15) is satisfied by $(\bar{k}, \bar{q}) = (0, 4)$, and H is α-concave–β-convex near $(\bar{k}, \bar{q})$ for $\alpha = \beta = 1$. The basic curvature condition is therefore fulfilled, assuming $0 < \rho < 2^{1/2}$.

We claim that $\phi(c)$ is finite for all $c \in R$, while the function $\tilde{\phi}$ obtained by dropping the boundedness restriction in the infimum defining $\phi(c)$ has in fact $\tilde{\phi}(c) = -\infty$ for all $c \in R$; thus, the boundedness restriction plays an essential role. To see this, consider first an arbitrary arc k over $[0, +\infty)$ with $k(0) = c$ and $e^{-\rho t}k(t)$ bounded. If $L(k(t), \dot{k}(t))\, e^{-\rho t}$ is majorized by a summable function over $[0, +\infty)$, Proposition 2(a) implies that $L_0(k(t), \dot{k}(t))\, e^{-\rho t}$ is summable over $[0, +\infty)$ (here $k(t) = x(t)$ because $\bar{k} = 0$), and furthermore $e^{-\rho t}k(t) \to 0$ as $t \to +\infty$. In consequence,

$$\begin{aligned} &\int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt \\ &\quad = \int^{+\infty} [r(k(t)) + r(\dot{k}(t))]\, e^{-\rho t}\, dt + 4\int_0^{\infty} (d/dt)\, e^{-\rho t}k(t)\, dt \\ &\quad \geqslant -4k(0) = -4c. \end{aligned} \tag{6.7}$$

This demonstrates that

$$\phi(c) \geqslant -4c > -\infty \qquad \text{for all} \quad c \in R. \tag{6.8}$$

On the other hand, the finiteness of L ensures that $\phi(c) < +\infty$. Therefore ϕ is indeed finite. Observe at the same time that $r(z) \leqslant |z|$ and, hence,

$$\begin{aligned} L(k, z) &\leqslant |k| + |z - \rho k + \rho k| + 4(z - \rho k) \\ &\leqslant (1 + \rho)|k| + |z - \rho k| + 4(z - \rho k). \end{aligned} \tag{6.9}$$

For the arc $k(t) = -e^{2\rho t}$ (which does *not* have $e^{-\rho t}k(t)$ bounded over $[0, +\infty)$), we obtain from (6.9) that

$$\begin{aligned} \int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt &\leqslant \int_0^{+\infty} [(1 + \rho)\, e^{2\rho t} + \rho e^{2\rho t} - 4\rho e^{2\rho t}]\, e^{-\rho t}\, dt \\ &= [1 - 2\rho] \int_0^{+\infty} e^{\rho t}\, dt = -\infty \end{aligned} \tag{6.10}$$

if $\rho > \frac{1}{2}$. In this case,

$$-\infty = \check{\phi}(k(0)) = \check{\phi}(-1),$$

and since $\check{\phi}$ is a convex function with $\check{\phi}(c) \leqslant \phi(c) < +\infty$, it follows that $\check{\phi}(c) = -\infty$ for all $c \in R$ [5, Theorem 7.2].

Thus if $\frac{1}{2} < \rho < 2^{1/2}$, we have $\phi(c) \neq \check{\phi}(c)$ for all c, even though the basic curvature assumption is fulfilled.

We conclude now with our theorem on the existence of stationary points for sufficiently small values of the discount rate ρ.

THEOREM 5. *Suppose that $(\bar{k}_0, \bar{q}_0)$ satisfies the stationary point condition* (1.15) *for $\rho = 0$ and that H is α-concave–β-convex in some neighborhood of $(\bar{k}_0, \bar{q}_0)$. Then for each $\rho > 0$ sufficiently small, there exists a unique pair $(\bar{k}_\rho, \bar{q}_\rho)$ satisfying the stationary point condition for that value of ρ, and one has*

$$\lim_{\rho \to 0} (\bar{k}_\rho, \bar{q}_\rho) = (0, 0). \tag{6.11}$$

Proof. We know that H is globally concave–convex (since it is related to the convex function L by (1.5)), and our hypothesis entails the finiteness of H near $(0, 0)$. Therefore, H is continuous near $(0, 0)$ [5, Theorem 35.1]. Let $C \times D$ be a compact convex neighborhood of $(\bar{k}_0, \bar{q}_0)$ such that, on $C \times D$, $H(k, q)$ is finite and continuous, α-concave in k and β-convex in q. In particular, for each $q' \in R^n$ and $\rho \geqslant 0$ the function

$$(k, q) \to H(k, q) - \rho k q' \tag{6.12}$$

is strictly concave–convex on $C \times D$ and has a unique saddle point relative to $C \times D$ (in the minimax sense), which, as is easy to see from the continuity of H and definition of "saddle point," must depend continuously on q' and ρ. Let

$$\Phi_\rho : C \times D \to C \times D \tag{6.13}$$

be the continuous mapping which assigns to each $(k', q') \in C \times D$ the saddlepoint (k'', q'') of the function (6.12) relative to $C \times D$. Then Φ_ρ has at least one fixed point. Such fixed points are the pairs $(\bar{k}, \bar{q}) \in C \times D$ such that

$$H(\bar{k}, \bar{q}) - \rho k \cdot \bar{q} - \rho k \cdot q \leqslant H(\bar{k}, \bar{q}) - \rho \bar{k} \cdot \bar{q} \qquad \text{for all} \quad k \in C, \tag{6.14}$$

$$H(\bar{k}, q) - \rho \bar{k} \cdot \bar{q} \geqslant H(\bar{k}, \bar{q}) - \rho \bar{k} \cdot \bar{q} \qquad \text{for all} \quad q \in D. \tag{6.15}$$

Of course, the term $\rho \bar{k} \cdot \bar{q}$ can be dropped from both sides of (6.15). The theory of the minimum of a convex function informs us that (6.15) holds if and only if there is a vector $\bar{m}$ such that

$$\bar{m} \cdot q \leqslant \bar{m} \cdot \bar{q} \qquad \text{for all} \quad q \in D, \tag{6.16}$$

$$H(\bar{k}, q) + \bar{m} \cdot q \geqslant H(\bar{k}, \bar{q}) + \bar{m} \cdot \bar{q} \qquad \text{for all} \quad q \in R^n \tag{6.17}$$

(cf. [5, Theorem 27.4]). Likewise, (6.14) holds if and only if there exists a vector $\bar{n}$ such that

$$k \cdot \bar{n} \leqslant \bar{k} \cdot \bar{n} \qquad \text{for all} \quad k \in C, \tag{6.18}$$

$$H(\bar{k}, \bar{q}) - \rho k \cdot \bar{q} - k \cdot \bar{n} \leqslant H(\bar{k}, \bar{q}) - \rho \bar{k} \cdot \bar{q} - \bar{k} \cdot \bar{n} \qquad \text{for all} \quad k \in R^n. \tag{6.19}$$

Observe that (6.17) and (6.19) can be combined into

$$(\rho \bar{q} + \bar{n}_0 - \bar{m}) \in \partial H(\bar{k}, \bar{q}). \tag{6.20}$$

If $\bar{n} = 0$ and $\bar{m} = 0$, as must be true by (6.16) and (6.17) if $(\bar{k}, \bar{q})$ is an interior point of $C \times D$, then (6.20) reduces to the stationary point condition (1.15).

We have seen that for each ρ there is at least one set of vectors $\bar{k}, \bar{q}, \bar{m}, \bar{n}$ with $(\bar{k}, \bar{q}) \in C \times D$ such that (6.16)–(6.19) are satisfied. To investigate this further, consider also another set of vectors $\bar{k}', \bar{q}', \bar{m}', \bar{n}'$ satisfying these conditions for a possibly different value ρ'. Since H is α-concave–β–convex on $C \times D$, (6.17) implies

$$H(\bar{k}, q) + \bar{m} \cdot q \geqslant H(\bar{k}, \bar{q}) + \bar{m} \cdot \bar{q} + \tfrac{1}{2}\beta \mid q - \bar{q} \mid^2 \qquad \text{for all} \quad q \in C, \tag{6.21}$$

while (6.19) implies

$$H(k,\bar{q}) - k\cdot(\rho\bar{q}+\bar{n}) \geqslant H(\bar{k},\bar{q}) - k\cdot(\rho\bar{q}+\bar{n}) - \tfrac{1}{2}\alpha\,|\,k-\bar{k}\,|^2 \quad \text{for all } k\in D. \tag{6.22}$$

Therefore,

$$H(\bar{k},\bar{q}') - H(\bar{k},\bar{q}) \geqslant \bar{m}\cdot(\bar{q}-\bar{q}') + \tfrac{1}{2}\beta\,|\,\bar{q}'-\bar{q}\,|^2, \tag{6.23}$$

$$H(\bar{k},\bar{q}) - H(\bar{k}',\bar{q}) \geqslant (\bar{k}-\bar{k}')\cdot(\rho\bar{q}+\bar{n}) + \tfrac{1}{2}\alpha\,|\,\bar{k}'-\bar{k}\,|^2, \tag{6.24}$$

and dually,

$$H(\bar{k}',\bar{q}) - H(\bar{k}',\bar{q}') \geqslant \bar{m}'\cdot(\bar{q}'-\bar{q}) + \tfrac{1}{2}\beta\,|\,\bar{q}-\bar{q}'\,|^2, \tag{6.25}$$

$$H(\bar{k}',\bar{q}') - H(\bar{k},\bar{q}') \geqslant (\bar{k}'-\bar{k})\cdot(\rho'\bar{q}'+\bar{n}') + \tfrac{1}{2}\alpha\,|\,\bar{k}-\bar{k}'\,|^2. \tag{6.26}$$

Adding the last four inequalities and using the fact that by (6.16) and (6.18),

$$\bar{m}\cdot(\bar{q}-\bar{q}') \geqslant 0, \qquad (\bar{k}-\bar{k}')\cdot\bar{n} \geqslant 0,$$
$$\bar{m}'\cdot(\bar{q}'-\bar{q}) \geqslant 0, \qquad (\bar{k}'-\bar{k})\cdot\bar{n}' \geqslant 0,$$

we obtain

$$0 \geqslant (\bar{k}-\bar{k}')(\rho\bar{q}-\rho'\bar{q}') + \alpha\,|\,\bar{k}'-\bar{k}\,|^2 + \beta\,|\,\bar{q}'-\bar{q}\,|^2, \tag{6.27}$$

or equivalently,

$$(\rho'-\rho)(\bar{k}-\bar{k}')\cdot\bar{q}' \geqslant \alpha\,|\,\bar{k}'-\bar{k}\,|^2 + \rho(\bar{k}'-\bar{k})\cdot(\bar{q}'-\bar{q}) + \beta\,|\,\bar{q}'-\bar{q}\,|^2. \tag{6.28}$$

Defining

$$\gamma = \max\{|(k-k')\cdot q'\,|\,|\,k\in C,\ k'\in C,\ q'\in D\}, \tag{6.29}$$

we can convert (6.28) into the bound

$$\gamma\,|\,\rho'-\rho\,| \geqslant \alpha\,|\,\bar{k}'-k\,|^2 - \rho\,|\,\bar{k}'-\bar{k}\,|\cdot|\,\bar{q}'-\bar{q}\,| + \beta\,|\,\bar{q}'-\bar{q}\,|^2$$
$$\geqslant \sigma_1[|\,\bar{k}'-\bar{k}\,|^2 + |\,\bar{q}'-\bar{q}\,|]^2, \tag{6.30}$$

where

$$\sigma_1 = (4\alpha\beta-\rho^2)/4(\alpha+\beta+\rho); \tag{6.31}$$

the proof of the second inequality in (6.30) is given at the end of the proof of Proposition 5. Let us suppose that $\rho^2 < 4\alpha\beta$, so that $\sigma_1 > 0$. Then two

conclusions are apparent from (6.30). First, if $\rho' = \rho$, then $\bar{k}' = \bar{k}$ and $\bar{q}' = \bar{q}$. In other words, $(\bar{k}, \bar{q})$ is *uniquely* determined by conditions (6.16)–(6.19). Second, taking $\rho' = 0$ and $(\bar{k}', \bar{q}') = (\bar{k}_0, \bar{q}_0)$, we see that as $\rho \downarrow 0$, the point $(\bar{k}, \bar{q})$ must tend to $(\bar{k}_0, \bar{q}_0)$. Hence, for ρ sufficiently small, $(\bar{k}, \bar{q})$ must be an interior point of $C \times D$, so that $\bar{m} = 0 = \bar{n}$ as already explained, and the stationary point condition (1.15) is actually satisfied. These two conclusions immediately yield the assertions in the theorem.

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Existence of Solutions to Hamiltonian Dynamical Systems of Optimal Growth

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1. INTRODUCTION

Cass and Shell [3] have shown how a wide range of problems in the theory of economic dynamics may be formulated in terms of Hamiltonian dynamical systems. In particular they have studied the system modeling consumption–optimal growth with discount rate $\rho > 0$:

$$\begin{aligned} \dot{k} &= \partial H(Q, k)/\partial Q, \\ \dot{Q} &= -(\partial H(Q, k)/\partial k) + \rho Q, \\ k(0) &= \bar{k} \geq 0, \\ \lim_{t \to +\infty} Q(t)\, e^{-\rho t} k(t) &= 0, \\ Q(t) &\geq 0, \\ k(t) &\geq 0, \end{aligned} \tag{1}$$

where

$k = (k_1, \ldots, k_n) =$ capital stocks,

$Q = (Q_1, \ldots, Q_n) =$ current prices of capital goods output,

and

$H(Q, k) =$ current value of imputed NNP.

The Hamiltonian function $H(Q, k)$ is defined on the basis of certain assumptions about technology. For a detailed discussion of the manner in which the Hamiltonian function "captures" the essence of the techno-

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logy, a discussion of the derivation and significance of the particular model (1), and an extensive bibliography we refer the reader to [3].

In this paper we present a set of conditions on the Hamiltonian function $H(Q, k)$ which are sufficient to guarantee that the system (1) has a nonnegative solution. These conditions, presented and briefly discussed in Section 2, include convexity–concavity and other properties that are consistent with or mimic the technological assumptions of Cass and Shell. Various implications of these conditions which are crucial to our existence analysis (and which are of some interest in their own right) are presented in Section 3. These include a priori bounds on solutions to a finite time problem associated with (1).

The existence proof, presented in Section 4, relies on a continuation principle from the theory of fixed points for multivalued mappings in infinite dimensional spaces and on the a priori information obtained in Section 3. The mathematical machinery we employ is described in the appendix.

The system (1) can be considered as necessary conditions arising from an associated optimization problem, i.e., the problem of maximizing the integral of discounted consumption over an infinite time horizon. For a study of (1) from this vantage point, we refer the reader to the excellent work of Rockafellar [8]. Using *only* convexity–concavity assumptions on $H(Q, k)$, Rockafellar obtains a remarkably complete qualitative theory for problem (1) including existence of solutions. He also shows that solutions to (1) are solutions to the optimization problem (if that problem is properly posed). Though we do not reproduce the machinery to state and analyze the optimization problem here, using the propositions developed by Rockafellar in [8, Section 2], it can be shown that *under our assumptions, solutions to* (1) (*whose* existence we establish) *are solutions to the optimization problem* (as posed by Rockafellar).

Our conditions described in Section 2 involve much weaker convexity–concavity on $H(Q, k)$ than Rockafellar's α-concave, β-convex condition, but on the other hand our methods require exploitation of structural features of $H(Q, k)$ (designed to mimic assumptions on the underlying technology) other than convexity–concavity. As a bonus from these additional assumptions we obtain various bounds on solutions; for example, nonnegativity of $(Q(t), k(t))$. We suspect, though to our knowledge the work has not been carried out, that our assumptions could be used in conjunction with the most general theorems of optimal control (see, for example, [6]) to prove existence of solutions to (1). (This observation was suggested by W. A. Brock.)

However, apart from our primary interest in developing an existence theory that exploits the underlying technological assumptions of Cass–

Shell and only those assumptions, our second primary objective is to develop machinery for investigating existence which is *independent of the associated optimization problem.* Our view is that (1) is a representative selected from a large class of models some of which are descriptive models where the associated optimization problem is only implicity defined, if it exists at all. The application of the kind of machinery developed here to those descriptive models is a subject for future research.

2. Sufficient Conditions for Existence

Existence of solutions to (1) will be established under the conditions (E1)–(E6) stated in this section. In the statements, if $x = (x_1, x_2, \ldots, x_n)$, then $\| x \| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. Inequalities stated for set-valued functions hold for every member of the set. The Hamiltonian function $H(Q, k)$ of Cass and Shell is defined on the nonnegative orthant; i.e., on $\{(Q, k): Q \geqslant 0, k \geqslant 0\}$. To simplify our analysis we will assume that $H(Q, k)$ can be extended so that its domain is all of $R^n \times R^n$. Moreover, we assume that the extended function satisfies

(E.1) Convexity–Concavity. *H is concave in k and convex in Q on $R^n \times R^n$.*

Under (E1) H possesses generalized gradients

$$\frac{\partial H(Q, k)}{\partial Q}, \frac{\partial H(Q, k)}{\partial k}$$

defined by

$$\frac{\partial H(Q, k)}{\partial Q} = \{z \in R^n: H(Q', k) \geqslant H(Q, k) + (Q' - Q) \cdot z, \forall Q' \in R^n\},$$

$$\frac{\partial H(Q, k)}{\partial k} = \{r \in R^n: H(Q, k') \leqslant H(Q, k) + (k' - k) \cdot r, \forall k' \in R^n\}.$$

It is these "set-valued derivatives" which appear in (1). Thus, to be completely accurate we should write in place of (1) the system

$$\begin{aligned}
&k \in (\partial H(Q, k)/\partial Q),\\
&\dot{Q} \in -(\partial H(Q, k)/\partial k) + \rho Q,\\
&\quad k(0) = \bar{k} \geqslant 0,\\
&\lim_{t \to +\infty} Q(t)\, e^{-\rho t} k(t) = 0,\\
&\quad Q(t) \geqslant 0,\\
&\quad k(t) \geqslant 0,
\end{aligned}$$

involving differential correspondences or contingent equations. By a *solution* to the first two correspondences we will mean a pair of absolutely continuous functions $(Q(t), k(t))$ which satisfies the inclusion requirement almost everywhere on some interval.

The assumption of convexity–concavity of the extended function $H(Q, k)$ on *all* of $R^n \times R^n$ is made for convenience and can be replaced by the weaker assumption of convexity–concavity on the nonnegative orthant. The extension to the weaker case can be made by considering an appropriate sequence $\{H_m(Q, k)\}$ of functions concave–convex on $R^n \times R^n$ which converge to a nonextendable $H(Q, k)$ on the nonnegative orthant, applying the existence theory to each $H_m(Q, R)$, and considering convergence of the resulting sequence of solutions $\{(Q_m(t), k_m(t)\}$. Since we deal only with nonnegative solutions anyway, the stronger assumption is used only as a simple way to guarantee regularity of the generalized gradients on the boundary of the nonnegative orthant.

(E2) Existence of a Nonnegative Stationary Point. *There exists* $(Q^*, k^*) \geqslant (0, 0)$ *such that*

$$0 \in \partial H(Q^*, k^*)/\partial k,$$

$$0 \in (\partial H(Q^*, k^*)/\partial Q) + \rho Q^*.$$

Cass and Shell [3] establish the existence of such a stationary point as a natural consequence of assumptions on technology.

(E3) Limitation by Primary Factors. *There exists* $K > 0$ *such that* $Q \geqslant 0$, $k \geqslant 0$, $(k - k^*) \cdot (Q - Q^*) \leqslant 0$, *and* $\| k \| \geqslant K$, *then*

$$(\partial H(Q, k)/\partial Q) \cdot k < 0.$$

Since $\partial H(Q, k)/\partial Q$ has the economic interpretation of instantaneous net investment, this condition says, roughly speaking, that if capital stocks are sufficiently large the corresponding net investments must be nonpositive. Thus, this is a mathematically convenient manner of expressing the spirit of the condition Cass and Shell call "necessity of primary factors" (see [3, Condition (T6) and Assumption (B)]).

(E4) Productivity of Technology. *There exists* $\gamma > 0$ *such that if* $Q \geqslant 0$, $k \geqslant 0$, $\| k \| \leqslant \max [\| \bar{k} \|, K] \equiv K_0$, $(k - k^*) \cdot (Q - Q^*) \leqslant 0$, *and* $\| Q - Q^* \| \geqslant \gamma$, *then* $(\partial H(Q, k)/\partial Q) \cdot (Q - Q^*) > 0$.

The vector $\partial H(Q, k)/\partial Q$ represents those levels of technologically feasible instantaneous net investment which maximize instantaneous

NNP (*given* capital stocks k and prices Q). This assumption asserts, roughly speaking, that if prices Q of capital goods outputs are sufficiently high ($\| Q - Q^* \| \geqslant \gamma$) and corresponding capital stocks are *not* too high ($(k - k^*) \cdot (Q - Q^*) \leqslant 0$), then NNP will be maximized by choosing positive investment levels. Thus, the assumption entails that certain capital stock levels can support positive net investment and is in the same spirit as [3, (T4)].

(E5) BOUNDED RATE OF DEPRECIATION. *There exists* $\lambda > 0$ *such that for* $Q \geqslant 0$ *and* $k \geqslant 0$,

$$\partial H(Q, k)/\partial Q \geqslant -\lambda k.$$

Since $\partial H(Q, k)/\partial Q$ represents an instantaneous net investment z, this is implied by [3, (T7)].

(E6) FREE DISPOSAL IN ALLOCATION. *For the same* $\lambda > 0$ *as in* (E5), $Q \geqslant 0$ *and* $k \geqslant 0$,

$$-(\partial H(Q, k)/\partial k) \leqslant \lambda Q.$$

This can be shown to be a consequence of the definition of $H(Q, k)$ and the free disposal assumption (T8) of Cass and Shell [3].

3. CONSEQUENCES OF THE EXISTENCE CONDITIONS: A PRIORI BOUNDS

We begin by stating some regularity results.

LEMMA 1. *Under assumption* (E1) the mapping

$$T : (Q, k) \mapsto (\partial H(Q, k)/\partial k, -(\partial H(Q, k)/\partial Q) + \rho Q)$$

(i) *nonempty, convex, compact-valued*;

(ii) *upper semi-continuous.*

Proof. This is essentially a result of Rockafellar [6, Lemma 2, p. 415].

LEMMA 2. *Under assumption* (E1), *if* $(Q(t), k(t))$ *is a pair of absolutely continuous functions on* $[a, b]$ *such that*

$$\dot{k}(t) \in \partial H(Q(t), k(t))/\partial Q, \text{ a.e. on } [a, b],$$

$$\dot{Q}(t) \in -(\partial H(Q(t), k(t))/\partial k) + \rho Q(t), \text{ a.e. on } [a, b],$$

then $H(Q(t), k(t))$ *is absolutely continuous on* $[a, b]$ *and*

$$d[H(Q(t), k(t))]/dt = \rho Q(t) \cdot \dot{k}(t) \text{ a.e. on } [a, b].$$

Proof. This can be obtained by slightly modifying another proof of Rockafellar [6, Theorem 3, p. 42].

Note that Lemma 2 has an immediate economic interpretation since $H(Q(t), k(t))$ represents NNP along the path $(Q(t), k(t))$ and $\dot{k}(t)$ is instantaneous net investment.

Our mathematical strategy will be to consider first the following problem on the finite time interval $[0, T]$:

$$\begin{aligned} \dot{k} &\in (\partial H(Q, k)/\partial Q), \\ \dot{Q} &\in -(\partial H(Q, k)/\partial k) + \rho Q, \\ k(0) &= \bar{k} \geqslant 0, \\ Q(T) &= Q^*. \end{aligned} \tag{2.T}$$

We will then obtain a solution to (1′) by considering sequences of solutions to (2.T) with $T \to +\infty$.

The following Lemmas yield a priori information concerning the location of solution trajectories for (2.T). The first lemma is a modification of an argument of Cass and Shell (see [3]).

LEMMA 3. *Under assumptions* (E1) *and* (E2), *if* $(Q(t), k(t))$ *is a solution to* (2.T), *then*

$$(k(t) - k^*)(Q(t) - Q^*) \leqslant 0, \qquad t \in [0, T].$$

Proof. Let

$$V(t) = (k(t) - k^*)(Q(t) - Q^*).$$

We have from (2.T) and (E1) that

$$\begin{aligned} \frac{d[e^{-\rho t}V(t)]}{dt} &= e^{-\rho t}[\dot{V}(t) - \rho V(t)] \\ &= e^{-\rho t}[\dot{k}(Q - Q^*) + (k - k^*)\dot{Q} - \rho(k - k^*)(Q - Q^*)] \\ &\in e^{-\rho t}\left[\frac{\partial H(Q, k)}{\partial Q}(Q - Q^*) + (k - k^*)\left(-\frac{\partial H(Q, k)}{\partial k} + \rho Q\right)\right. \\ &\qquad \left. - \rho(k - k^*)(Q - Q^*)\right] \\ &\geqslant e^{-\rho t}[H(Q, k) - H(Q^*, k) + H(Q, k^*) - H(Q, k) \\ &\quad + \rho(k - k^*)Q^*] \\ &\geqslant e^{-\rho t}[H(Q, k^*) - H(Q^*, k) + \rho(k - k^*)Q^*] \end{aligned}$$

where all the relationships hold almost everywhere on $[0, T]$. Since (Q^*, k^*) is a stationary point, we have from (E1) and (E2) that

$$0 \in (Q - Q^*)\frac{\partial H(Q^*, k^*)}{\partial Q} \leqslant H(Q, k^*) - H(Q^*, k^*),$$

$$0 \in \left[\rho Q^* - \frac{\partial H(Q^*, k^*)}{\partial Q}\right](k - k^*) \leqslant \rho Q^*(k - k^*) + H(Q^*, k^*) - H(Q^*, k).$$

But then

$$H(Q^*, k) - \rho Q^* k \leqslant H(Q^*, k^*) - \rho Q^* k^* \leqslant H(Q, k^*) - \rho Q^* k^*.$$

Thus

$$d[e^{-\rho t}V(t)]/dt \geqslant 0.$$

Since $V(T) = (k(T) - k^*)(Q^* - Q^*) = 0$, we have $e^{-\rho t}V(t) \leqslant 0$ on $[0, T]$.

LEMMA 4. *Under assumptions* (E5) *and* (E6), *if* $(Q(t), k(t))$ *is a solution to* (2.T), *then*

$$k(t) \geqslant \bar{k}e^{-\lambda t};$$

and

$$Q(t) \geqslant Q^* e^{-(\lambda+\rho)(T-t)}$$

on $[0, T]$.

Proof. We have that $k(t)$ is absolutely continuous on $[0, T]$ and

$$\dot{k}(t) \in \partial H(Q(t), k(t))/\partial Q \geqslant -\lambda k(t)$$

almost everywhere on $[0, T]$. But then

$$\dot{k}(t)\, e^{\lambda t} + \lambda K(t)\, e^{\lambda t} \geqslant 0 \qquad \text{a.e. on } [0, T].$$

Thus

$$d[e^{\lambda t}k(t)]/dt \geqslant 0, \qquad \text{a.e. on } [0, T],$$

and we have

$$e^{\lambda t}k(t) - \bar{k} \geqslant 0$$

or

$$k(t) \geqslant \bar{k}e^{-\lambda t} \qquad \text{a.e. on } [0, T].$$

LEMMA 5. *Under assumptions* (E1)–(E3), (E5), *and* (E6), *if* $(Q(t), k(t))$ *is a solution to* (2.T), *then*

$$\| k(t)\| \leqslant \max [\| \bar{k} \|, K] = K_0.$$

Proof. Suppose

$$\max_{[0,T]} \|k(t)\| = \| k(t_0)\| > K_0 .$$

Then $t_0 \neq 0$, since $k(0) = \bar{k}$. Since $\| k(t_0)\| > K$, there exists an interval $[t_0 - \Delta, t_0] \subset [0, T]$ such that

$$\| k(t)\| > K$$

on $[t_0 - \Delta, t_0]$. Moreover, by Lemmas 3 and 4

$$(k(t) - k^*)(Q(t) - Q^*) \leqslant 0,$$

and

$$k(t) \geqslant 0, Q(t) \geqslant 0$$

on $[t_0 - \Delta, t_0]$. Thus, by (2.T) and (E3),

$$d[\| k(t)\|^2]/dt = 2k(t) \cdot \dot{k}(t) \in 2k(t) \cdot (\partial H(Q(t), k(t))/\partial Q) < 0$$

a.e. on $[t_0 - \Delta, t_0]$. But then

$$\| k(t)\| > \| k(t_0)\|$$

on $[t_0 - \Delta, t_0)$, which contradicts our choice of t_0.

Before we obtain a similar bound on $\| Q(t)\|$ we need the following technical lemma.

LEMMA 6. *Under assumptions* (E1)–(E6), *if* $k \geqslant 0$, $Q \geqslant 0$, $\| Q - Q^* \| \geqslant \gamma$, $\| k \| \leqslant K_0$, *and* $(k - k^*)(Q - Q^*) \leqslant 0$, *then*

$$H(Q, k) \geqslant h_0 + ((\| Q - Q^* \|/\gamma) - 1)\mu$$

where

$$C_0 = \{(Q', k): \| Q' - Q^* \| = \gamma, \| k \| \leqslant K_0, (k - k^*)(Q - Q^*) \leqslant 0\},$$

$$h_0 = \min\{H(Q', k): (Q', k) \in C_0\},$$

$$\mu = \min\{(\partial H(Q', k)/\partial Q)(Q' - Q^*): (Q', k) \in C_0\} > 0.$$

Proof. By Lemma 1, $(\partial H(Q, k)/\partial Q)\, Q$ is upper semicontinuous. Hence, the set appearing in the definition of μ is compact. Thus from (E4), $\mu > 0$.

Let (Q, k) satisfy the hypothesis of the lemma and let

$$Q' = (\gamma/\| Q - Q^* \|)(Q - Q^*) + Q^*.$$

Then

$$\| Q' - Q^* \| = \gamma,$$

and

$$(k - k^*)(Q' - Q^*) = (\gamma/\| Q - Q^* \|)(Q - Q^*)(k - k^*) \leqslant 0.$$

From (E1)

$$H(Q, k) \geqslant H(Q', k) + (Q - Q')\,(\partial H(Q', k)/\partial Q).$$

We have

$$Q - Q' = [(\| Q - Q^* \|/\gamma) - 1](Q' - Q^*),$$

thus

$$H(Q, k) \geqslant h_0 + [(\| Q - Q^* \|/\gamma) - 1]\,\mu.$$

LEMMA 7. *Under assumptions* (E1) *through* (E6), *if* $(Q(t), k(t))$ *is a solution to* (2.T), *then*

$$\| Q(t) \| \leqslant Q_0 = \| Q^* \| + \gamma\{(1/\mu)[H_0 + 2\rho \| Q^* \| K_0 - h_0] + 1\}$$

where μ *and* h_0 *are as in Lemma* 6 *and*

$$H_0 = \max\{H(Q, k): \| Q - Q^* \| \leqslant \gamma, \| k \| \leqslant K_0, (k - k^*)(Q - Q^*) \leqslant 0, \\ k \geqslant 0, Q \geqslant 0\}.$$

Proof. We first show that

$$H(Q(t), k(t)) - \rho Q^* k(t) \leqslant H_0 + \rho \| Q^* \| K_0.$$

Suppose not; i.e., suppos

$$\max_{[0,T]} [H(Q(t), k(t)) - \rho Q^* k(t)] \\ = H(Q(t_0), k(t_0)) - \rho Q^* k(t_0) > H_0 + \rho \| Q^* \| K_0.$$

By Lemmas 3, 4, and 5 we have

$$\| k(t) \| \leqslant K_0,$$
$$(k(t) - k^*)(Q(t) - Q^*) \leqslant 0,$$

and

$$k(t) \geqslant 0, Q(t) \geqslant 0.$$

Thus we must have, from the definition of H_0,

$$\| Q(t_0) - Q^* \| > \gamma.$$

Moreover, by definition of H_0, since $Q(T) = Q^*$, we must have $t_0 < T$. Thus, there exists an interval $[t_0, t_0 + \Delta]$ on which

$$H(Q(t), k(t)) - \rho Q^* k(t) > H_0 + \rho \| Q^* \| K_0$$

and

$$\| Q(t) - Q^* \| > \gamma.$$

From Lemma 2 and (E4) we have

$$\frac{d[H(Q(t), k(t)) - \rho Q^* k(t)]}{dt}$$
$$= \rho Q(t)\, k(t) - \rho Q^* k(t) \in \rho(Q(t) - Q^*) \frac{\partial H(Q(t), k(t))}{\partial Q} > 0$$

a.e. on $[t_0, t_0 + \Delta]$. This contradicts our choice of t_0. By Lemma 6, if $\| Q(t) - Q^* \| \geqslant \gamma$, then

$$H_0 + \rho \| Q^* \| K_0 \geqslant H(Q(t), k(t)) - \rho Q^* k(t)$$
$$\geqslant h_0 + ((\| Q(t) - Q^* \|/\gamma) - 1)\, \mu - \rho \| Q^* \| K_0.$$

Thus,

$$\| Q(t) - Q^* \| \leqslant \gamma\{(1/\mu)[H_0 + 2\rho \| Q^* \| K_0 - h_0] + 1\}$$

and the desired inequality follows.

4. Existence Proof

Our preliminary esistence theorem dealing with problem (2.T) is based on the following principle.

Continuation Principle. *Suppose* (E1) *is satisfied. Suppose there exist* $K_0 > 0$ *and* $Q_0 > 0$ *such that any solution to*

$$\begin{aligned} \dot{k} &\in \sigma \frac{\partial H(Q, k)}{\partial Q}, \\ \dot{Q} &\in -\sigma \frac{\partial H(Q, k)}{\partial k} + \sigma \rho Q, \qquad (2.\mathrm{T}.\sigma) \\ k(0) &= \bar{k} \geqslant 0, \\ Q(T) &= Q^* \end{aligned}$$

for $\sigma \in (0, 1)$ *satisfies* $\| k(t) \| \leqslant K_0$, $\| Q(t) \| \leqslant Q_0$ *for* $t \in [0, T]$. *Then* (2.T) *has at least one solution* $(k(t), Q(t))$ *satisfying* $\| k(t) \| \leqslant K_0$ *and* $\| Q(t) \| \leqslant Q_0$ *for* $t \in [0, T]$.

This principle is an adaptation to our problem of Leray–Schauder degree theory extended to multifunctions. We outline the proof in an appendix.

As a consequence of Section 3 and the continuation principle we have

THEOREM 1. *Under assumptions* (E1)–(E6), *problem* (2.T) *has at least one solution* $(Q(t), k(t))$ *satisfying* $\| k(t)\| \leqslant K_0$, $\| Q(t)\| \leqslant Q_0$, $k(t) \geqslant 0$, *and* $Q(t) \geqslant 0$.

Proof. We need only observe that (2.T.σ) is a dynamical system corresponding to the Hamiltonian function

$$H_\sigma(Q, k) = \sigma H(Q, k).$$

This new Hamiltonian satisfies hypotheses (E1) through (E4) (with the new discount $\sigma\rho$) and new depreciation rate $\sigma\lambda$ with the same (Q^*, k^*) and the same K and γ.

We may apply Lemma 5 to problem (2.T.σ) obtaining the same bound K_0 for all σ. In Lemmas 6 and 7 the constants μ, h_0, and H_0 must be replaced by

$$\sigma\mu, \sigma h_0, \sigma H_0.$$

Since ρ is replaced by $\sigma\rho$, Lemma 7 yields the same Q_0 for all values of σ. Thus the continuation principle yields the existence of at least one solution.

We now are in a position to prove the central theorem of this paper.

THEOREM 2. *Under assumptions* (E1)–(E6), *problem* (1)′ *has at least one solution satisfying* $\| Q(t)\| \leqslant Q_0$, $\| k(t)\| \leqslant K_0$, $k(t) \geqslant 0$, *and* $Q(t) \geqslant 0$ *on* $[0, +\infty)$.

Proof. Let $\{T_n\} \to +\infty$ and let $(Q(t; T_n), k(t; T_n))$ be a solution to (2.T$_n$). Define

$$\begin{aligned} k(t; T_n) &= k(T_n; T_n), \qquad t \geqslant T_n, \\ Q(t; T_n) &= Q(T_n; T_n) = Q^*, \qquad t \geqslant T_n. \end{aligned} \tag{3}$$

Since $\| k(t; T_n)\| \leqslant K_0$ and $\| Q(t; T_n)\| \leqslant Q_0$ on $[0, T_n]$, upper semi-continuity (Lemma 1) implies that there exist $\dot{K}_0$, $\dot{Q}_0$ (independent of $k(t; T_n)$ and $Q(t; T_n)$) such that

$$\begin{aligned} \| k(t; T_n)\| &\in \left\| \frac{\partial H(Q(t; T_n), k(t; T_n))}{\partial Q} \right\| \leqslant \dot{K}_0, \\ \| \dot{Q}(t; T_n)\| &\in \left\| - \frac{\partial H(Q(t; T_n), k(t; T_n))}{\partial k} + \rho Q(t; T_n) \right\| \leqslant \dot{Q}_0 \end{aligned} \tag{4}$$

almost everywhere on $[0, T_n]$. Since the extended functions defined by (3) are constant for $t \geqslant T_n$, the inequalities (4) hold on $[0, \infty)$ almost everywhere.

We have by absolute continuity [9, p. 107],

$$\begin{aligned} \| k(t; T_n) - k(s; T_n) \| &= \left\| \int_s^t \dot{k}(\tau; T_n)\, d\tau \right\| \\ &\leqslant \left| \int_s^t \| \dot{k}(\tau; T_n) \|\, d\tau \right| \\ &\leqslant | t - s |\, \dot{K}_0 . \end{aligned}$$

Similarly,

$$\| Q(t; T_n) - Q(s; T_n) \| \leqslant | t - s |\, \dot{Q}_0 .$$

Thus, the sequence $\{(Q(t; T_n), k(t; T_n))\}$ is uniformly bounded and equicontinuous on $[0, +\infty)$. By the Ascoli–Arzela theorem [10, p. 179] there exists a subsequence, call it $\{(Q(t; T_k), k(t; T_k))\}$ for simplicity, which converges uniformly on compact subintervals of $[0, +\infty)$ to some $(Q_0(t), k_0(t))$.

It is a consequence of a theorem of Zaremba [13, Theorem II.6] (or also of the Bebernes–Shuur–Kamke theorem [1, p. 275]) that the limit function $(Q_0(t), k_0(t))$ is a solution to the differential correspondence in (1′) on $[0, \infty)$; i.e.,

$$\begin{aligned} \dot{k}_0(t) &\in (\partial H(Q_0(t), k_0(t))/\partial Q \qquad \text{a.e. on } [0, +\infty), \\ \dot{Q}_0(t) &\in -(\partial H(Q_0(t), k_0(t))/\partial k) + \rho Q_0(t) \qquad \text{a.e. on } [0, +\infty). \end{aligned}$$

Moreover, it is immediate that $k_0(0) = \bar{k}$, $\| Q_0(t) \| \leqslant Q_0$ on $[0, \infty)$, $\| k_0(t) \| \leqslant K_0$ on $[0, \infty)$, $k_0(t) \geqslant 0$ on $[0, \infty)$, and $Q_0(t) \geqslant 0$ on $[0, \infty)$. It follows that

$$\lim_{t \to +\infty} Q_0(t)\, e^{-\rho t} k_0(t) = 0.$$

APPENDIX. The Continuation Principle

In this appendix we outline the mathematical justification for the continuation principle stated at the beginning of Section 4. Our first step in this direction is to rewrite problem (2.T) as a fixed point problem in a function space setting. Let

$$C[0, T] = \{(x(t), y(t)) : x(t) \text{ and } y(t) \text{ are continuous functions from } [0, T] \text{ into } R^n\}$$

with the accompanying norm

$$\|(x, y)\| = \max_{[0,T]} \|x(t)\| + \max_{[0,T]} \|y(t)\|.$$

For $(x, y) \in C[0, T]$ we may consider the problem

$$\begin{aligned} &\dot{k}(t) \in \partial H(x(t), y(t))/\partial Q, \\ &\dot{Q}(t) \in -(\partial H(x(t), y(t))/\partial k) + \rho x(t), \\ &\qquad k(0) = \bar{k}, \\ &\qquad Q(T) = Q^*. \end{aligned} \tag{A1}$$

We define a mapping $\mathscr{K}: C[0, T] \to 2^{C[0,T]}$ (the subsets of $C[0, T]$), by

$$\mathscr{K}(x, y) = \{(Q, k): (Q, k) \text{ is a solution to (A1)}\}.$$

(Note that the functions in $\mathscr{K}(x, y)$ are actually absolutely continuous on $[0, T]$). The mapping $\mathscr{K}(x, y)$ may be given a somewhat more explicit description.

LEMMA A1. *For any* $(x, y) \in C[0, T]$,

$$\mathscr{K}(x, y) = \left\{\left(\bar{k} + \int_0^t z(s)\, ds,\ Q^* - \int_t^T r(s)\, ds\right): (z(t), r(t)) \in N_0(x, y)\right\}$$

where $N_0(x, y)$ *is the set of pairs* $(z(t), r(t))$ *which are Lebesgue-integrable on* $[0, T]$ *and such that*

$$\begin{aligned} z(t) &\in \frac{\partial H(x(t), y(t))}{\partial Q} \qquad \text{a.e. in } [0, T] \\ r(t) &\in -\frac{\partial H(x(t), y(t))}{\partial k} + \rho x(t) \qquad \text{a.e. in } [0, T]. \end{aligned} \tag{A2}$$

Proof. If $(Q(t), k(t)) \in \mathscr{K}(x, y)$ then $(Q(t), k(t))$ is a solution to (A1) and $Q(t)$ and $k(t)$ are absolutely continuous on $[0, T]$. Thus $\dot{k}(t)$ and $\dot{Q}(t)$ are Lebesgue-integrable [10, p. 107] and $(\dot{k}(t), \dot{Q}(t)) \in N_0(x, y)$. Moreover,

$$k(t) = k(0) + \int_0^t \dot{k}(s)\, ds = \bar{k} + \int_0^t \dot{k}(s)\, ds,$$

$$Q(t) = Q(T) - \int_t^T \dot{Q}(s)\, ds = Q^* - \int_t^T \dot{Q}(s)\, ds.$$

Conversely, if

$$k(t) = \bar{k} + \int_0^t z(s)\, ds,$$

$$Q(t) = Q^* - \int_t^T r(s)\, ds,$$

where $(z(s), r(s)) \in N_0(x, y)$, then since $(z(s), r(s))$ are Lebesgue-integrable

$$\begin{aligned} k(t) &= z(t) \in \partial H(x(t), y(t))/\partial Q \qquad \text{a.e. on } [0, T], \\ \dot{Q}(t) &= r(t) \in -(\partial H(x(t), y(t))/\partial k) + \rho x(t) \qquad \text{a.e. on } [0, T], \\ k(0) &= \bar{k}, \\ Q(T) &= Q^*. \end{aligned}$$

The importance of the mapping $\mathscr{H}$ is expressed in the following lemma which follows immediately from the definition of $\mathscr{H}$ and the form of (2.T).

LEMMA A2. *A pair $(Q(t), k(t))$ is a solution to* (2.T) *if and only if*

$$(Q, k) \in \mathscr{H}(Q, k).$$

In order to establish that (A3) has at least one solution we apply Leray–Schauder degree theory for multifunctions as developed by Webb [12]. A complete discussion of this theory is beyond the scope of this appendix. For a concise introduction to degree theory the reader is referred to Rouche and Mawhin [9, pp. 173–185]. In particular, we will employ the following two fundamental properties.

1. *Existence. Let X be a Banach space and $\Omega \subset X$ be an open bounded subset of X. Suppose $\mathscr{T}: \bar{\Omega} \to 2^X$ is such that*:

(a) *$\mathscr{T}(x)$ is convex, compact and nonempty for each $x \in \bar{\Omega}$;*

(b) *$\mathscr{T}$ is upper-semicontinuous on $\bar{\Omega}$;*

(c) *$\mathscr{T}(\Omega)$ is compact;*

(d) *$x \notin \mathscr{T}(x)$ for $x \in \partial\Omega$ (the boundary of Ω);*

(e) *$d[I - \mathscr{T}, \Omega, 0] \neq 0$ (the Leray–Schauder degree is nonzero).*

Then there exists $x \in \bar{\Omega}$ such that $x \in \mathscr{T}(x)$.

2. *Invariance under Homotopy. Let X be a Banach space and $\Omega \subset X$ be an open bounded subset of X. Suppose $\mathscr{T}(x, \sigma): \Omega \times [0, 1] \to 2^X$ is such that*:

(a) *$\mathscr{T}(x, \sigma)$ is convex, compact and nonempty for $(x, \sigma) \in \Omega \times [0, 1]$;*

(b) *$\mathscr{T}(x, \sigma)$ is upper semicontinuous on $\bar{\Omega} \times [0, 1]$;*

(c) *$\mathscr{T}(\Omega, \sigma)$ is compact for each $\sigma \in [0, 1]$;*

(d) *$x \notin \mathscr{T}(x, \sigma)$ for $(x, \sigma) \in \partial\Omega \times [0, 1]$.*

Then

$$d[I - \mathscr{T}(x, 0), \Omega, 0] = d[I - \mathscr{T}(x, 1), \Omega, 0].$$

In order to apply these results in our context we need the following technical lemma.

LEMMA A3. *Under assumption* (E1) *for any* $\bar{k}$ *and* Q^*:

(a) $\mathscr{K}(x, y)$ *is convex, nonempty, and compact for each* $(x, y) \in C[0, 1]$;

(b) $\mathscr{K}$ *is upper semicontinuous on* $C[0, 1]$;

(c) $\mathscr{K}(\Omega)$ *is compact for any open bounded set* Ω.

Proof. These properties have been established for a general class of mappings containing our map $\mathscr{K}$ by Lasota and Opial [5] (see also the paper of Grandolfi [4]). We will not include the details here. We observe only in passing that it is the properties stated in Lemma 1 which are crucial to the proof of Lemma A3.

Proof of the Continuation Principle. We define

$$\Omega = \{(Q, k) \in C[0, 1]: \max_{[0,T]} \| k(t)\| < K_0 + \epsilon, \max_{[0,T]} \| Q(t)\| < Q_0 + \epsilon\}$$

where $\epsilon > 0$ is an arbitrary positive number. Then Ω is an open bounded subset of $C[0, T]$. Define

$$\mathscr{T}(x, y, \sigma) = \{(Q, k): (Q, k) \text{ is a solution to (A4)}\}$$

where

$$\begin{aligned} k &= \sigma(\partial H(x(t), y(t))/\partial Q), \\ \dot{Q} &= -\sigma(\partial H(x(t), y(t))/\partial k) + \sigma\rho x(t), \\ k(0) &= \bar{k}, \\ Q(T) &= Q^*, \end{aligned} \tag{A4}$$

Note that

$$\begin{aligned} \mathscr{T}(x, y, 1) &= \mathscr{K}(x, y), \\ \mathscr{T}(x, y, 0) &= (\bar{k}, Q^*)(\text{the constant mapping}). \end{aligned}$$

Moreover,

$$\mathscr{T}(x, y, \sigma) = (\bar{k}, Q^*) + \sigma\mathscr{K}_0(x, y) \tag{A5}$$

where $\mathscr{K}_0(x, y)$ is the mapping defined as in Lemma A1 with $\bar{K} = Q^* = 0$.

We first apply the invariance under homotopy property. That hypotheses (a), (b), (c) of Property 2 are satisfied is a consequence of Lemma A3 applied to $\mathscr{K}_0$ and the representation (A5). The inclusion

$$x \in \mathscr{T}(x, \sigma)$$

is equivalent to (2.T.σ). In the statement of the continuation principle we require that solutions to (2.T.σ) satisfy $\| k(t)\| \leqslant K_0$ and $\| Q(t)\| \leqslant Q_0$. Thus, $(Q, k) \notin \partial\Omega$ and hypothesis (d) of Property 2 is satisfied. Thus, we may conclude that

$$d[I - \mathscr{H}, \Omega, 0] = d[I - \mathscr{T}(x, y, 0), \Omega, 0] = d[I - (\bar{k}, Q^*), \Omega, 0] \neq 0,$$

the nonzero degree of the last mapping following from standard arguments of degree theory.

By Property 1 we need only show that

$$d[I - \mathscr{T}(x, y, 0), \Omega, 0] \neq 0.$$

But it is easily shown that

$$d[I - \mathscr{T}(x, y, 0), \Omega, 0] = d[I - (\bar{k}, Q^*), \Omega, 0] \neq 0.$$

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A Characterization of the Normalized Restricted Profit Function*

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A duality theorem between production functions and normalized restricted profit functions is derived under the assumptions of joint biconvexity and independence of the production possibilities sets. Biconvexity is a natural generalization of the concept of convexity to the case of overall increasing returns which maintains at the same time the conventional property of diminishing marginal rates of transformation (substitution) amongst certain subsets of commodities.

Specializing to the twice differentiable, locally strongly convex case, certain identities linking the Hessians of the production function and the normalized restricted profit function are derived. Using these identities, Le Chatelier's principle is demonstrated.

1. INTRODUCTION

The purpose of this paper is to provide a complete characterization of the normalized restricted profit function via the conjugate duality approach and under conditions more general than those customarily assumed in the literature.[1] A normalized restricted profit function is defined as the maximized value of the normalized profits given the values of the normalized prices of the variable commodities and the quantities of the fixed commodities. It may be identified with the Hamiltonian function employed by

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[1] The term "restricted profit function" is due to McFadden [26]. Other terminology includes "gross profit function" [12], "partial profit function" [19], and "variable profit function" [6]. The concept of a normalized profit function is traceable to Hotelling [13], the terminology is due to Jorgenson and Lau [17]. "Normalized restricted profit function" seems a more appropriate terminology because the concept of a normalized restricted profit function also applies to cases in which some commodities are restricted to vary within a closed convex set in addition to the case in which some commodities are restricted to be fixed.

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Cass [3] and Cass and Shell [4] in the analysis of competitive dynamical systems if the variable commodities are interpreted as the outputs and the fixed commodities are interpreted as the input stocks.

Restricted profit functions have been analyzed and characterized by Samuelson [31], who is an early pioneer of the concept, and by Gorman [12], McFadden [26], and Diewert [6, 7]. Jacobsen [14, 15], McFadden [26], Rosse [28], and Shephard [33] have also examined the properties of multiple-output cost and multiple-input revenue functions, which may be regarded as special cases of the restricted profit function. None of the above treatments is based on the conjugacy correspondence of closed, proper, convex functions.[2] And while Jorgenson and Lau [16, 17] have developed a theory of production with six alternative but equivalent characterizations of the technology (production possibilities set, production function, marginal productivity correspondence, normalized supply correspondence, normalized profit function, and normalized price and profits possibilities set) based on the conjugacy correspondence, they have not analyzed the normalized restricted profit function explicitly.

In this paper, we analyze the properties of the normalized restricted profit function within the framework of conjugacy correspondence. In addition, technological assumptions are relaxed in various directions. First, all commodities (except one) are treated symmetrically. There is no artificial distinction between commodities that are net outputs and commodities that are net inputs. The theory of the normalized restricted profit function thus developed will not only accommodate normalized cost functions (all net outputs fixed) and normalized revenue functions (all net inputs fixed) as special cases, but also all the mixed cases in between (some net outputs and some net inputs fixed; for example, when a firm has long-term fixed output delivery as well as material supply contracts). An analytically simple and unified treatment is hence available for all these related concepts. Second, there is no assumption of free disposal except for the numeraire commodity. The assumption of free disposal in all commodities is unduly restrictive because it rules out the possibility that some of the jointly produced net outputs may be undesirable, that is, may have negative normalized prices. Air pollution is an obvious example. From a practical point of view, allowing the existence of nonfree disposal may be important in many situations. For example, a firm may have to maximize profits subject to a government restriction that its pollution is less than or equal to a given level. Third, the assumption that the production possibilities set is a convex cone is relaxed. It does not need to be a

[2] The conjugacy correspondence was first discovered by Fenchel [9, 10]. A modern development with many extensions can be found in Rockafellar [27].

convex cone, or in fact, even convex. Instead, a new assumption of biconvexity, which is weaker than convexity, is introduced. Biconvexity allows the existence of overall increasing returns while preserving the properties of diminishing marginal rates of transformation (substitution) amongst certain subsets of commodities. Fourth, reversibility in production is allowed in a limited sense, not all production plans are reversible, but some production plans may be reversible. In the short run, reversibility in the variable commodities for some subset of production plans is not an unrealistic possibility. Finally, the normalized prices of the commodities are allowed to be either positive, zero, or negative. Only the price of the numeraire commodity is required to be positive.

Normalized restricted profit functions have applications in the analysis of competitive dynamical systems as Hamiltonian functions, as demonstrated by Cass [3] and Cass and Shell [4]. They also have applications in the analysis of production behavior under circumstances in which some commodities are variable and other commodities are fixed over the period of production. Econometric applications of the normalized restricted profit function have been made by Lau and Yotopoulos and their collaborators in their extensive studies of agricultural production.[8] Econometric applications of the cost function are much more numerous. One may refer to Diewert's [7] excellent survey article.

In Section 2, duality theorems relating the production function and the normalized restricted profit functions under alternative assumptions on the technology are given. A proof of the generalized Hotelling's [13] or Shephard's [32] lemma is also included. In Section 3, the results derived in Section 2 are applied to the case of twice differentiable, locally strongly convex technologies, leading to certain identities linking the Hessians of the production function, the normalized restricted profit function, and the normalized (unrestricted) profit function. These identities enable us to deduce the supply elasticities of normalized prices and/or fixed commodities under one environment from a knowledge of their values under another environment. In particular, they allow us to compute the long-run supply elasticities given the short-run elasticities and vice versa. They also allow the direct translation of conditions on the Hessian matrix of the production function into conditions on the Hessian matrix of the normalized restricted profit function (or alternatively the Hamiltonian function) or into conditions on the Hessian matrix of the normalized (unrestricted) profit function, or vice versa. Thus, for instance, the sufficient conditions on the Hamiltonian function for global asymptotic stability of the optimal control systems obtained by Brock and Scheinkman

[8] See, for example [22, 23, 21, 24, 35, 36].

[1] may be directly interpreted as conditions on the production function. Finally, based on the identities, an alternative demonstration of Le Chatelier's principle, first introduced into economics by Samuelson [29], is given.[4]

A brief summary and conclusion is given in Section 4.

2. Duality Theorems

2.1. *Conventions and Definitions*

A commodity can be either a net output or a net input. Each commodity is measured as if it were a net output. Thus, if the quantity of a commodity is positive, it is a net output; if the quantity is negative, it is a net input; if the quantity is zero, it is neither a net output nor a net input.

It is assumed that there are $(n + 1 + m)$ commodities. The quantities of the first $(n + 1)$ commodities, denoted $y_1, y_2, \ldots, y_{n+1}$, are assumed to be variable, and the quantities of the last m commodities, denoted $K_1, K_2, \ldots, K_m$, are assumed to be fixed. They are collectively referred to as vectors $[y' y_{n+1}]'$ and K, respectively.[5] The n-dimensional vector formed by the quantities of the first n commodities is referred to as y.

All prices are measured relative to the price of the $(n + 1)$st commodity. The price of the $(n + 1)$st commodity is always one. Thus the price system is referred to as the normalized price system, and the prices as the normalized prices. Except for the price of the numeraire commodity, the prices, and hence, the normalized prices, of the commodities can be either positive, zero, or negative. Thus the normalized price system is not restricted to the nonnegative orthant. A tax on a commodity such as pollution may be treated as a negative price. The normalized prices of the variable commodities are denoted $y_1^*, y_2^*, \ldots, y_n^*$; the normalized prices of the fixed commodities are denoted $K_1^*, K_2^*, \ldots, K_m^*$.

The production possibilities set is the set of all feasible production plans. The production function is defined by the supremum of the quantity of the $(n + 1)$st commodity given the quantities of all the other commodities, such that the production plan is contained in the production possibilities set. The set of production plans $[y' y_{n+1} K']'$ contained in the

[4] Samuelson [29] first enunciated this principle and indicated its application in equilibrium systems in economics. See also [30]. This version of the principle is referred to as the "strong" Le Chatelier–Samuelson principle by Eichhorn and Oettli [8].

[5] A prime signifies the transposition operation. Whenever a prime follows a square bracket, it implies the transpose of the row vector enclosed within the square brackets.

production possibilities set for given y and K may be empty. We adhere to the convention that the supremum of an empty set is equal to $-\infty$;

$$\sup\{\varnothing\} = -\infty,$$

a convention which we shall appeal to repeatedly.

The vector $[y'y_{n+1}K']'$ is referred to as a producrion plan. The vector $[y'y_{n+1}]'$ for given K is referred to as a short-run production plan, or where no ambiguity arises, simply as a production plan. The vector y alone is sometimes used to refer to the short-run production plan $[y'y_{n+1}]'$ where y_{n+1} is the maximum value of y_{n+1} such that $[y'y_{n+1}K']'$ is contained in the production possibilities set. If there does not exist any value for y_{n+1} such that $[y'y_{n+1}K']'$ is contained in the production possibilities set, then y refers to the nonfeasible production plan $[y' -\infty]'$.

2.2. *Convex Production Functions*

With these conventions, we may proceed to define the production possibilities set formally. The production possibilities set is referred to as T. It is a subset of R^{n+1+m} and is assumed to have the following properties.

T.1. *Origin.* $0 \in T$.

T.2. *Closure.* T is closed.

T.3. *Convexity.* T is convex.

T.4. *Monotonicity.* If $[y'y_{n+1}K']' \in T$, then $[y'y^{*}_{n+1}K']' \in T$, $\forall y^{*}_{n+1} \leqslant y_{n+1}$.

T.5. *Nonproducibility.* If $[y'y_{n+1}K']' \in T$, $y_{n+1} \leqslant 0$.

T.1 through T.3 are standard assumptions and require no discussion. T.4 requires only that one variable commodity be freely disposable. This represents a substantial weakening of the conventional monotonicity assumption and opens up the possibility of analyzing many production processes in which part of the joint output may be considered undesirable, that is, negatively priced. In most production processes, one can usually find a commodity, for example, unskilled labor, that may be considered freely dispoable if found in excess of technical requirements.[6] T.5 implies that there is at least one variable and freely disposable commodity which can only be a net input to the production process, that is, it is a primary

[6] Strictly speaking, such an assumption is not necessary. It is introduced for technical convenience. Monotonicity assures that there is a one-to-one correspondence between the production possibilities set and the production function, to be defined below. Otherwise, two production functions may be necessary to characterize one production possibilities set completely. See the example given by Jorgenson and Lau [17].

factor of production.[7] Unskilled labor is an obvious example of a commodity satisfying T.5, because it is not generally considered a producible commodity. Note that nonproducibility is a much weaker assumption than essentiallity, which implies that none of the elements of $[y'y_{n+1}]'$ and K can be positive unless y_{n+1} is negative. T.5 also implies that if a feasible production plan uses any positive quantity of the $(n+1)$ st commodity as a net input, the production plan with all quantities reversed in sign is not feasible, that is, the production plan is not reversible. However, feasible production plans with y_{n+1} equal to zero may be reversible. Finally, T.5 replaces the conventional boundedness assumption on the production possibilities set.

Corresponding to such a production possibilities set T, Jorgenson and Lau [16, 17][8] have shown that there exists an extended-real-valued production function $F(y, K)$ defined on R^n given by

$$F(y, K) = -\sup\{y_{n+1} \mid [y'y_{n+1}K']' \in T\}^9$$

with the convention that $\sup\{\varnothing\}$ where $\{\varnothing\}$ is the empty set is $-\infty$, that is, the negative of the maximum of y_{n+1}, given y and K, such that $[y'y_{n+1}K']'$ is a feasible production plan.[10] The production function has the following properties.

F.1. *Domain.* The effective domain (that is, the set of $[y'K']'$ such that $F(y, K)$ is finite) of $F(y, K)$ is a convex set containing the origin. $F(0, 0) = 0$.

F.2. *Closure.* $F(y, K)$ is closed in $\{y, K\}$.

F.3. *Convexity.* $F(y, K)$ is convex in $\{y, K\}$.

F.4. *Nonnegativity.* $F(y, K)$ is nonnegative.

These assumptions on the production function are again standard, with the possible exception of nonnegativity. We note that the fact the

[7] Again, this assumption may also be relaxed. However, the resulting technical complications, especially on the specification of the properties of the marginal productivity correspondences and normalized supply correspondences, are quite substantial. See [17] for a treatment of the duality between production and normalized profit functions without this assumption. A second, and less substantial, reason for this assumption is that it preserves the formal symmetry between the production function and the normalized profit function.

[8] Actually, Jorgenson and Lau [16, 17] do not treat the case considered here explicitly. However, the results used here may be obtained as a convex combination of the results contained in their two papers.

[9] This is the negative of the conventional definition. With this sign convention, the production function is convex rather than concave.

[10] Because of closure of the production possibilities set, the supremum and maximum may actually be used interchangeably here.

production function takes the value zero at the origin together with nonnegativity imply that the production function achieves its minimum at the origin.

Given the production function $F(y, K)$, the production possibilities set may be reconstructed as

$$T^* = \{[y' y_{n+1} K']' \mid [y' K']' \in R^{n+m}, \quad y_{n+1} \in R \text{ and } y_{n+1} \leqslant -F(y, K)\}.$$

Under the assumptions T.1 through T.5 on the production possibilities set and F.1 through F.4 on the production function there is a one-to-one correspondence between T and $F(y, K)$, so that T^* is actually equal to T, the original production possibilities set from which $F(y, K)$ is derived in the first place.[11]

The purpose of productive activity is assumed to be maximization of short-run profits, or equivalently, normalized restricted profits, defined as the sum of the values of the net outputs in terms of the numeraire commodity. The producer maximizes

$$P = \langle y^*, y \rangle + y_{n+1}$$

where $\langle\ ,\ \rangle$ denotes the inner product operation,[12] with respect to y for given K. Since $y_{n+1} = -F(y, K)$, one may write the maximand equivalently as $P = \langle y^*, y \rangle - F(y, K)$. The maximized value of P, for given y^* and K, considered a function of y^* and K, is precisely the normalized restricted profit function, which will be denoted $H(y^*, K)$. It may be written as

$$H(y^*, K) = \sup_{y} \{\langle y^*, y \rangle - F(y, K)\}.^{13}$$

Similarly, the normalized (unrestricted) profit function is given by

$$F^*(y^*, K^*) = \sup_{y, K} \{\langle y^*, y \rangle + \langle K^*, K \rangle - F(y, K)\}$$

where K^* is the normalized price of K. We note that this maximization may be decomposed into two stages, that is,

$$\begin{aligned} F^*(y^*, K^*) &= \sup_{K} \sup_{y} \{\langle y^*, y \rangle + \langle K^*, K \rangle - F(y, K)\} \\ &= \sup_{K} \{\langle K^*, K \rangle + H(y^*, K)\}. \end{aligned}$$

[11] Diewert [6] first derived the one-to-one correspondence between the production possibilities set and the production function. His assumptions are different from the ones used here.

[12] $$\langle x, y \rangle \equiv \sum_i x_i y_i .$$

[13] Supremum is used rather than maximum because there may be an upper bound to normalized restricted profits which is not attainable by any finite production plan.

The duality relations between $F(y, K)$ and $F^*(y^*, K^*)$ are well known.[14] We shall concentrate on the duality relations among $F(y, K)$, $H(y^*, K)$, and $F^*(y^*, K^*)$.[15]

The normalized restricted profit function $H(y^*, K)$ is an extended-real-valued function with the following properties.

H.1. *Domain.* The effective domain of $H(y^*, K)$ is a convex set containing the origin. $H(0, 0) = 0$.

H.2. *Closure.* $H(y^*, K)$ is lower-closed.[16]

H.3. *Convexity–concavity.* $H(y^*, K)$ is convex in y^* for every K and concave in K for every y^*.[17]

H.4. *Nonnegativity–nonpositivity.* $H(y^*, 0)$ is nonnegative; $H(0, K)$ is nonpositive.

By conjugate duality, one has

$$H^*(y, K) = \sup_{y^*} \{\langle y^*, y \rangle - H(y^*, K)\} - F(y, K).$$

We now state the first duality theorem.

THEOREM 1. *There is a one-to-one correspondence between the set of production functions $F(y, K)$ satisfying properties* F.1 *through* F.4 *and the set of normalized restricted profit functions $H(y^*, K)$ satisfying properties* H.1 *through* H.4.

This theorem is based on Rockafellar [27].[18] A proof is attached in the Appendix.

Thus H.1 through H.4 are the properties that a normalized restricted profit function must satisfy if it is to be generated from a production function satisfying F.1 through F.4 above, or equivalently, from a production possibilities set T satisfying T.1 through T.5 above. Gorman [12] and

[14] See, for instance [25, 26, 19, 6, 16, 17].

[15] One can also consider the symmetric case of $H(y, K^*)$, but the results will be identical to the case of $H(y^*, K)$ with y and K interchanged.

[16] $H(y^*, K)$ is lower-closed if $\mathrm{cl}_{y^*}\mathrm{cl}_K H(y^*, K) = H(y^*, K)$ where cl_x denotes the closure operation with respect to the set of variables x. See [27, p. 357]. If it is assumed that $H(y^*, K)$ is finite on R^{n+m}; this assumption may be strengthened to $H(y^*, K)$ is closed.

[17] Convex–concave functions are sometimes known as *saddle functions*. See [27, pp. 350–358].

[18] See [27, pp. 357–358, Theorem 3].

Diewert [6] have proved the duality between T and $H(y^*, K)$ directly, using different methods and employing somewhat different assumptions than the ones used here.[19]

Note that essentially the same theorem goes through if there is a third set of commodities (or factors), denoted t, over which no convexity assumptions are made, so that $F(y, K, t)$ is convex in $\{y, K\}$ for all t. In this case $H(y^*, K, t)$ is convex–concave in $\{y^*, K\}$ for all t. Examples of possible t variables include time, locational variables, and entrepreneurial ability, among others. Note also that because of the definition of $H(y^*, K)$ as a supremum, it is always convex and closed in y^* independently of the properties of $F(y, K)$.[20]

In fact, because $H(y^*, K)$ is a lower-closed, convex–concave function, it is endowed with some exceptional continuity properties. Rockafellar [27] has shown that every finite convex–concave function is jointly continuous in both of its arguments on any product set $C \times D$, where $C \subseteq R^n$, $D \subseteq R^m$ are relatively open convex sets, contained in its effective domain.[21] Moreover, if in addition there is a third set of arguments, say t, $t \in R^l$, and $H(y^*, K, t)$ is a convex–concave function for all t, and continuous in t for all y^* and K, then $H(y^*, K, t)$ is jointly continuous in y^*, K and t on $C \times D \times R^l$.[22]

To specialize to the case of the normalized (unrestricted) profit function, we assume that there are no fixed commodities. To specialize to the case of the negative of the normalized cost function we replace T.5 by T.5′.

T.5′. *Input–output nonswitchability.* $[y'y_{n+1}K']' \in T$ implies that $[y'y_{n+1}]' \leqslant 0$ and $K \geqslant 0$, or equivalently, F.1 by F.1′.

F.1′. *Domain.* The effective domain of $F(y, K)$ is a convex set containing the origin. It is contained in $R_{\ominus}^{n} \times R_{\oplus}^{m}$ where $R_{\ominus}^{n}$ and

[19] See Gorman [12, pp. 152–153]. His proof is not based on the conjugacy correspondence between the production function and the normalized restricted profit function as is the proof here. See also [6].

[20] This is a well-known fact. Let y_1, y_2, y_λ be profit maximizing plans at normalized prices $y_1^* y_2^*$ and y_λ^* respectively, where y_λ^* is a convex combination of y_1^* and y_2^*, that is, $y_\lambda^* = (1-\lambda)y_1^* + \lambda y_2^*$, $0 \leqslant \lambda \leqslant 1$. Consider

$$\begin{aligned}(1-\lambda)H(y_1^*, K)+\lambda H(y_2^*, K) &\geqslant (1-\lambda)[\langle y_1^*, y_\lambda\rangle - F(y_\lambda, K)]+\lambda[\langle y_2^*, y_\lambda\rangle - F(y_\lambda, K)]\\ &\geqslant \langle y_\lambda^*, y_\lambda\rangle - F(y_\lambda, K)\\ &\geqslant H(y_\lambda^*, K).\end{aligned}$$

Closure follows from the fact that the epigraph of $H(y^*, K)$ for every K contains all its boundary points.

[21] See [27, p. 370, Theorem 35.1].

[22] See [27, p. 371, Theorem 35.3].

$R_{\oplus}{}^m$ are the *nonpositive* orthant and *nonnegative* orthant of R^n and R^m respectively. $F(0, 0) = 0$. Or equivalently, H.1 by H.1′ and H.5′.

H.1′. *Domain.* The effective domain of $H(y^*, K)$ is a convex set containing the origin. It is contained in $R^n \times R_{\oplus}{}^m$. $H(0, 0) = 0$.

H.5′. *Monotonicity.* $H(y^*, K)$ is nonincreasing in y^* for all K.

The cost function is thus given by

$$C(p, K) = -p_{n+1}H\left(\frac{p_1}{p_{n+1}}, \ldots, \frac{p_n}{p_{n+1}}; K\right)$$

where p is the vector of money prices of the variable commodities and K is the vector of net outputs.

To specialize to the case of normalized revenue function, we replace T.5 by T.5″ and add assumption T.6″.

T.5″. *Input–output nonswitchability.* $[y'y_{n+1}K']' \in T$ implies that $[y'y_{n+1}]' \geqslant 0$ and $K \leqslant 0$.

T.6″. *Boundedness.* For at least one variable commodity which is freely disposable and always a net output, say y_{n+1}, if a production plan is feasible, the level of net output of that commodity corresponding to finite levels of net outputs of all other commodities is bounded above.

Or, equivalently we replace F.1 by F.1″, F.4 by F.4″, and add assumption F.5″.

F.1″. *Domain.* The effective domain of $F(y, K)$ is a convex set containing the origin. It is contained in $R_{\oplus}{}^n \times R_{\ominus}{}^m$ where $R_{\oplus}{}^n$ and $R_{\ominus}{}^m$ are the nonnegative orthant and nonpositive orthant of R^n and R^m respectively.

F.4″. *Nonpositivity.* $F(y, K)$ is nonpositive.

F.5″. *Boundedness.* $F(y, K)$ is bounded below. (It nowhere takes the value minus infinity.)

Or equivalently we may replace H.1 by H.1″ and add assumptions H.5″ and H.6″.

H.1″. *Domain.* The effective domain of $H(y^*, K)$ is a nonempty convex set containing the origin. It is contained in $R^n \times R_{\ominus}{}^m$.

H.5″. *Monotonicity.* $H(y^*, K)$ is nondecreasing in y^* for all K.

H.6″. *Boundedness.* $H(0, 0)$ is bounded above.

The revenue function is thus given by

$$R(p, K) = p_{n+1}H\left(\frac{p_1}{p_{n+1}}, \frac{p_2}{p_{n+1}} \cdots \frac{p_n}{p_{n+1}}; K\right)$$

where p is the vector of money prices of the variable commodities and K is the vector of net inputs.

A convex function is differentiable almost everywhere in the relative interior of its effective domain. Where it is not differentiable a "generalized" derivative always exists, called a *subgradient*. A vector y^* is said to be a *subgradient* of the convex function $F(y)$ at y if

$$F(z) \geqslant F(y) + \langle y^*, z - y\rangle$$

for all z. For a convex function $F(y)$ which is finite at y, the linear function

$$G(z) = F(y) + \langle y^*, z - y\rangle$$

is a nonvertical supporting hyperplane to the epigraph of $F(y)$ (the set of production possibilities) at the point $(y, F(y))$. Corresponding to each y, the subgradient need not be unique. The set of all subgradients y^* at y is said to be the *subdifferential* of $F(y)$ at y. The correspondence which associates with each y, the subdifferential of $F(y)$ at y, is known as the subdifferential of the function $F(y)$, and is denoted $\partial F(y)$.[23]

A concave function is the negative of a convex function. Thus a "generalized" derivative also exists for a concave function in the relative interior of its effective domain. Thus, if $F(y)$ is a concave function, $-F(y)$ is convex, and, hence, in the relative interior of its effective domain there exists $-y^*$, a subgradient of the convex function $-F(y)$, such that

$$-F(z) \geqslant -F(y) + \langle -y^*, z - y\rangle$$

for all z. We note, therefore, y^* satisfies, for all z, the inequality

$$F(z) \leqslant F(y) + \langle y^*, z - y\rangle.$$[24]

We refer to y^* as the supergradient of the concave function $F(y)$ at y. The set of all supergradients y^* at y is said to be the *superdifferential* of the

[23] Thus a subdifferential is a generalization of the concept of a gradient to a point where a function is not differentiable. McFadden [26] first employed the concept of a subdifferential in the theory of production.

[24] Note that this is precisely the subgradient inequality with the sense of the inequality reversed. Diewert [6, 7] refers to such a vector also as a subgradient. In order to avoid confusion, we use the term supergradient, it being the coefficients of the supporting hyperplane of $F(y)$ at y that lies completely *above* $F(y)$.

concave function $F(y)$ at y. The correspondence which associates with each y the superdifferential of $F(y)$ at y is known as the superdifferential of the function $F(y)$, and denoted $\partial F(y)$.[25] Wherever the supergradient is unique, that is, the superdifferential is single-valued, it coincides with the ordinary gradient of the function $F(y)$.

It is well known that the subdifferential of $H(y^*, K)$ with respect to y^* at $[y^{*\prime}K']'$ contains the set of profit maximizing production plans at $[y^{*\prime}K']'$. This is true, again, independently of the properties of $F(y, K)$, as long as a profit-maximizing production plan exists. Under the additional assumptions of F.1 through F.4, the subdifferential actually coincides with the set of profit-maximizing plans. In the case of differentiable normalized restricted profit functions, this fact is variously referred to as either Hotelling's [13] lemma or Shephard's [32] lemma. Also, in the case of differentiable functions, the gradient of $H(y^*, K)$ at $[y^{*\prime}K']'$ with respect to K is known to coincide with minus the gradient of $F(y, K)$ with respect to K at $[y'K']'$ where y is set equal to the profit-maximizing production plan at $[y^{*\prime}K']'$. There is an appropriate generalization of this fact to the general nondifferentiable case. However, there does not appear to be a generally available proof in the literature of these two propositions. We shall refer to them collectively as the Generalized Hotelling's lemma and give a simple proof in the Appendix.

GENERALIZED HOTELLING'S LEMMA. *Under the assumptions* F.1 *through* F.4, *the subdifferential of* $H(y^*, K)$ *at* $[y^{*\prime}K']'$ *with respect to* y^* *coincides with the set of profit-maximizing production plans; the superdifferential of* $H(y^*, K)$ *with respect to* K *at* $[y^{*\prime}K']'$ *coincides with the negative of the last m components of the subdifferential of* $F(y, K)$ *at* $[y'K']'$ *where* y *is a profit-maximizing production plan at* $[y^{*\prime}K']'$.

From the Generalized Hotelling's lemma, one deduces immediately that $F(y, K)$ is nondecreasing (nonincreasing) in K if and only if $H(y^*, K)$ is nonincreasing (nondecreasing) in K.

The supergradient of $H(y^*, K)$ has the interpretation of the set of normalized shadow prices or normalized rental prices of the fixed commodities. It measures the effect on the normalized restricted profit function of a marginal increase in the net output of the fixed commodities. If $K \geqslant 0$, then the supergradient gives the negative of the normalized marginal cost. If $K \leqslant 0$, then the supergradient gives the negative of the normalized shadow prices or normalized rental prices of the net inputs.

[25] One can always tell from the context whether that refers to a subdifferential or a superdifferential.

2.3. *Convex–Quasi-convex and Order-Independent Production Functions*

It is sometimes desirable to drop the assumption of convexity of the production function. This is necessary, for instance, if one wants to allow the existence of increasing returns to scale in the overall technology. However, one often wishes to maintain the assumption of a diminishing marginal rate of transformation, at least among some subset of commodities. In the one-output, multiple-input case, the traditional solution is to assume that

$$K = f(y, y_{n+1})$$

is a quasiconcave function where K is identified as the output and $[y'y_{n+1}]'$ the vector of inputs. This is equivalent to, in our notation, the specification that

$$-y_{n+1} = F(y, K)$$

where F is convex in y for all K. In addition, if $F(y, K)$ is nondecreasing in K, as will be the case if $f(y, y_{n+1})$ is nondecreasing in y_{n+1}, $F(y, K)$ is also quasiconvex in K for all y. We note also that given this monotonicity assumption on K, from the point of view of the production possibilities set, the sets $T^K = \{[y'y_{n+1}]' \mid [y'y_{n+1}K']' \in T\}$ and $T^{[y'y_{n+1}]'} = \{K \mid [y'y_{n+1}K']' \in T\}$ are all convex. Thus, a natural generalization to the case of many fixed commodities is the concept of biconvexity.

A production possibilities set is *disjointly biconvex* if the set T^K is convex for all K and the set $T^{[y'y_{n+1}]'}$ is convex for all $[y'y_{n+1}]'$, where the sets of commodities $[y'y_{n+1}]'$ and K are disjoint. We thus replace the assumptions on the production possibilities set by the following.

T.1A. *Origin.* $0 \in T$.

T.2A. *Closure.* T is closed.

T.3A. *Disjoint biconvexity.* The sets $T^K = \{[y'y_{n+1}]' \mid [y'y_{n+1}K']' \in T\}$ are convex for every K; the sets $T^{[y'y_{n+1}]'} = \{K \mid [y'y_{n+1}K']' \in T\}$ are convex for every $[y'y_{n+1}]'$.

T.4A. *Monotonicity.* If $[y'y_{n+1}K']' \in T$, then $[y'y^*_{n+1}K']' \in T$, $\forall y^*_{n+1} \leqslant y_{n+1}$.

T.5A. *Nonproducibility.* If $[y'y_{n+1}K']' \in T$, $y_{n+1} \leqslant 0$.

For such a production possibilities set T, it can be shown that there exists a production function $F(y, K)$ defined as before, with the following properties.

F.1A. *Origin.* $F(0, 0) = 0$.

F.2A. *Closure.* $F(y, K)$ is closed.

F.3A. *Convexity–quasiconvexity.* $F(y, K)$ is convex in y for every K, and quasiconvex in K for every y.[26]

F.4A. *Nonnegativity.* $F(y, K) \geqslant 0$.

Note that no monotonicity assumption is made concerning the production function, just as before. Thus, y and K can be either net outputs or net inputs, or combinations of both. Formally, we state Theorem 2.

THEOREM 2. *There is a one-to-one correspondence between the set of production possibilities sets T satisfying properties* T.1A *through* T.5A *and the set of production functions $F(y, K)$ satisfying properties* F.1A *through* F.4A.

A proof is sketched in the Appendix.

We note that convexity of T implies disjoint biconvexity of T but not vice versa.[27] Similarly, convexity of $F(y, K)$ in $\{y, K\}$ implies convexity of $F(y, K)$ in y for every K and quasiconvexity of $F(y, K)$ in K for every y, but not vice versa. Economically, biconvexity means there is not necessarily overall generalized dimininishing returns. But each subset of commodities taken by themselves show generalized diminishing returns. Biconvex production possibilities sets may be useful in other areas of economic analysis such as externalities, public goods, and regulation.

A natural question which arises is: What are the properties of a normalized restricted profit function $H(y^*, K)$ which corresponds to a convex–quasiconvex production function, or more specifically, to a production function satisfying properties F.1A through F.4A? It turns out that these properties are rather complicated, although it can be shown in a straightforward manner that a normalized restricted profit function $H(y^*, K)$ is quasiconcave in K for every y^* only if the production function $F(y, K)$ is quasiconvex in K for every y. The converse, unfortunately, is not true. However, under an additional assumption of independence on the production possibilities set, it is possible to establish a one-to-one correspondence between the set of convex–quasiconvex production functions satisfying an independence assumption and the set of convex–

[26] Let $y_\lambda = (1 - \lambda)y_1 + \lambda y_2$. A function $F(y)$ is quasiconvex if

$$\max\{F(y_1), F(y_2)\} \geqslant F(y_\lambda), \qquad \forall\lambda, 0 \leqslant \lambda \leqslant 1.$$

It is quasiconcave if

$$\min\{F(y_1), F(y_2)\} \leqslant F(y_\lambda), \qquad \forall\lambda, 0 \leqslant \lambda \leqslant 1.$$

[27] In fact, convexity of T is necessary and sufficient for the disjoint biconvexity of T in all possible partitions of the set of commodities.

quasiconcave normalized restricted profit functions satisfying an independence assumption.

First, we give a symmetric definition of the concept of independence which follows Jorgenson and Lau [18]. A production possibilities set T is said to be *independent* in the $[y'y_{n+1}]' - K$ partition if

$$[y'y_{n+1}K_1']' \in T \qquad \text{and} \qquad [y'y_{n+1}K_2']' \notin T$$

for some y, y_{n+1}, K_1, K_2 imply that

$$T^{K_2} = \{[y'y_{n+1}]' \mid [y'y_{n+1}K_2']' \in T\}$$
$$\subset T^{K_1} = \{[y'y_{n+1}]' \mid [y'y_{n+1}K_1']' \in T\}.$$

It has been shown by Jorgenson and Lau [18] that this definition is symmetric with respect to the two groups of commodities $[y'y_{n+1}]'$ and K.[28]

If we add the independence assumption on the production possibilities set as T.6A,

T.6A. *Independence.* If $[y'y_{n+1}K_1']' \in T$ and $[y'y_{n+1}K_2']' \in T$ for some y, y_{n+1}, K_1, K_2, then

$$T^{K_2} = \{[y'y_{n+1}]' \mid [y_{n+1}K_2']' \in T\}$$
$$\subset T^{K_1} = \{[y'y_{n+1}]' \mid [y'y_{n+1}K_1']' \in T\},$$

it induces a corresponding independence assumption on the production function.

A production function $F(y, K)$ is said to be *order-independent* in K if

$$F(y, K_1) > F(y, K_2)$$

for some y, K_1, K_2 implies that

$$F(y^*, K_1) \geqslant F(y^*, K_2) \qquad \forall y^*.$$

We therefore add the order-independence assumption on the production function as F.5A.

F.5A. *Order-independence.* If $F(y, K_1) > F(y, K_2)$ for some y, K_1, K_2, then $F(y^*, K_1) \geqslant F(y^*, K_2)$, $\forall y^*$.

Corresponding to a production possibilities set T satisfying T.1A through T.6A, there exists a production function $F(y, K)$ satisfying F.1A through F.5A, and vice versa. Formally, we state Theorem 3.

[28] See [18].

THEOREM 3. *There is a one-to-one correspondence between the set of production possibilities sets T satisfying properties* T.1A *through* T.6A *and the set of production functions satisfying properties* F.1A *through* F.5A.

A proof is sketched in the Appendix.

Corresponding to a production function satisfying F.1A through F.5A above, there exist a normalized restricted profit function with the following properties.

H.1A. *Domain.* The effective domain of $H(y^*, K)$ is a convex set containing the origin. $H(0, 0) = 0$.

H.2A. *Closure.* $H(y^*, K)$ is lower-closed.

H.3A. *Convexity–quasiconcavity.* $H(y^*, K)$ is convex in y^* for every K and quasiconcave in K for every y^*.

H.4A. *Nonnegativity–nonpositivity.* $H(y^*, 0)$ is nonnegative; $H(0, K)$ is nonpositive.

H.5A. *Order-independence.* If $H(y^*, K_1) \geq H(y^*, K_2)$ for some y^*, K_1, K_2, then $F(y^{**}, K_1) \geq F(y^{**}, K_2), \forall y^{**}$.

By conjugate duality, one has

$$F^{**}(y, K) = \sup_{y^*} \{\langle y^*, y\rangle - H(y^*, K)\} = F(y, K).$$

We now state our second duality theorem between production functions and normalized restricted profit functions.

THEOREM 4. *There is a one-to-one correspondence between the set of production functions satisfying properties* F.1A *through* F.5A *above and the set of normalized restricted profit functions satisfying* H.1A *through* H.5A *above.*

This theorem does not appear to have been proved elsewhere. A simple proof is given in the Appendix.

We note that the same theorem goes through if there is a third set of commodities (or factors), denoted t, over which no convexity assumptions are made, so that $F(y, K, t)$ is convex in y for all $\{K, t\}$ and quasiconvex in K for all $\{y, t\}$. In this case $H(y^*, K, t)$ is convex in y^* for al $\{K, t\}$ and quasiconcave in K for all $\{y^*, t\}$. Examples of possible t variables include time, locational variables, and entrepreneurial ability, among others.

We note, in passing, that order-independence of $F(y, K)$ in K implies and is implied by order-independence of $H(y^*, K)$ in K. This is true independently of any assumption on the convexity properties of $F(y, K)$ or $H(y^*, K)$ with respect to K. Moreover, strict monotonicity of $F(y, K)$ in K implies and is implied by strict monotonicity of $H(y^*, K)$ in K. This is true, again, independently of any assumption on the convexity properties of $F(y, K)$ or $H(y^*, K)$ with respect to K.

If we assume, in addition to F.1A through F.5A, that $F(y, K)$ is continuous in $[y'K']'$ and strictly monotonic in K, then it can be shown that $F(y, K)$ has the form $F(y, G(K))$ where $F(y, G)$ is continuous in $[y'G]'$ and strictly monotonic in G, and G is a continuous and strictly monotonic real-valued function of K. The corresponding normalized restricted profit function has the form $H(y^*, K) = H(y^*, G(K))$ with the same $G(K)$ function. In this form, the property of order-independence is also referred to as *separability*.

An example of an order-independent convex–quasiconvex production function is the following.

$$-y_{n+1} = F(y, G(K))$$

where $K \geqslant 0$, and $G(K)$, net output aggregate, is convex in K. The normalized restricted profit function is given by

$$H(y^*, G(K)),$$

and is quasiconcave in K (and, hence, the cost function is quasiconvex in K).

The concept of disjoint biconvexity may be generalized to that of disjoint multiconvexity in an obvious manner. Theorem 2 will apply to these cases with minimum modifications. It may be generalized in another direction to that of joint biconvexity, that is, the subsets of commodities $[y'y_{n+1}]'$ and K contain an overlapping subset of commodities. More specifically, a production possibilities set T is jointly biconvex if the set of $[y'y_{n+1}]'$ contained in T is convex for all K and the set of $[y_{n+1}K']'$ contained in T is convex for all y. Note that y_{n+1} overlaps in the two partitions. It can be shown that such a production possibilities set always generates a production function $F(y, K)$ that is convex in y for every K and convex in K for every y and vice versa. We refer to such a function as a convex–convex function.[29]

[29] We can call this a biconvex function, but that will be confusing because a biconvex function is not equivalent to biconvexity of T.

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3. Hessian Matrix Identities

3.1. *Hessian Identities*

In this section we strengthen our assumptions on T so that both $F(y, K)$ and $F^*(y^*, K^*)$, the normalized unrestricted profit function are twice differentiable everywhere. This implies that both $F(y, K)$ and $F^*(y^*, K^*)$ are *locally strongly convex* everywhere, and that $H(y^*, K)$ is twice differentiable in $\{y^*, K\}$, locally strongly convex everywhere in y^* for every K, and locally strongly concave everywhere in K for every y^*.[30] A twice differentiable function defined on an open convex set contained in R^n is said to be *locally strongly convex* everywhere if its Hessian matrix is positive definite everywhere. It is said to be *strongly convex* if there exists a positive constant α such that the matrix formed by subtracting α times the identity matrix from the Hessian matrix is a positive semidefinite matrix everywhere. A twice differentiable strongly convex function is locally strongly convex everywhere; however, the converse is not true. Under the assumptions of twice differentiability and local strong convexity, the relationships among the Hessian matrices of the production function, the normalized restricted profit function (or Hamiltonian function), and the normalized (unrestricted) profit function may be derived. The purpose of establishing the relationship between the various Hessians is to enable us to derive the comparative statics results in one environment given the comparative statics result in another environment. For example, if we know the supply elasticities (with respect to both normalized prices and quantities of fixed commodities) of a commodity, we should be able to deduce the supply elasticities when some fixed commodities become variable. The Hessian matrix is a first step in the computation of the supply elasticities. Given the Hessian matrix, the supply elasticities can be obtained directly and vice versa. These Hessian matrix identities also enable us to translate conditions on the production function into conditions on the normalized restricted profit function (or Hamiltonian function) and vice versa.

The relationship between the Hessians of $F(y, K)$ and $F^*(y^*, K^*)$ is well known. They are related by

$$\begin{bmatrix} \dfrac{\partial^2 F^*}{\partial y^{*2}} & \dfrac{\partial^2 F^*}{\partial y^* \partial K^*} \\ \dfrac{\partial^2 F^*}{\partial K^* \partial y^*} & \dfrac{\partial^2 F^*}{\partial K^{*2}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 F}{\partial y^2} & \dfrac{\partial^2 F}{\partial y \partial K} \\ \dfrac{\partial^2 F}{\partial K \partial y} & \dfrac{\partial^2 F}{\partial K^2} \end{bmatrix}^{-1}.$$ [31]

[30] See [16] for a discussion.

[31] See, for instance [19].

We consider the relationship between the Hessians of $F(y, K)$ and $H(y^*, K)$. The first-order conditions of restricted profit maximization are

$$y^* - \frac{\partial F}{\partial y}(y, K) = 0; \tag{1}$$

$$\frac{\partial H(y^*, K)}{\partial y^*} = y, \tag{2}$$

and

$$\frac{\partial H(y^*, K)}{\partial K} = \frac{-\partial F(y, K)}{\partial K}. \tag{3}$$

By differentiating Eq (1) with respect to y^*, we obtain

$$I - \left[\frac{\partial^2 F}{\partial y^2}\right] \frac{\partial y}{\partial y^*} = 0. \tag{4}$$

By differentiating Eqs. (2) and (3) with respect to y^* and K, we obtain

$$\left[\frac{\partial^2 H}{\partial y^{*2}}\right] = \frac{\partial y}{\partial y^*}, \tag{5}$$

$$\left[\frac{\partial^2 H}{\partial K\, \partial y^*}\right] = \frac{\partial y}{\partial K}, \tag{6}$$

$$\left[\frac{\partial^2 H}{\partial y^*\, \partial K}\right] = -\left[\frac{\partial y}{\partial y^*}\right]' \left[\frac{\partial^2 F}{\partial y\, \partial K}\right], \tag{7}$$

$$\left[\frac{\partial^2 H}{\partial K^2}\right] = -\left[\frac{\partial^2 F}{\partial K^2}\right] - \left[\frac{\partial y}{\partial K}\right]' \left[\frac{\partial^2 F}{\partial y\, \partial K}\right]. \tag{8}$$

Solving these equations, we have

$$\left[\frac{\partial^2 H}{\partial y^{*2}}\right] = \left[\frac{\partial^2 F}{\partial y^2}\right]^{-1}, \tag{9}$$

$$\left[\frac{\partial^2 H}{\partial y^*\, \partial K}\right] = -\left[\frac{\partial^2 F}{\partial y^2}\right]^{-1} \left[\frac{\partial^2 F}{\partial y\, \partial K}\right], \tag{10}$$

$$\left[\frac{\partial^2 H}{\partial K^2}\right] = -\left[\frac{\partial^2 F}{\partial K^2}\right] - \left[\frac{\partial^2 F}{\partial K\, \partial y}\right]\left[\frac{\partial^2 F}{\partial y^2}\right]^{-1} \left[\frac{\partial^2 F}{\partial y\, \partial K}\right]. \tag{11}$$

We note that Eqs. (9) through (11) are true even if $F(y, K)$ is only convex in y rather than convex in $[y'K']'$.

Next, we derive a similar relationship between $[H_{ij}]$ and $[F^*_{ij}]$. The point of departure is the fact that

$$F^*(y^*, K^*) = \sup_K \{\langle K^*, K\rangle + H\langle y^*, K\rangle\}.$$

Under the assumptions of differentiability, one has immediately at the profit-maximizing equilibrium,

$$K^* + \frac{\partial H(y^*, K)}{\partial K} = 0, \tag{12}$$

$$\frac{\partial F^*(y^*, K^*)}{\partial K^*} = K, \tag{13}$$

$$\frac{\partial F^*(y^*, K^*)}{\partial y^*} = \frac{\partial H(y^*, K)}{\partial y^*}. \tag{14}$$

By differentiating Eq (12) with respect to K^*, we obtain

$$I + \left[\frac{\partial^2 H(y^*, K)}{\partial K^2}\right] \frac{\partial K}{\partial K^*} = 0,$$

$$\therefore, \qquad \frac{\partial K}{\partial K^*} = -\left[\frac{\partial^2 H}{\partial K^2}\right]^{-1}.$$

By differentiating Eqs. (13) and (14) with respect to y^* and K^*, we obtain

$$\left[\frac{\partial^2 F^*(y^*, K^*)}{\partial y^* \partial K^*}\right] = \frac{\partial K}{\partial y^*}, \tag{15}$$

$$\left[\frac{\partial^2 F^*(y^*, K^*)}{\partial K^{*2}}\right] = \frac{\partial K}{\partial K^*}, \tag{16}$$

$$\left[\frac{\partial^2 F^*(y^*, K^*)}{\partial y^{*2}}\right] = \left[\frac{\partial^2 H(y^*, K)}{\partial y^{*2}}\right] + \left[\frac{\partial K}{\partial y^*}\right]' \left[\frac{\partial^2 H(y^*, K)}{\partial K \partial y^*}\right], \tag{17}$$

$$\left[\frac{\partial^2 F^*(y^*, K^*)}{\partial K^* \partial y^*}\right] = \left[\frac{\partial K}{\partial K^*}\right]' \left[\frac{\partial^2 H(y^*, K)}{\partial K \partial y^*}\right]. \tag{18}$$

Solving these equations, we obtain

$$\left[\frac{\partial^2 H}{\partial K^2}\right] = -\left[\frac{\partial^2 F^*}{\partial K^{*2}}\right]^{-1}, \tag{19}$$

$$\left[\frac{\partial^2 H}{\partial y^* \partial K}\right] = \left[\frac{\partial^2 F^*}{\partial y^* \partial K^*}\right]\left[\frac{\partial^2 F^*}{\partial K^{*2}}\right]^{-1}, \tag{20}$$

$$\left[\frac{\partial^2 H}{\partial y^{*2}}\right] = \left[\frac{\partial^2 F^*}{\partial y^{*2}}\right] - \left[\frac{\partial^2 F^*}{\partial y^* \partial K^*}\right]\left[\frac{\partial^2 F^*}{\partial K^{*2}}\right]^{-1}\left[\frac{\partial^2 F^*}{\partial K^* \partial y^*}\right]. \tag{21}$$

Of course, all these identities hold only when K is actually equal to the last m components of the long-run profit-maximizing production plan at normalized prices $[y^{*\prime} K^{*\prime}]'$. Thus, given the values of one Hessian matrix,

we can evaluate the other Hessian matrices in the neighborhood of the long-run equilibrium.

We know that $[\partial^2 H/\partial y^{*2}]$ must be positive semidefinite because $H(y^*, K)$ is convex in y^* for all K. One can reach the same conclusion by using the fact that the matrix $(A_{22} - A_{12}'A_{11}^{-1}A_{12})$ is always positive semidefinite whenever $\begin{bmatrix} A_{11} A_{12} \\ A_{12}' A_{22} \end{bmatrix}$ is positive semidefinite, and that $[F_{ij}^*]$ is positive semidefinite.[32] Similarly, we know that $[\partial^2 H/\partial K^2]$ must be negative semidefinite.

3.2. *Le Chatelier's Principle Revisited*

Samuelson [29] first enunciated Le Chatelier's principle for the behavior of equilibrium economic systems. Basically, Le Chatelier's principle may be stated as follows. Let $y_1 = g_1(y^*)$ be the values of y for which the function $G(y, y^*) = \langle y^*, y \rangle - F(y)$ is at a maximum with respect to y for given values of y^*. Let $y_2 = g_2(y^*)$ be the values of y for which the function $G(y, y^*)$ is at a maximum with respect to y for given values of y^*, where y is subject to additional constraints. If $g_1(y^*)$ and $g_2(y^*)$ are both differentiable, then

$$0 \leqslant \frac{\partial g_{2i}^*(y^*)}{\partial y_i^*} \leqslant \frac{\partial g_{1i}^*(y^*)}{\partial y_i^*}, \qquad \forall i.$$

In other words, the effect on the maximizing y_i of an increase in y_i^* is larger for all i when y is unconstrained than when y is subject to constraints. Diewert [7] derived a local version of this principle using the restricted profit function.

We shall demonstrate the principle, using the Hessian matrix identities we have developed and without resorting to determinants. We note first of all from Eq. (21) that the matrix

$$\left[\frac{\partial^2 H}{\partial y^{*2}}\right] = \left[\frac{\partial^2 F^*}{\partial y^{*2}}\right] - \left[\frac{\partial^2 F^*}{\partial y^* \partial K^*}\right]\left[\frac{\partial^2 F^*}{\partial K^{*2}}\right]\left[\frac{\partial^2 F^*}{\partial K^* \partial y^*}\right]$$

[32] Consider the quadratic form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} A_{11} A_{12} \\ A_{12}' A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1' A_{11} x_1 + 2x_1' A_{12} x_2 + x_2' A_{22} x_2 \geqslant 0, \qquad \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

By letting

$$\begin{aligned} x_1 &= -A_{11}^{-1} A_{12} x_2 \\ &= x_2' A_{12}' A_{11}^{-1} A_{11} A_{11}^{-1} A_{12} x_2 - 2x_2' A_{12}' A_{11}^{-1} A_{12} x_2 + x_2' A_{22}\, x \geqslant 0, \qquad \forall x_2 \\ &= x_2'(A_{22} - A_{12}' A_{11}^{-1} A_{12}) x_2 \geqslant 0, \qquad \forall x_2 . \end{aligned}$$

is the difference between two positive semidefinite matrices. Thus, we conclude immediately that

$$\frac{\partial^2 H}{\partial y_i^{*2}} \leqslant \frac{\partial^2 F^*}{\partial y_i^{*2}},$$

or, equivalently,

$$\frac{\partial x_i^K}{\partial y_i^*} \leqslant \frac{\partial x_i^*}{\partial y_i^*};$$

that is, the own price effect with all commodities variable is greater than the own price effect with some commodities fixed.

Next, we investigate the effect of fixing one more commodity. We refer to the vector of $(n-1)$ normalized prices of variable commodities as y_-^* and the additional fixed commodity as y_n. By substituting H^{K+} for H and H for F^* in Eq (21) where K^+ denotes the fact that one more commodity is fixed, we obtain

$$\left[\frac{\partial^2 H^{K+}}{\partial y_-^{*2}}\right] = \left[\frac{\partial^2 H}{\partial y_-^{*2}}\right] - \left[\frac{\partial^2 H}{\partial y_-^* \partial y_n}\right]\left[\frac{\partial^2 H}{\partial y_n^{*2}}\right]^{-1}\left[\frac{\partial^2 H}{\partial y_n \partial y_-^*}\right];$$

we conclude again that

$$0 \leqslant \frac{\partial x_i^{K+}}{\partial y_i^*} \leqslant \frac{\partial x_i^K}{\partial y_i^*} \leqslant \frac{\partial x_i^*}{\partial y_i^*}. \tag{22}$$

Thus, we have Samuelson's result that the own price effect of the remaining variable commodities decreases monotonically as the quantities of additional commodities are fixed. And as Samuelson has pointed out, these results are completely independent of whether the commodities are complements or substitutes. These inequalities become equalities if and only if the normalized restricted profit function is additive in the commodities. Similarly, we obtain from Eq. (19),

$$\frac{\partial^2 H^{K+}}{\partial y_n^2} = -\left[\frac{\partial^2 H}{\partial y_n^{*2}}\right]^{-1}.$$

Now suppose we perform the reverse experiment, that is, we allow y_n to become variable again. We have

$$H(y^*, K) = \sup_{y_n} \{\langle y_n^*, y_n\rangle + H^{K+}(y_-^*, y_n, K)\}.$$

By substituting H^{K+} for F in Eq. (11), we obtain

$$\left[\frac{\partial^2 H}{\partial K^2}\right] = \left[\frac{\partial^2 H^{K+}}{\partial K^2}\right] - \left[\frac{\partial^2 H^{K+}}{\partial K \partial y_n}\right]\left[\frac{\partial^2 H^{K+}}{\partial y_n^2}\right]^{-1}\left[\frac{\partial^2 H^{K+}}{\partial y_n \partial K}\right].$$

Since

$$\left[\frac{\partial^2 H}{\partial K^2}\right], \left[\frac{\partial^2 H^{K+}}{\partial K^2}\right] \quad \text{and} \quad \left[\frac{\partial^2 H^{K+}}{\partial K\, \partial y_n}\right]\left[\frac{\partial^2 H^{K+}}{\partial y_n{}^2}\right]^{-1}\left[\frac{\partial^2 H^{K+}}{\partial y_n\, \partial K}\right]$$

are all negative semidefinite matrices, by an argument similar to the one used earlier we conclude that the own effect of changes in the quantities of the fixed commodities on their implicit normalized shadow prices decreases monotonically as the quantities of an additional number of commodities are fixed; that is,

$$0 \geqslant \frac{\partial(\partial H/\partial K_i)}{\partial K_i} \geqslant \frac{\partial(\partial H^{K+}/\partial K_i)}{\partial K_i}. \tag{23}$$

Again, these inequalities become equalities if and only if the normalized restricted profit function is additive in the commodities.

4. Concluding Remarks

We have shown that there is a one-to-one correspondence between production functions and normalized restricted profit functions under alternative sets of conditions. Thus, for the purposes of economic analysis, theoretical or empirical, one may just as well start with the appropriate normalized restricted profit function. In many short-run situations, the normalized restricted profit function is often the most suitable concept for analysis and interpretation. Given a normalized profit function, we can always recover the original production function by the conjugacy operation.

More duality results linking the production function and the normalized restricted profit function can be found in Lau [19].

We have provided formulae for the computation of the Hessian matrices of the normalized restricted profit functions under alternative specifications of the environment given knowledge of one environment. Thus, one can deduce long-run behavior from short-run behavior or vice versa. From these Hessians, the matrix of supply elasticities can be easily calculated.

For econometric purposes, various functional forms are available. Diewert [7] has proposed several, including the generalized linear and transcendental logarithmic functions. In view of the formulae linking the Hessian matrices, however, the quadratic functional form seems to have special appeal, the Hessian matrices of all the normalized profit functions corresponding to the same production possibilities set are constant

matrices.[33] Hence, given the parameters for one, the parameters of the others can be computed directly using our Hessian matrix identities. For the other functional forms, the computation is likely to be involved.

APPENDIX

1. *Proof of Theorem* 1

We note that the closure assumptions assure that the dual of the dual of a convex function is the primal function itself. We shall not repeat Rockafellar's [27] proof here. We shall, however, show equivalence of the other properties.

A. *Domain.* By definition,

$$H(0, 0) = \sup_{y} \{\langle 0, y \rangle - F(y, 0)\}$$
$$= 0,$$

since $F(y, K) \geq 0$. Conversely,

$$F(0, 0) = \sup_{y^*} \{\langle y^*, 0 \rangle - H(y^*, 0)\}$$
$$= 0,$$

since $H(y^*, 0) \geq 0$.

B. *Convexity–Concavity.* Convexity of $H(y^*, K)$ in y^* is well known and will not be repeated here. It remains to be shown that $H(y^*, K)$ is concave in K. Let y_1, y_2 be the profit-maximizing production plans at $[y^{*\prime} K_1']'$ and $[y^{*\prime} K_2']'$, respectively. Let

$$y_\lambda = (1 - \lambda) y_1 + \lambda y_2;$$
$$K_\lambda = (1 - \lambda) K_1 + \lambda K_2, \qquad 0 < \lambda < 1.$$

Then,

$$\begin{aligned} H(y^*, K_\lambda) &\geq \langle y^*, y_\lambda \rangle - F(y_\lambda, K_\lambda) \\ &\geq \langle y^*, (1 - \lambda) y_1 + \lambda y_2 \rangle - [(1 - \lambda) F(y_1, K_1) + \lambda F(y_2, K_2)] \\ &\qquad \text{by convexity of } F(y, K) \\ &\geq (1 - \lambda)[\langle y^*, y_1 \rangle - F(y_1, K_1)] + \lambda[\langle y^*, y_2 \rangle - F(y_2, K_2)] \\ &\geq (1 - \lambda) H(y^*, K_1) + \lambda H(y^*, K_2). \end{aligned}$$

Thus, $H(y^*, K)$ is concave in K.

[33] Lau [20] first proposed the use of the quadratic function as a normalized profit function and production function.

Conversely, if $H(y^*, K)$ is a convex–concave function, we shall show that

$$F(y, K) = \sup_{y^*} \{\langle y^*, y\rangle - H(y^*, K)\}$$

is convex. Let $y_\lambda{}^*$ by the production function maximizing prices at $[y_\lambda{}' K_\lambda{}']'$. Consider

$$\begin{aligned}(1-\lambda)F(y_1, K_1) + \lambda F(y_2, K_2) &\geqslant (1-\lambda)[\langle y_\lambda{}^*, y_1\rangle - H(y_\lambda{}^*, K_1)] \\ &\quad + \lambda[\langle y_\lambda{}^*, y_2\rangle - H(y_\lambda{}^*, K_2)] \\ &\geqslant \langle y_\lambda{}^*, y_\lambda\rangle - H(y_\lambda{}^*, K_\lambda), \\ &\qquad \text{by concavity of } H(y^*, K) \text{ in } K \\ &\geqslant F(y_\lambda, K_\lambda).\end{aligned}$$

Thus, $F(y, K)$ is convex.

D. *Nonnegativity*

$$\begin{aligned}H(y^*, 0) &= \sup_{y} \{\langle y^*, y\rangle - F(y, 0)\} \\ &= 0\end{aligned}$$

since $F(0, 0) = 0$.

$$\begin{aligned}H(0, K) &= \sup_{y} \{-F(y, K)\} \\ &\leqslant 0\end{aligned}$$

since $F(y, K) \geqslant 0$. Conversely,

$$\begin{aligned}F(y, K) &= \sup_{y^*} \{\langle y^*, y\rangle - H(y^*, K)\} \\ &\geqslant 0,\end{aligned}$$

since $H(0, K) \leqslant 0$. Q.E.D.

2. *Proof of Generalized Hotelling's Lemma*

First, we show that a profit-maximizing production plan at $[y^{*\prime} K']'$ must be a subgradient of $H(y^*, K)$ with respect to y^* at $[y^{*\prime} K']'$. Let y be the profit-maximizing production plan at $[y^{*\prime} K']'$. Thus, $H(y^*, K) = \langle y^*, y\rangle - F(y, K)$. By definition, $H(y^{**}, K) \geqslant \langle y^{**}, y\rangle - F(y, K), \forall y^{**}$. Thus, $H(y^{**}, K) \geqslant H(y^*, K) + \langle y, y^{**} - y^*\rangle, \forall y^{**}$.

Conversely, if y is a subgradient of $H(y^*, K)$ at $[y^{*\prime} K']$ with respect to y^*, one has, by the subgradient inequality,

$$H(y^{**}, K) \geqslant H(y^*, K) + \langle y, y^{**} - y^*\rangle, \qquad \forall y^{**},$$

which may be rewritten as

$$\begin{aligned} \langle y^*, y\rangle - H(y^*, K) &\geqslant \langle y^{*\#}, y\rangle - H(y^{*\#}, K), \qquad \forall y^{*\#} \\ &\geqslant \sup_{y^{*\#}} \{\langle y^{*\#}, y\rangle - H(y^{*\#}, K)\} \\ &\geqslant F(y, K).^{34} \end{aligned}$$

On the other hand,

$$\begin{aligned} F(y, K) &= \sup_{y^*} \{\langle y^*, y\rangle - H(y^*, K)\} \\ &\geqslant \langle y^*, y\rangle - H(y^*, K). \end{aligned}$$

Hence, one has

$$F(y, K) \geqslant \langle y^*, y\rangle - H(y^*, K) \geqslant F(y, K),$$

which implies $H(y^*, K) = \langle y^*, y\rangle - F(y, K)$. Hence, y is indeed a profit-maximizing production plan at $[y^{*\prime}K']'$.

Second, if K^* is a supergradient of $H(y^*, K)$ with respect to K, one needs to show that $[y^{*\prime}K^{*\prime}]'$ is a subgradient of $F(y, K)$ at $[y'K']'$, where y is a profit-maximizing plan at $[y^{*\prime}K']'$. From the supergradient inequality one has

$$H(y^*, K^\#) \leqslant H(y^*, K) + \langle K^*, K^\# - K\rangle, \qquad \forall K^\#.$$

y is a profit-maximizing production plan implies

$$H(y^*, K) = \langle y^*, y\rangle - F(y, K).$$

Adding the second equation to the first inequality and rearranging the terms, one has

$$H(y^*, K^\#) \leqslant \langle y^*, y\rangle - F(y, K) + \langle K^*, K^\# - K\rangle, \qquad \forall K^\#.$$

But $H(y^*, K^\#) \geqslant \langle y^*, y^\#\rangle - F(y^\#, K^\#), \forall y^\#$. Hence, $\langle y^*, y^\#\rangle - F(y^\#, K^\#) \leqslant \langle y^*, y\rangle - F(y, K) + \langle K^*, K^\# - K\rangle, \forall K^\#$, which finally yields

$$F(y^\#, K^\#) \geqslant F(y, K) + \langle [y^{*\prime} - K^{*\prime}]', [y^{\#\prime}K^{\#\prime}]' - [y'K']'\rangle, \quad \forall [y^{*\prime}K^{\#\prime}]',$$

where $-K^{*\prime}$ is indeed the last m component of a subgradient of $F(y, K)$.

[34] Note that $\sup_{y^{*\#}}\{y^{*\#}, y - H(y^{*\#}, K)\} = F(y, K)$ if and only if $F(y, K)$ is convex and closed in $\{y\}$. That is why assumptions F. 4 through F. 4 have to be satisfied for the coincidence of the subdifferential and the set of profit-maximizing production plans.

Conversely, if $[y^{*\prime} - K^{*\prime}]'$ is a subgradient of $F(y, K)$ where y is a profit-maximizing production plan at $[y^{*\prime}K']'$, one has, by the subgradient inequality,

$$F(y^{\#}, K^{\#}) \geqslant F(y, K) + \langle [y^{*\prime} - K^{*\prime}]', [y^{\#\prime}K^{\#\prime}]' - [y'K']' \rangle, \quad \forall [y^{\#\prime}K^{\#\prime}]',$$
$$H(y^*, K) = \langle y^*, y \rangle - F(y, K).$$

By adding the second equation to the first inequality and rearranging terms, one has

$$F(y^{\#}, K^{\#}) + H(y^*, K) \geqslant \langle y^*, y \rangle + \langle y^*, y^{\#} - y \rangle - \langle K^*, K^{\#} - K \rangle, \quad \forall y^{\#}, K^{\#},$$

or

$$H(y^*, K) + \langle K^*, K^{\#} - K \rangle \geqslant \langle y^*, y^{\#} \rangle - F(y^{\#}, K^{\#}), \qquad \forall y^{\#}, K^{\#},$$
$$\geqslant H(y^*, K^{\#}), \qquad \forall K^{\#}.$$

Hence, K^* is a supergradient of $H(y^*, K)$ with respect to K at $[y^{*\prime}K']'$. Q.E.D.

3. *Proof of Theorem* 2

We need only show that T.3A is equivalent to F.3A. Convexity of T^K for all K immediately implies that $F(y, K)$ is convex in y for all K. It remains to be shown that $F(y, K)$ is quasiconvex in K for all y. Let

$$y^{K_1}_{n+1} \equiv -F(y, K_1); \qquad y^{K_2}_{n+1} \equiv -F(y, K_2).$$

Then by monotonicity,

$$\underset{K_1}{[\min}^{y} \{y^{K_1}_{n+1}, y^{K_2}_{n+1}\}] \in T \quad \text{and} \quad \underset{K_2}{[\min}^{y} \{y^{K_1}_{n+1}, y^{K_2}_{n+1}\}] \in T.$$

By convexity,

$$\underset{K_\lambda}{[\min}^{y} \{y^{K_1}_{n+1}, y^{K_2}_{n+1}\}] \in T,$$

where

$$K_\lambda = (1 - \lambda) K_1 + \lambda K_2, \qquad 0 < \lambda < 1.$$

Then

$$-F(y, K_\lambda) = \sup\{y_{n+1} \mid [y' y_{n+1} K_\lambda']' \in T\}$$
$$\geqslant \min\{y^{K_1}_{n+1}, y^{K_2}_{n+1}\}$$
$$\geqslant -\max\{F(y, K_1), F(y, K_2)\}.$$

Reversing the signs, we have

$$\max\{F(y, K_1), F(y, K_2)\} \geqslant F(y, K_\lambda), \qquad 0 < \lambda < 1.$$

Conversely, if $F(y, K)$ is convex-quasiconvex, we need to show that the production possibilities set T^* defined by

$$T^* = \{[y' y_{n+1} K']' \mid y_{n+1} \leqslant -F(y, K)\}$$

is convex in $[y' y_{n+1}]'$ for every K, and convex in K for every $[y' y_{n+1}]'$. Let $[y_1' y_{n+1,1} K']'$ and $[y_2' y_{n+1,2} K']' \in T^*$. Let $y_\lambda = (1 - \lambda) y_1 + \lambda y_2$; $y_{n+1,\lambda} = (1 - \lambda) y_{n+1,1} + \lambda y_{n+1,2}$; $\forall y$, $0 < \lambda < 1$. Consider

$$\begin{aligned} -F(y_\lambda, K) &\geqslant (1 - \lambda)[-F(y_1, K)] + \lambda[-F(y_2, K)], \\ &\qquad \text{by concavity of } -F(y, K), \\ &\geqslant (1 - \lambda) y_{n+1,1} + \lambda y_{n+1,2} \\ &\geqslant y_{n+1,\lambda}, \qquad \forall\lambda, \quad 0 < \lambda < 1. \end{aligned}$$

Thus, $[y_\lambda' y_{n+1,\lambda} K']' \in T^*$. Now let $[y' y_{n+1} K_1']'$ and $[y' y_{n+1} K_2']' \in T^*$. Let $K_\lambda = (1 - \lambda) K_1 + \lambda K_2$; $\forall\lambda$, $0 < \lambda < 1$. By quasiconcavity of $-F(y, K)$,

$$\begin{aligned} -F(y, K_\lambda) &\geqslant \min\{F(y, K_1), -F(y, K_2)\} \\ &\geqslant y_{n+1}, \qquad \text{by the definition of } -F(y, K). \end{aligned}$$

Thus, $[y' y_{n+1} K_\lambda']' \in T^*$. Q.E.D.

4. *Proof of Theorem* 3

It only needs to be shown that independence of the production possibilities set T in the $[y' y_{n+1}]' - K$ partition implies and is implied by order-independence of the production function $F(y, K)$.

Suppose T is independent. Let $F(y, K_1) > F(y, K_2)$ for some y, K_1, K_2 ; then, by the definition of the production function and monotonicity of the production possibilities set,

$$\begin{bmatrix} y \\ -F(y, K_2) \\ K_2 \end{bmatrix} \in T \qquad \text{and} \qquad \begin{bmatrix} y \\ -F(y, K_2) \\ K_1 \end{bmatrix} \notin T.$$

Thus, independence of T implies that $T^{K_1} \subset T^{K_2}$. Hence, if $[y^{*\prime} y_{n+1}^* K_1']' \in T$, $[y^{*\prime} y_{n+1}^* K_2']' \in T$, $\forall [y^{*\prime} y_{n+1}^*]'$. Therefore,

$$\begin{aligned} F(y^*, K_1) &\equiv -\sup\{y_{n+1}^* \mid [y^{*\prime} y_{n+1}^* K_1']' \in T\} \\ &\geqslant F(y^*, K_2) \equiv -\sup\{y_{n+1}^* \mid [y^{*\prime} y_{n+1}^* K_2']' \in T\}, \qquad \forall y^*. \end{aligned}$$

Now suppose $F(y, K)$ is order-independent. Let $[y'y_{n+1}K_1']' \in T$ and $[y'y_{n+1}K_2']' \notin T$, then $F(y, K_2) > F(y, K_1)$ by monotonicity of the production possibilities set. Hence, $F(y^*, K_2) \geqslant F(y^*, K_1)\ \forall y^*$. This implies that $[y^{*\prime}y^*_{n+1}K_2']' \in T$ implies that $[y^{*\prime}y^*_{n+1}K_1']' \in T$ or $T^{K_2} \subset T^{K_1}$. Hence, T is independent in the $[y'y_{n+1}]' - K$ partition. Q.E.D.

5. *Proof of Theorem* 4

We first show that F.5A is equivalent to H.5A. Suppose $F(y, K)$ is order-independent. Let $H(y^*, K_1) > H(y^*, K_2)$ for some y^*, K_1, K_2. By the definition of $H(y^*, K)$, the inequality implies that there is y such that

$$\begin{aligned} H(y^*, K_1) &= \langle y^*, y\rangle - F(y, K_1) \\ &> H(y^*, K_2) \\ &\geqslant \langle y^*, y\rangle - F(y, K_2), \end{aligned}$$

and, thus, $F(y, K_2) > F(y, K_1)$. It then follows from order-independence of $F(y, K)$ in K that

$$F(y^*, K_2) \geqslant F(y^*, K_1), \qquad \forall y^*.$$

Consider now for any y^{**},

$$\begin{aligned} H(y^{**}, K_1) &\geqslant \langle y^{**}, y^*\rangle - F(y^*, K_1), \qquad \forall y^* \\ &\geqslant (y^{**}, y^*\rangle - F(y^*, K_2), \qquad \forall y^*, \\ &\quad \text{since} \quad F(y^*, K_2) \geqslant F(y^*, K_1) \qquad \forall y^*, \\ &\geqslant \sup_{y^*} \{\langle y^{**}, y^*\rangle - F(y^*, K_2)\} \\ &= H(y^{**}, K_2). \end{aligned}$$

Thus $H(y^*, K)$ is order-independent in K.

Conversely, suppose $H(y^*, K)$ is order-independent. Let $F(y, K_1) > F(y, K_2)$ for some y, K_1, and K_2. By the fact that the dual of the dual is the primal function itself, there is y^* such that

$$\begin{aligned} F(y, K_1) &= \langle y^*, y\rangle - H(y^*, K_1) \\ &> F(y, K_2) \\ &\geqslant \langle y^*, y\rangle \qquad H(y^*, K_2), \end{aligned}$$

and, thus, $H(y^*, K_2) > H(y^*, K_1)$. It then follows from order-independence of $H(y^*, K)$ in K that

$$H(y^{**}, K_2) \geqslant H(y^{**}, K_1), \qquad \forall y^{**}.$$

Consider now for any y^{**},

$$\begin{aligned} F(y^{**}, K_1) &\geq \langle y^{***}, y^{**} \rangle - H(y^{***}, K_1), \qquad \forall y^{***} \\ &= \langle y^{***}, y^{**} \rangle - H(y^{***}, K_2), \qquad \forall y^{***}, \\ &\quad \text{since} \quad H(y^{***}, K_2) \geq H(y^{***}, K_1), \qquad \forall y^{***} \\ &\geq \sup_{y^{***}} \{\langle y^{***}, y^{**} \rangle - H(y^{***}, K_2)\} \\ &= F(y^{**}, K_2). \end{aligned}$$

Thus, $F(y, K)$ is order-independent in K.

It remains to be shown that under the assumptions of F.1A through F.5A, the corresponding $H(y^*, K)$ is quasiconcave in K for every y^*, and that similarly, under the assumptions of H.1A through H.5A, the corresponding $F(y, K)$ is quasiconcave in K for every y.

Consider y_1, y_2 which are optimal at some y^* and K_1, K_2, respectively. Let y_λ and K_λ be any convex combination of y_1, y_2 and K_1, K_2, respectively. Then

$$\begin{aligned} H(y^*, K_\lambda) &\geq \langle y^*, y_\lambda \rangle - F(y, K_\lambda), \qquad \forall y \\ &\geq \langle y^*, y_\lambda \rangle - \max[F(y, K_1), F(y, K_2)], \qquad \forall y \\ &\quad \text{by quasiconvexity of } F(y, K) \text{ in } K \\ &= -\max[F(y, K_1) - \langle y^*, y_\lambda \rangle, F(y, K_2) - \langle y^*, y \rangle], \qquad \forall y \\ &\quad \text{by subtraction of an identical quantity from both} \\ &\quad \text{arguments of the max function} \\ &= \min[\langle y^*, y_\lambda \rangle - F(y, K_1), \langle y^*, y_\lambda \rangle - F(y, K_2)], \qquad \forall y \\ &\geq \sup_y \{\min[\langle y^*, y \rangle - F(y, K_1), \langle y^*, y \rangle - F(y, K_2)]\}. \end{aligned}$$

But by order-independence, if $\langle y^*, y_\lambda \rangle - F(y, K_1) \geq \langle y^*, y \rangle - F(y, K_2)$ for some y (or vice versa), the sense of the inequality is not reversed for all y. Thus, one may interchange the order of $\sup_y$ and min so that

$$\begin{aligned} H(y^*, K_\lambda) &\geq \min[\sup_y \{\langle y^*, y \rangle - F(y, K_1)\}, \sup_y \{\langle y^*, y \rangle - F(y, K_2)\}] \\ &\geq \min[H(y^*, K_1), H(y^*, K_2)]. \end{aligned}$$

This is true for all λ and all y^*, K_1, K_2. Thus, $H(y^*, K)$ is quasiconcave in K.

Conversely, if $H(y^*, K)$ is quasiconcave and order-independent in K, then consider

$$\begin{aligned}
&\max[F(y, K_1), F(y, K_2)] \\
&\quad = \max[\sup_{y^*} \{\langle y^*, y\rangle - H(y^*, K_1)\}, \sup_{y^*} \{\langle y^*, y\rangle - H(y^*, K_2)\}] \\
&\quad = \sup_{y^*}\{\max[\langle y^*, y\rangle - H(y^*, K_1), \langle y^*, y\rangle - H(y^*, K_2)]\} \\
&\qquad \text{because } \max_{K_1, K_2} \sup_{y^*} = \sup_{y^*} \max_{K_1, K_2} \\
&\quad \geqslant \sup_{y^*} \{\langle y^*, y\rangle - H(y^*, K_\lambda)\} \\
&\qquad \text{by quasiconvexity of } \langle y^*, y\rangle - H(y^*, K) \text{ in } K \\
&\quad \geqslant F(y, K_\lambda).
\end{aligned}$$

Thus $F(y, K)$ is quasiconvex in K.

We conclude therefore order-independence and quasiconvexity in K of $F(y, K)$ implies order-independence and quasiconcavity in K of $H(y^*, K)$ and vice versa. Q.E.D.

Note that in the proof of quasiconvexity of $F(y, K)$ in K, we do not need to use the order-independence property. Thus, in general, quasiconcavity of $H(y^*, K)$ in K implies quasiconvexity of $F(y, K)$. But the converse is not true.

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Global Asymptotic Stability of Optimal Control Systems with Applications to the Theory of Economic Growth*

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1

The qualitative study of optimal economic growth has attracted the attention of economic theorists for some number of years. One major focus of this research has been to find sufficient conditions on models of economic growth for the convergence of growth paths to a steady state. In this symposium volume of the *Journal of Economic Theory*, Cass and Shell [16] present the Hamiltonian formulation of competitive dynamical systems that arise in capital theory. In this paper, we present a set of sufficient conditions on the Hamiltonian for such dynamical systems to converge to a steady state as time tends to infinity.

We refer the reader to the Cass-Shell paper for an introduction to competitive dynamical systems, and a complete survey of the literature. Only a very brief survey, that is useful for leading the reader into our approach, will be given here.

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A general formulation of capital theory or optimal growth problems is

$$\text{Max} \int_0^\infty e^{-\rho t} u[k(t), \dot{k}(t)]\, dt$$

subject to

$$k(0) = k_0 .$$

Here $u: B \subset \mathring{R}_+^n \times R^n \to R$ is usually assumed C^2 and concave. Here $\mathring{R}_+^n = \{x \in R^n / x_i > 0;\ i = 1,..., n\}$, and B is convex with nonempty interior.

$$R(k_0) = \text{Max} \int_0^\infty e^{-\rho t} u[k(t), \dot{k}(t)]\, dt$$

is called the value of the initial stock k_0.

A capital theory model generates a capital–price differential equation by using the Hesteness–Pontriagin [22, 35] maximal principle to write down necessary conditions for an optimal solution. This process generates a type of differential equation system that we will call a "modified Hamiltonian dynamical system." The adjective "modified" appears because it is a certain type of perturbation peculiar to economics of the standard Hamiltonian system.

Since the standard Hamiltonian case will be a special case of our problem, our results will be of independent interest to mathematicians working in the field of Hamiltonian dynamical systems.

In order to discuss our approach, we need a definition. A modified Hamiltonian dynamical system (call it an MDHS for short) is a differential equation system of the form

$$\begin{aligned} \dot{q}_j &= \rho q_j - \frac{\partial H}{\partial k_j}(q, k), \qquad H(q, k) = \underset{k:(k,\dot{k}) \in B}{\text{maximum}} \{u(k, \dot{k}) + q\dot{k}\}, \\ \dot{k}_j &= \frac{\partial H}{\partial q_j}(q, k), \qquad j = 1, 2,..., n. \end{aligned} \tag{1.1}$$

Here H: $\mathring{R}_+^n \times \mathring{R}_+^n \to R$, and $\rho \in R_+$. In economics, k_j is stock of capital good j and q_j is the price of capital good j. The function H is called a Hamiltonian, and it is well defined on $\mathring{R}_+^n \times \mathring{R}_+^n$ for many economic problems. However, what follows only depends on H being defined on an open convex subset of R^{2n}, provided the obvious changes are made. H turns out to be the current value of national income evaluated at prices q. The number ρ is a discount factor on future welfare arising from the structure of social preferences. See Cass and Shell for a complete interpretation of (1.1).

Clearly not all solutions of (1.1) which satisfy $k(0) = k_0$ will, in general,

be optimal. We will call a solution $[q(t), k(t)]$ of (1.1) optimal if $k(t)$ is the optimal solution of the optimal growth problem when $k(0) = k_0$.

The problem that we shall address in this paper may now be defined.

PROBLEM 1. Find sufficient conditions on optimal solutions ϕ_t of (1.1) such that $\phi_t \to (\bar{q}, \bar{k})$, $t \to \infty$. Also find sufficient conditions such that the steady state $(\bar{q}, \bar{k})$ is independent of the initial condition (q_0, k_0).

The literature on Problem 1 has two main branches: (1) analysis of the local behavior of (1.1) in a neighborhood of a steady state, and (2) analysis of global behavior of solutions of (1.1).

The first branch of the literature is fairly complete. It studies the linear approximation of (1.1) in a neighborhood of a rest point. Eigenvalues have a well-known symmetric structure that determines the local behavior. Since we have nothing new to contribute to this branch of study, we, therefore, cite some representative references and move on [27, 41, 28]. To the global problem we now turn.

The literature on Problem 1 is very large yet there are no general results on global stability. Representative literature on existing global stability results for optimal paths follows. The case $n = 1$ (the one good optimal growth model) is well understood. See [11, 14, 25, 27]. Ryder and Heal [39] analyze a case of (1.1) for $n = 2$. They generate a variety of examples of different qualitative behavior of optimal paths. Burmeister and Graham [12] present an analysis of a model where there is a set S containing the steady state capital $\bar{k}$ such that for $k \in S$, the value of q along an optimal path is independent of k. See [12, Theorem 2, p. 149].

The case $\rho = 0$ is the famous Ramsey problem, studied first by Ramsey for the one good model, then by Gale [18], McKenzie [33], McFadden [32], and Brock [4] for the n goods model in discrete time and by Rockafellar [36] for continuous time. These results state, roughly speaking, that if $H(q, k)$ is strictly convex in q and strictly concave in k and $\rho = 0$, then all solution paths of (1.1) that are optimal converge to a unique steady state $(\bar{q}, \bar{k})$ as $t \to \infty$ independently of (q_0, k_0).

Scheinkman [42] has recently proved a result that shows that the qualitative behavior for $\rho = 0$ is preserved for small changes in ρ near $\rho = 0$.

Until very recently no general results on the convergence of optimal solutions of (1.1) were available. In fact, little was known about sufficient conditions for the uniqueness of steady states of (1.1). Recently, papers of Brock [5] and Brock and Burmeister [46] gave a fairly general set of sufficient conditions for uniqueness of the steady state. There was nothing done in the Brock paper on convergence, however.

In this paper we will present new results that build on the work of Cass

and Shell [16], Rockafellar [38], and Hartman and Olech [21]. We start by discussing the three basic types of results obtained and their relation to the literature.

The following problem has been analyzed extensively in the differential equations literature. Let

$$\dot{x} = f(x),\ x(0) = x_0$$

be a differential equation system. Let $\bar{x}$ satisfy $f(\bar{x}) = 0$. Under what conditions is the solution $x(t) \equiv \bar{x}$ globally asymptotically stable for all x_0?

DEFINITION 1.1. The solution $x(t) \equiv \bar{x}$ of $\dot{x} = f(x)$ is *globally asymptotically stable* (GAS) if for all x_0 the solution $\phi_t(x_0) \to \bar{x}$, $t \to \infty$.

DEFINITION 1.2. The solution $x(t) \equiv \bar{x}$ of $\dot{x} = f(x)$ is *locally asymptotically stable* (LAS) if there is $\epsilon > 0$ such that $| x_0 - \bar{x} | < \epsilon$ implies $\phi_t(x_0) \to \bar{x}$, $t \to \infty$.

If f is a modified Hamiltonian, new problems arise. In this case, $x(t) \equiv \bar{x}$ is usually never even locally asymptotically stable in a neighborhood of $\bar{x}$. This is so because if λ is an eigenvalue of the linear approximation so also is $-\lambda + \rho$ (see [27], e.g.). Thus, a natural question to pose is our Problem 1 for MHDS.

DEFINITION 1.3. $\phi_t(q_0, k_0)$ will be called *a bounded solution of* (1.1) if there exists a compact set $K \subseteq \mathring{R}_+^n \times \mathring{R}_+^n$ such that $\phi_t(q_0, k_0) \subseteq K$ for all t.

Note that our definition of bounded solutions requires not only boundedness on the (q, k) space, but also that there exists $\epsilon > 0$ such that

$$q_i(t) \geqslant \epsilon, \qquad k_i(t) \geqslant \epsilon, \qquad i = 1,\ldots, n; \qquad 0 \leqslant t < \infty.$$

In many optimal growth problems, "Inada"-type conditions guarantee that in fact optimal solutions will satisfy our boundedness condition. For this reason, we concentrate on the convergence of bounded solutions.

DEFINITION 1.4. The steady state solution $(q, k) = (\bar{q}, \bar{k})$ of (1.1) is said to be *globally asymptotically stable* for bounded solutions of (1.1) [i.e., those who satisfy Definition 1.3] if for all (q_0, k_0) such that $\phi_t(q_0, k_0)$ is bounded, we have

$$\phi_t(q_0, k_0) \to (\bar{q}, \bar{k}),\ t \to \infty.$$

DEFINITION 1.5. The steady state solution $(q, k) = (\bar{q}, \bar{k})$ of (1.1) is said to be *locally asymptotically stable* for bounded solutions of (1.1) if there is $\epsilon > 0$ such that $|(q_0, k_0) - (q, k)| < \epsilon$ implies $\phi_t(q_0, k_0) \to (\bar{q}, \bar{k})$, $t \to \infty$, provided that $\phi_t(q_0, k_0)$ is bounded.

For MHDS, the words "GAS" and "LAS" will always apply to bounded solutions alone in this paper. We will sometimes call MHDS "saddle-point" systems when we want to emphasize their saddle-point structure.

Let us denote by $(\bar{q}, \bar{k})$ some rest point of (1.1); then we can rewrite (1.1) as

$$\begin{aligned} \dot{z}_1 &= F_1(z_1, z_2), \\ \dot{z}_2 &= F_2(z_1, z_2), \end{aligned} \tag{1.2}$$

where $F_i: A \subseteq R^{2n} \to R^n$ where A is open and convex, by letting

$$q - \bar{q} = z_1, k - \bar{k} = z_2, F_1(z) = -H_2[z + (\bar{q}, \bar{k})] + \rho(z_1 + \bar{q}), F_2(z) = H_1[z + (\bar{q}, \bar{k})].$$

Here again a "bounded" solution will mean a solution which is contained in a compact set $K \subset A$.

Most of our assumptions will refer to the "curvature" matrix

$$Q(z) = \begin{bmatrix} H_{11}(z) & \rho/2\, I_n \\ \rho/2\, I_n & -H_{22}(z) \end{bmatrix}$$

where $H_{11}(z) = (\partial H_1/\partial z_1)(z + (\bar{q}, \bar{k}))$, $H_{22}(z) = (\partial H_2/\partial z_2)(z + (\bar{q}, \bar{k}))$, and I_n is the $n \times n$ identity matrix.

In Section 2, we show that *if* $F(z)\, Q(z)\, F(z) \geq 0$ *for all* z *such that* $F(z) \neq 0$, *then every bounded trajectory converges to a rest point.* This is the "Hamiltonian version" of the well known result in differential equations which states that if $f: R^n \to R$ and $J(x) = (\partial f(x)/\partial x)$, then $f^T(x)\, J(x)\, f(x) < 0$ for all x with $f(x) \neq 0$ implies that all solutions of $\dot{x} = f(x)$ converge to a rest point. Our result is obtained by using the "Lyapunov" function $F_1^T(z)\, F_2(z)$.

In Section 3, we show that *if* $z_1^T F_2(z) + z_2^T F_1(z) = 0$ *implies* $z^T Q(z)\, z > 0$ *and if* $Q(0)$ *is positive definite, then all bounded trajectories of* (1.2) *converge to the origin.* This result is related to a result by Hartman and Olech [21] that states that if $w^T J(x)\, w < 0$ for all w such that $|w| = 1$ and $w^T f(x) = 0$, then every solution to $\dot{x} = f(x)$ converges to the origin provided that 0 is LAS. The proof is inspired by the elegant proof of a Hartman–Olech type of result obtained by Mas-Colell [31]. As a by-product of the proof, we show that the above conditions are sufficient for the function $z_1^T z_2$ to be monotonically increasing along trajectories, which is the Cass and Shell hypothesis. Note that the assumptions of Sections 2 and 3 are somewhat complementary.

The method of proof of Section 3 does not, unfortunately, generalize to prove results analogous to Hartman and Olech's [21] most general results. For this reason, in Section 4 we outline the method of proof of a more general theorem. This method is similar to the method used by

Hartman and Olech of constructing an "orthogonal" field of trajectories to the trajectories generated by a system of differential equations $\dot{z} = F(z)$, and placing conditions on the Jacobian matrix of F so that all trajectories of the original field come together monotonically as $t \to \infty$, in the metric induced by the arc length measure along the "orthogonal" field of trajectories. The complete proof is quite messy and the reader is referred to our paper [8].

Furthermore, the results in this section have a nice geometrical interpretation in terms of quasi-convexity and quasi-concavity. In particular, we show that *for the case* $\rho = 0$, *if the Hamiltonian is quasi-concave in the state variable, then GAS holds* (although in this case "optimality" may not make sense). A notion of α-quasi-convexity is introduced to provide a geometric interpretation for the case where $\rho > 0$.

Curvature Matrix Q: Geometric and Economic Meaning

The curvature matrix Q is a natural economic and geometric quantity. As Cass and Shell [16] point out, the Hamiltonian is convex in q and concave in k for optimal control problems with a concave objective function. Hence H_{11}, $-H_{22}$ are positive semidefinite matrices.

Geometric Content of Q

For the one-dimensional case Q is positive definite provided that

$$(H_{11})(-H_{22}) > \rho^2/4.$$

This suggests that if the smallest eigenvalue α of H_{11} and the smallest eigenvalue β of $-H_{22}$ satisfy

$$\alpha\beta > \rho^2/4, \tag{R}$$

then Q is positive definite. It is easy to show that (R), which is Rockafellar's [38] basic stability hypothesis, does indeed imply that Q is positive definite.[1]

[1] Let the minimum eigenvalue of $H_{11}(q, k)$ be $\alpha(q, k)$ and the minimum eigenvalue of $-H_{22}(q, k)$ be $\beta(q, k)$. Then if $\alpha\beta > \rho^2/4$, it follows that

$$Q(q, k) = \begin{bmatrix} H_{11}(q, k), & \rho/2\, I_n \\ \rho/2\, I_n, & -H_{22}(q, k) \end{bmatrix}$$

is positive definite.

To prove this, examine

$$\begin{aligned} x^T Qx &= x_1^T H_{11} x_1 + x_2^T(-H_{22})x_2 + \rho x_1^T x_2 \\ &\geq \alpha x_1^T x_1 + \beta x_2^T x_2 + \rho x_1^T x_2 \\ &\geq \alpha x_1^T x_1 + \beta x_2^T x_2 - \rho\, |\, x_1^T x_2\, | \\ &\geq \alpha \| x_1 \|^2 + \beta \| x_2 \|^2 - \rho \| x_1 \| \| x_2 \| \end{aligned}$$

where $\| z \| \equiv (z^T z)^{1/2}$ for any vector z.

Economic Content of Q

It is well known that the Hamiltonian $H(q, k)$ can be interpreted as "shadow profit" when q is the shadow price of investment, all with utility as numeraire. Cass and Shell [16] develop in detail the economic meaning of the Hamiltonian. They show, in particular,

$$\partial H/\partial q = \text{optimum investment level},$$

and

$$\partial H/\partial k = \text{marginal value product of } k.$$

Therefore, $\partial H/\partial q$ is an "internal supply curve for investment," and $\partial H/\partial k$ is a "Marshallian" demand curve for capital services. Thus,

$$H_{11} = \partial^2 H/\partial q^2,\ H_{22} = \partial^2 H/\partial k^2,$$

are generalized slopes of supply and demand curves for investment and capital services. The matrix Q is just a convenient way of tabulating information on supply, demand, and the "interest" rate ρ, that is important for stability analysis.

More specifically, a sufficient condition for stability is that

$$Q = \begin{bmatrix} H_{11} & \rho/2\, I_n \\ \rho/2\, I_n & H_{22} \end{bmatrix}$$

be positive definite. An intuitive way of putting this is that the slopes of the supply curves for investment and the demand curves for capital services are large relative to the interest rate ρ, and that cross terms are small relative to own terms.

Let us expand upon the economics here. Along an optimum path, $q(t)$

Now,

$$\alpha \| x_1 \|^2 + \beta \| x_2 \|^2 - \rho \| x_1 \| \| x_2 \| \cdot \alpha \| x_1 \|^2 + \beta \| x_2 \|^2 - (4\alpha\beta)^{1/2} \| x_1 \| \| x_2 \|.$$

So our problem reduces to showing that

$$\alpha \| x_1 \|^2 + \beta \| x_2 \|^2 - (4\alpha\beta)^{1/2} \| x_1 \| \| x_2 \| \geq 0.$$

But this last quantity is just

$$(\alpha^{1/2} \| x_1 \| - \beta^{1/2} \| x_2 \|)^2$$

which is nonnegative. This ends the proof.

is the current value of the demand price for capital goods. Thus, $q(t)$ is a Marshallian demand curve for capital equipment; i.e.,

$$q(t) = \frac{\partial}{\partial k}\left\{\text{Maximum} \int_t^\infty e^{-\rho(s-t)}u(y, \dot{y})\, ds, \qquad \text{s.t.} \quad y(t) = k(t)\right\}$$
$$= \frac{\partial R[k(t)]}{\partial k} = R'[k(t)]$$

if $\partial R/\partial k$ exists. Like any demand curve, the demand for k should be downward-sloping. For $u(y, \dot{y})$ concave the Hessian of R, $R''(k)$, is a negative semidefinite matrix which exists for almost every k [24, p. 405].

Look at the "reduced form" and its equation of first variation,

$$\dot{k}(t) = H_1(R'[k(t)], k(t)), \tag{1.3}$$

$$\ddot{k} = (H_{11}R'' + H_{12})\, k. \tag{1.4}$$

Stabilizing forces are forces that lead to increased negative feedback in (1.3). An "increase" in H_{11} is stabilizing because R'' is negative semidefinite.

It is a little more difficult to explain why an increase in $-H_{22}$ is stabilizing. For an increase in $-H_{22}$ is clearly a destabilizing force for q as can be seen intuitively by examining

$$\dot{q} = \rho q - H_2 .$$

But since along an optimum path, q decreases in k, a destabilizing force for q is stabilizing for k.

It is also intuitively clear that moving ρ closer to 0 is stabilizing. This is so because if $\rho = 0$, the system strives to maximize long-run static profit since the future is worth as much as the present.

Inspection of (1.4) hints that an increase in H_{12} is destabilizing. This source of instability is not exposed by the Q matrix. The quantity H_{12} represents a shift in the internal supply curve of investment when capital stock is increased. Therefore, an *increase* in H_{12} represents a type of increase of nonnormality. I.e., since H_{12} is the derivative of the internal supply curve w.r.t. k, therefore, it is likely that H_{12} will be negative in some sense. For an increase in the number of machines (an increase in k) is likely to lead to a decrease in new machines supplied by the "firm" to itself when q increases, if some sort of diminishing returns to capital services and substitutability between investment goods and capital goods is present.

We would expect an increase in H_{12} to contribute to instability because an increase in k leads to more new machines, which leads to yet larger k.

See [6] for stability results that focus on the role of H_{12}, and that are based on a different class of Lyapunov functions than what is presented here.

2. A First Result on G.A.S.

The work in this section is closely related to work by Cass and Shell [16], a paper by Rockafellar [38], papers by Arrow and Hurwicz [3]; Arrow, Block, and Hurwicz [2]; Hartman [19]; and Markus and Yamabe [30]. Consider the modified Hamiltonian system

$$\begin{aligned} \dot{q} - \rho q &= -H_k(q, k), \\ \dot{k} &= H_q(q, k), \qquad k(0) = k_0, \qquad q(0) = q_0. \end{aligned} \tag{2.1}$$

Let $(\bar{q}, \bar{k})$ be a rest point of (2.1) and let $z_1 = q - \bar{q}$, $z_2 = k - \bar{k}$, $F_1(z) = \rho(z_1 + \bar{q}) - H_2 F_2(z + (\bar{q}, \bar{k}))(z)$, $= H_1(z + (\bar{q}, \bar{k}))$. Then (2.1) becomes

$$\begin{aligned} \dot{z}_1 &= F_1(z), \\ \dot{z}_2 &= F_2(z). \end{aligned} \tag{2.2}$$

In [30] and in [3], sufficient conditions for global stability of differential equations, $\dot{x} = f(x)$ are given. Let $J(x) = \partial f/\partial x$. Arrow and Hurwicz and Markus and Yamabe prove (roughly speaking) that the negative definiteness of $J^T + J$ is sufficient for global stability of $\dot{x} = f(x)$ by differentiating $(\dot{x})^T \dot{x} = W$ with respect to t and showing, thereby, that $x(t) \to 0$, $t \to \infty$ along trajectories. Here x^T denotes x transposed. We are going to present an analog of this type of result for Hamiltonian systems.

Consider a trajectory $z(t) = \phi_t(q_0, k_0)$ of (2.1) where q_0 is chosen so that $\phi_t(q_0, k_0)$ is bounded. Optimal growth paths will have this boundedness property under reasonable conditions. Cass and Shell [16][2] prove global stability of such a trajectory by differentiating the Lyapunov function $V = z_1^T z_2$ w.r.t. t. It is, therefore, natural to ask what may be obtained by differentiating the closely related Lyapunov function $F_1^T F_2$.

We will make use of the following result, which is basically the result in Hartman's book [20, p. 539].

Lemma 2.1. *Let $F(z)$ be continuous on an open set $E \subseteq R^m$, and such that solutions of*

$$\dot{z} = F(z) \tag{*}$$

[2] Cass and Shell are the first to obtain GAS results for (2.1) by use of the Lyapunov function $V \equiv z_1^T z_2$ for the discounted case. Their methods provided inspiration for many of our results. Rockafellar [37] and Samuelson [41] have used the same function to investigate stability for the case $\rho = 0$. Magill [45] used V to obtain GAS results for a discounted linear quadratic problem.

are uniquely determined by initial conditions. Let $W(z)$ be a real-valued function on E with the following properties:

(a) W is C^1 on E,

(b) $0 \leqslant \dot{W}(z)$ [*where $\dot{W}(z)$ is the trajectory derivative of $W(z)$ for any $z \in E$*].

Let $z(t)$ be a solution of (*) for $t \geqslant 0$. Then the limit points of $z(t)$ for $t \geqslant 0$, in E, if any, are contained in the set $E_0 = \{z \mid \dot{W}(z) = 0\}$.

Proof. Let $t_n < t_{n+1} \to \infty$, $z(t_n) \to z_0$ as $n \to \infty$, and $z_0 \in E$. Then $W[z(t_n)] \to W(z_0)$ as $n \to \infty$, and by (b) $W[z(t)] \leqslant W(z_0)$ for $t \geqslant 0$. Suppose $z_0 \notin E_0$ so that $\dot{W}(z_0) > 0$ by (b). Let $z_0(t)$ be a solution of (*) satisfying $z_0(0) = z_0$. Since W is C^1 and $\dot{W}(z_0) > 0$, there exists by the mean value theorem $\epsilon > 0$ and $\delta > 0$ such that for $0 \leqslant t \leqslant \epsilon$,

$$W[z_0(t)] - W(z_0) > \delta t,$$

and, in particular,

$$W[z_0(\epsilon)] - W(z_0) > \delta\epsilon. \tag{2.3}$$

Since $z(t_n) \to z_0$, and solutions are continuously dependent on initial values, given any $\eta > 0$, there exists $N(\eta)$ such that for $n \geqslant N(\eta)$,

$$\| z(t + t_n) - z_0(t)\| < \eta \qquad \text{for} \quad 0 \leqslant t \leqslant \epsilon.$$

And in particular,

$$\| z(\epsilon + t_n) - z_0(\epsilon)\| < \eta \qquad \text{for} \quad n \geqslant N(\eta).$$

The continuity of W guarantees that for η sufficiently small,

$$| W[z(\epsilon + t_n)] - W[z_0(\epsilon)]| < \delta\epsilon/2. \tag{2.4}$$

Inequalities (2.3) and (2.4) imply that $W[z(\epsilon + t_n)] > W(z_0)$, and this contradicts $W[z(t)] \leqslant W(z_0)$. Q.E.D.

We can now prove

THEOREM 2.1. *Let*

$$Q(z) = \begin{bmatrix} H_{11}(z) & \rho/2\, I_n \\ \rho/2\, I_n & -H_{22}(z) \end{bmatrix}$$

where $H_{11} = \partial F_2/\partial z_1$, $H_{22} = -\partial F_1/\partial z_2$, and I_n denotes the $n \times n$ identity matrix. If $F^T(z)\, Q(z)\, F(z) > 0$ for all z with $F(z) \neq 0$, and if the rest points

of (2.2) *are isolated, then given any* z_0 *such that* $\phi_t(z_0)$ *is bounded, there exists a rest point* $\bar{z}$, *which may depend on* z_0, *such that* $\lim_{t\to\infty} \phi_t(z_0) = \bar{z}$.

Proof. Let $\gamma^+ = \{z \in R^n / z = \phi_t(z_0)$ for some $t \geqslant 0\}$, and $\Omega(\gamma^+) = \{z \in R^n /$ there exists increasing sequence $\{t_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \phi_{t_n}(z_0) = z\}$.

Since γ^+ has compact closure on the domain of F, $\Omega(\gamma^+)$ is nonempty, compact and connected [20, Theorem 1.1, p. 145].

Let $W(z) = F_1^T(z) F_2(z)$. Then $\dot{W}(z) = \dot{F}_1^T(z) F_2(z) + F_1^T(z) \dot{F}_2(z) = \rho F_1^T(z) F_2(z) + F_1^T(z) H_{11}(z) F_1(z) - F_2^T(z) H_{22}(z) F_2(z) = F^T(z) Q(z) F(z) \geqslant 0$. By the previous lemma, if $\bar{z} \in \Omega(\gamma^+)$, then $F^T(\bar{z}) Q(\bar{z}) F(\bar{z}) = 0$, and, hence, $F(\bar{z}) = 0$; i.e., $\bar{z}$ is a rest point. Since the rest points are isolated and $\Omega(\gamma^+)$ is connected, $\Omega(\gamma^+) = \{\bar{z}\}$. Hence, $\lim_{t\to\infty} \phi_t(z_0) = \bar{z}$.

Remark 1. For MHDS derived from optimal growth problems, it would be useful to replace the condition "$\phi_t(z_0)$ is bounded" by "$\phi_t(z_0)$ is optimal," since it is possible to find models in which optimal paths are not bounded. One can, however, bound the Lyapunov function W by assuming regularity and concavity conditions on the so-called "value function." In fact, consider an optimal growth problem

$$\text{Max} \int_0^{\infty} e^{-\rho t} u[k(t), \dot{k}(t)]\, dt \tag{2.5}$$

given $k(0) = k_0$.

Here u: $R^{2n} \to R$ is usually assumed concave and C^2.

Let $k^*(t, k_0)$ be the optimal solution and $R(k_0) = \int_0^{\infty} e^{-\rho t} u[k^*(t, k_0), \dot{k}^*(t, k_0)]\, dt$; i.e., R is the value of the objective function along the optimal path (the value function).

If we assume that R is C^2, then one can show that $R'(k) = q$ where (q, k) solve the MHDS corresponding to (2.5). Benveniste and Scheinkman [47] provide a general set of conditions on u that imply that R' exists.

Hence, $\dot{q} = (d/dt)\, R'(k) = R''(k)\, \dot{k}$. The concavity of u implies that R is concave, and, hence, $R''(k)$ is seminegative definite. If one assumes that in fact $R''(k)$ is negative definite, then $k^T \dot{q} = k^T R''(k)\, \dot{k} < 0$ along any optimal path provided $k \neq 0$. Thus, $W = k^T \dot{q}$ is bounded above on optimal paths.

Remark 2. The Lyapunov function $V = z_1^T z_2 = (q - \bar{q})^T (k - \bar{k})$ amounts to

$$(q - \bar{q})^T (k - \bar{k}) = [R'(k) - R'(\bar{k})]^T (k - \bar{k}) \tag{i}$$

Since the value function $R(\cdot)$ is concave,

$$[R'(k) - R'(\bar{k})](k - \bar{k}) \leqslant 0 \tag{ii}$$

for all k. Inequality (ii) is well known for concave functions. It holds with strict inequality for strictly concave functions. Thus, it is natural to search for sufficient conditions on the Hamiltonian that imply V is increasing on trajectories, and the matrix Q plays an important role in such sufficient conditions.

3

A General Result on Convergence of Bounded Trajectories

In this section, we shall present a general theorem that will generate a result that is closely related to Hartman and Olech's basic theorem [21, p. 157, Theorem 2.3], our result of Theorem 3.1, and many other results, all as simple corollaries. Furthermore, the general theorem will be stated and proved in such a way as to highlight a general Lyapunov method that is especially useful for the stability analysis of optimal paths generated by optimal control problems arising in capital theory.

THEOREM 3.1. *Let $f: E \subset R^m \to R^m$ be C^2, E open and convex. Consider the differential equation system*

$$\dot{x} = f(x). \tag{3a.1}$$

Assume there is x such that $f(x) = 0$ (W.L.O.G. put $x = 0$) such that there is $V: E^m \to R$ satisfying:

(a) *for all $x \neq 0$, $x^T \nabla^2 V(0)[J(0)\, x] < 0$,*

(b) $\nabla V(0) = 0$,

(c) *for all $x \neq 0$, $[\nabla V(x)]^T f(x) = 0$ implies $x^T \nabla^2 V(x) f(x) = 0$,*

(d) *for all $x \neq 0$, $x^T \nabla^2 V(x) f(x) = 0$ implies $[\nabla V(x)]^T J(x)\, x < 0$.*

Then

(α) *$[\nabla V(x)]^T f(x) < 0$ for all $x \neq 0$,*

(β) *all trajectories that remain "bounded" (i.e., are contained in a compact set $k \subset E$) for $t \geqslant 0$ converge to 0.*

Proof. Let $x \neq 0$, and put

$$g(\lambda) \equiv [\nabla V(\lambda x)]^T f(\lambda x). \tag{3a.2}$$

We shall show that $g(1) < 0$ in order to obtain (α). We do this by showing that $g(0) = 0$, $g'(0) = 0$, $g''(0) < 0$, and $g(\bar{\lambda}) = 0$ implies $g'(\bar{\lambda}) < 0$ for $\bar{\lambda} > 0$. (At this point, the reader will do well to draw a graph of $g(\lambda)$ in

order to convince himself that the above statements imply $g(1) < 0$.) Calculating, we get

$$g'(\lambda) = x^T \nabla^2 V(\lambda x) f(\lambda x) + [\nabla V(\lambda x)]^T J(\lambda x)\, x, \tag{3a.3}$$

$$g''(\lambda) = x^T \left[\frac{d}{d\lambda} \nabla^2 V(\lambda x)\right] f(\lambda x) + x^T \nabla^2 V(\lambda x)[J(\lambda x)\, x] + x^T \nabla^2 V(\lambda x)[J(\lambda x)\, x] + [\nabla V(\lambda x)]^T \left[\frac{d}{d\lambda} J(\lambda x)\right] x. \tag{3a.4}$$

Now $\lambda = 0$ implies $f(\lambda x) = 0$, so $g(0) = 0$. Also $g'(0) = 0$ from $f(0) = 0$, and (b). Furthermore, $f(0) = 0$, (b) imply

$$g''(0) = 2x^T \nabla^2 V(0)[J(0)\, x] \tag{3a.5}$$

But this is negative by (a). By continuity of g'' in λ, it must be true that there is $\epsilon_0 > 0$ such that $g(\lambda) < 0$ for $\lambda \epsilon [0, \epsilon_0]$. Suppose now that there is $\lambda > 0$ such that $g(\lambda) = 0$. Then there must be a smallest $\bar{\lambda} > 0$ such that $g(\bar{\lambda}) = 0$. Also, $g'(\bar{\lambda}) \geqslant 0$. Let us calculate $g'(\bar{\lambda})$, show that $g'(\bar{\lambda}) < 0$, and get an immediate contradiction. From (3a.3),

$$g'(\bar{\lambda}) = x^T \nabla^2 V(\bar{\lambda} x) f(\bar{\lambda} x) + [\nabla V(\bar{\lambda} x)]^T J(\bar{\lambda} x)\, x. \tag{3a.6}$$

Now $g(\bar{\lambda}) = 0$ implies $[\nabla V(\bar{\lambda} x)]^T f(\bar{\lambda} x) = 0$. But this, in turn, implies that $\bar{\lambda} x^T \nabla^2 V(\bar{\lambda} x)\, f(\bar{\lambda} x) = 0$ by (c). Finally, (d) implies that $[\nabla V(\bar{\lambda} x)]^T \times J(\bar{\lambda} x)(\bar{\lambda} x) < 0$. Thus, $g'(\bar{\lambda}) < 0$, a contradiction to $g'(\bar{\lambda}) \geqslant 0$. Therefore,

$$[\nabla V(x)]^T f(x) < 0 \qquad \text{for all} \quad x \neq 0. \tag{3a.7}$$

By Lemma 2.1, all the rest points of $\phi_t(x_0)$ satisfy $[\nabla V(x)]^T f(x) = 0$, and, hence, $x = 0$ is the only candidate. But if $\phi_t(x_0)$ is bounded, $\phi_t(x_0)$ must have a limit point. Hence, $\lim_{t \to \infty} \phi_t(x_0) = 0$. Q.E.D.

Note that to get global asymptotic stability results for bounded trajectories, all one needs to do is find a V that is monotone on bounded trajectories and assume that $E_0 = \{x \mid [\nabla V(x)]^T f(x) = 0\} = \{0\}$. This result is important for global asymptotic stability analysis of optimal paths generated by control problems arising in capital theory. Also, Hartman–Olech [21] type results emerge as simple corollaries. Let us demonstrate the power of the theorem by extracting some corollaries.

COROLLARY 3.1. *Let $f: R^m \to R^m$. Consider the ordinary differential equations $\dot{x} = f(x)$, $f(0) = 0$. If $J(x) + J^T(x)$ is negative definite for each x, then 0 is globally asymptotically stable.*

Proof. Put $V = x^T x$. Then $\nabla V(x) = 2x\nabla^2 V(x) = 2I$ where I is the $n \times n$ identity matrix. Assumption (a) becomes

$$x^T J(0)\, x < 0 \qquad \text{for all} \quad x \neq 0.$$

But this follows because

$$2x^T J(0)x = x^T[J(0) + J^T(0)]\, x < 0.$$

Assumption (b) trivially holds since $\nabla V(x) = 2x$. Assumption (c) amounts to $2xf(x) = 0$ implies $x^T(2I) f(x) = 0$ which obviously holds. Assumption (d) obviously holds because $2x^T J(x)\, x < 0$ for *all* $x \neq 0$. Thus, all bounded trajectories converge to 0, as $t \to \infty$. It is easy to use $V = x^T x$ decreasing in t in order to show that all trajectories are bounded. This ends the proof.

The following corollary is a stronger result than Hartman and Olech [21] in one way and weaker in another.We will explain the difference in more detail below.

COROLLARY 3.2. (*Mas-Colell* [31]). *Consider* $\dot{x} = f(x)$, $f(0) = 0$. *Assume that* $x^T[J(0) + J^T(0)]\, x < 0$ *for all* $x \neq 0$, *and*

$$x^T f(x) = 0 \text{ implies } x^T[J(x) + J^T(x)]\, x < 0 \qquad \text{for all} \quad x \neq 0. \tag{3a.8}$$

Then 0 *is globally asymptotically stable.*

Proof. Let $V = x^T x$. We show that

$$\frac{dV}{dt} = 2x^T f(x) < 0 \qquad \text{for} \quad x \neq 0. \tag{3a.9}$$

Assumptions (a–d) of the theorem are trivially verified.[8] Therefore, $dV/dt < 0$, and the rest of the proof proceeds as in Corollary 3.1.

[8] The condition

$$x^T f(x) < 0 \qquad \text{for} \quad x \neq 0 \tag{i}$$

has a natural geometric interpretation when $\dot{x} = f(x)$ is a "gradient" flow; i.e., there is a "potential" F: $R^n \to R$ such that

$$\nabla F(x) = f(x) \tag{ii}$$

for all x. A function F: $R^n \to R$ is said to be pseudoconcave at x_0 if

$$F(x_0) > F(x_1) \qquad \text{implies} \quad \nabla F(x_1)^T(x_0 - x_1) > 0. \tag{iii}$$

But put $x_0 = 0$, assume that F is maximum at $x_0 = 0$, and note that (iii) is just

$$x_1^T f(x_1) < 0.$$

Thus, condition (i) amounts to pseudoconcavity of the potential F.

This type of result is reported in Hartman and Olech [21] and in Hartman's book [20]. In [20] and [21], 0 is assumed to be the only rest point, and it is assumed to be locally asymptotically stable. On the one hand, Mas-Colell puts the stronger assumption: $x[J(0) + J^T(0)]\, x < 0$ for $x \neq 0$ on the rest point. It is well known that negative real parts of the eigenvalues of $J(0)$ does not imply negative definiteness of $J(0) + J^T(0)$, but negative definiteness of $J(0) + J^T(0)$ does imply negative real parts for $J(0)$.

But on the other hand, Hartman and Olech [21] make the assumption: for all $x \neq 0$, $w^T f(x) = 0$ implies $w^T[J(x) + J^T(x)]\, w \leqslant 0$ for all vectors w. Note that Mas-Colell only assumes $x^T f(x) = 0$ implies $x^T[J(x) + J^T(x)]\, x < 0$. So he places the restriction on a much smaller set of w, but he requires the strong inequality. Furthermore, the proof of the Mas-Colell result is much simpler than that of Hartman and Olech.

It is possible to obtain general results of Hartman and Olech type from the theorem. For example,

COROLLARY 3.3. *Let G be a positive definite symmetric matrix, and let 0 be the unique rest point of $\dot{x} = f(x)$. Assume that*

$$x^T[GJ(0)]\, x < 0 \qquad \text{for all} \quad x \neq 0,$$

and

$$x^T G f(x) = 0 \text{ implies } x^T[GJ(x)]\, x < 0 \qquad \text{for all} \quad x \neq 0.$$

Then $x = 0$ is globally asymptotically stable for bounded trajectories.

Proof. Let $V(x) = x^T G x$. Then,

$$\nabla V(x) = 2Gx.$$

Also,

$$\nabla^2 V(x) = 2G.$$

The rest of the proof is now routine.

Corollary 3.3 is closely related to Hartman and Olech's [21, Theorem 2.3, p. 157] and to a theorem in Hartman's book [20, Theorem 1.4, p. 549]. Hartman and Olech also treat the case of G depending on x. We have not been able to obtain their result for nonconstant G as a special case of our theorem. Thus, their different methods of proof yield theorems that our methods presented in this section are unable to obtain. This leads us to believe that the original method of proof developed in [8] and outlined above is necessary for developing Hartman and Olech type generalizations

for nonconstant G for modified Hamiltonian dynamical systems. We turn now to an application of the general theorem to modified Hamiltonian dynamical systems.

Convergence of Bounded Trajectories of MHDS

In this section, we apply the results obtained in Section 3a to MHDS systems. We will assume that the MHDS has a singularity $(\bar{q}, \bar{k})$, and rewrite it as

$$\begin{aligned} \dot{z}_1 &= \rho(z_1 + \bar{q}) - H_2 z + (\bar{q}, \bar{k}) \equiv F_1(z), \qquad z \equiv (q, k) - (\bar{q}, \bar{k}), \\ \dot{z}_2 &= H_1(z + (\bar{q}, \bar{k})) \equiv F_2(z). \end{aligned} \tag{3b.1}$$

We may now state and prove

THEOREM 3.2. *Let*

$$Q(z) = \begin{bmatrix} H_{11}(z) & \rho/2\, I \\ \rho/2\, I & -H_{22}(z) \end{bmatrix} \tag{3b.2}$$

where I is the $n \times n$ identity matrix. Assume

(a) $0 = F(0)$ *is the unique rest point of* $\dot{z} = F(z)$,

(b) *for all* $z \neq 0$,

$$z_1^T F_2(z) + z_2^T F_1(z) = 0 \qquad \text{implies } z^T Q(z)\, z > 0, \tag{3b.3}$$

(c) *for all* $w \neq 0$, $w^T Q(0)\, w > 0$.

Then all trajectories that are bounded for $t \geqslant 0$ converge to 0 as $t \to \infty$.

Proof. Let $V = z^T A z$ where

$$A = -\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

where I is the $n \times n$ identity matrix. Note that $z^T A z = -2 z_1^T z_2$. Since $\nabla^2 V(0) = A + A^T = 2A$ and $(w^T A)^T (J(0)\, w) = -w^T Q(0)\, w$, we have that (c) implies (a) of Theorem 3.1. Also $[\nabla V(z)]^T = z^T(A + A^T) = 2 z^T A$, and, hence, $\nabla V(0) = 0$. Hence, (b) of Theorem 3.1 follows. Now (c) of Theorem 3.1 amounts to $[\nabla V(z)]^T F(z) \equiv 2 z^T A F(z) = 0$ implies $z^T \nabla^2 V(z)\, F(z) \equiv 2 z^T A F(z) = 0$, which is trivially true. Furthermore, (d) amounts to

$$2 z^T A F(z) = 0 \text{ implies } 2(z^T A)\, J(z)\, z > 0. \tag{3b.4}$$

But (3b.4) is identical to (3b.3), as an easy calculation will immediately show. Thus, $\dot{V} < 0$ except at the rest point 0. The rest of the proof is routine by now. Q.E.D.

The proof of Theorem 3.2 also yields the result that the Lyapunov function $z_1^T z_2 = (q - \bar{q})^T (k - \bar{k})$ is monotonically increasing along trajectories. Theorem 3.2 is a "local" sufficient condition for the hypothesis of the Cass and Shell stability theorem to hold. This is so because Theorem 3.2 gives conditions for the trajectory derivative of $V = z_1^T z_2$ to be positive for all $z \neq 0$, and that is the Cass and Shell[4] hypothesis.

Sufficient conditions (of local form such as our Q condition of Theorem 3.3) for positive trajectory derivative are useful for computations. Applications and economic interpretations of "local" conditions for stability to the adjustment cost literature are discussed in [10], which will be appearing in the Vienna 1974 Conference volume.

There is a neat sufficient condition for the positive definiteness of Q. If H is convexo-concave, the matrix Q is clearly positive semidefinite for $\rho = 0$. Furthermore, if the minimum eigenvalue of H_{11} is larger than α and the maximum eigenvalue of H_{22} is less than $-\beta$ where $\alpha\beta > \rho^2/4$, then Q is positive definite (cf. Footnote 1, Section 1). The hypothesis $\alpha\beta > \rho^2/4$ is the basic curvature assumption in Rockafellar's [38] analysis.

The positive definiteness of Q is also related to the Burmeister and Turnovsky [13] regularity condition.[5] Let $[\bar{q}(\rho), \bar{k}(\rho)]$ be the steady state associated with ρ. It solves

$$\begin{aligned} 0 &= \rho q - H_2(q, k) \\ 0 &= H_1(q, k). \end{aligned} \tag{i}$$

[4] It is worth pointing out here that the hypothesis

$$\dot{V}(z) > 0 \quad \text{for} \quad z \neq 0 \tag{a}$$

has a geometric interpretation for the case $\rho = 0$. Viz., it implies pseudoconvexity, pseudoconcavity at $(\bar{q}, \bar{k})$ of $H(q, k)$ in q, k respectively. For the special case

$$H(q, k) = f_1(q) + f_2(k),$$

inequality (a) is equivalent to pseudoconcavity at $(\bar{q}, \bar{k})$ of

$$G(q, k) = -f_1(q) + f_2(k).$$

These statements may be easily checked by referring to the definition of pseudoconcavity in Mangasarian [29, p. 147].

[5] We thank Ed Burmeister for this observation.

Differentiate (i) w.r.t. ρ to obtain

$$0 = \bar{q} + \rho\bar{q}' - H_{21}\bar{q}' - H_{22}\bar{k}',$$

$$0 = H_{11}\bar{q}' + H_{12}\bar{k}',$$

where

$$(\bar{q}', \bar{k}') = \left(\frac{d\bar{q}}{d\rho}, \frac{d\bar{k}}{d\rho}\right).$$

Rewrite this as

$$-\begin{pmatrix}\bar{q}\\ 0\end{pmatrix} = \begin{bmatrix}\rho - H_{21} & -H_{22}\\ H_{11} & H_{12}\end{bmatrix}\begin{pmatrix}\bar{q}'\\ \bar{k}'\end{pmatrix}.$$

Multiply both sides of this by the row vector $(\bar{k}', \bar{q}')$ to get

$$-\bar{q}^T\bar{k}' = (\bar{k}', \bar{q}')\begin{bmatrix}\rho - H_{21} & -H_{22}\\ H_{11} & H_{12}\end{bmatrix}\begin{pmatrix}\bar{q}'\\ \bar{k}'\end{pmatrix}$$

$$= (\bar{q}', \bar{k}')^T Q[\bar{q}(\rho), \bar{k}(\rho)](\bar{q}', \bar{k}').$$

But the quantity

$$\Theta = \bar{q}^T\bar{k}'$$

is the Burmeister–Turnovsky [13] "regularity" quantity. Burmeister and Turnovsky use the quantity Θ as an aggregate measure of capital deepening response. Thus, the positive definiteness of Q in the directions $(\bar{q}', \bar{k}')$ as ρ varies is equivalent to capital deepening response in the Burmeister–Turnovsky sense for each value of ρ.

4. A More General Result

In this section, we present a result, Theorem 4.2 below, on convergence of bounded trajectories of MHDS that is related to a theorem of Hartman and Olech [21, p. 549]. We start by presenting a sketch of the proof of a result, Theorem 4.1, that is, in fact, almost contained in Theorem 3.2. The method of proof, however, is the same as the one used in [8] to prove the more general result of Theorem 4.2. The reader is referred to [8] for a complete proof. Furthermore, Corollary 4.1 below gives us a nice geometric interpretation of our hypothesis in terms of quasi-convexity and quasi-concavity of the Hamiltonian function.

A Sketch of the Proof of the Hartman–Olech Type of Result for MHDS

Let us reconsider the system (3b.1),

$$\dot{z}_1 = F_1(z),$$

$$\dot{z}_2 = F_2(z).$$

Put

$$F(z) = [F_1(z), F_2(z)] \tag{4.1}$$

Assume $F(0) = 0$, $F(z) \neq 0$ for $z \neq 0$, $F \in C^1$. Let $\phi_t(z_0)$ be the solution of (4.1) given z_0; let W_s be the stable manifold of (4.1), i.e., $W_s = \{\bar{z}_0 \mid \phi_t(\bar{z}_0) \to 0, t \to \infty\}$; and let Λ be the bounded manifold of (4.1), i.e., $\Lambda = \{\bar{z}_0 \mid$ there is $M > 0$ such that for all $t \geqslant 0 \| \phi_t(\bar{z}_0)\| \leqslant M\}$. Assume that for each $z_{20} \in R^n$ there is a unique z_{10} such that $(z_{10}, z_{20}) \in \Lambda$. Write

$$z_{10} = g(z_{20}) \tag{4.2}$$

for this functional relation, and assume that g is differentiable.[6] (We can get by with less, but why mess up the picture with generality at this stage?) We are after sufficient conditions to guarantee that $\phi_t[g(z_{20}), z_{20}] \to 0$, $t \to \infty$ for all $z_{20} \in R^n$.

We could just apply the Hartman–Olech result to the "reduced form"

$$\dot{z}_2 = F_2[g(z_2), z_2], \tag{4.3}$$

but this requires knowledge of $\partial g/\partial z_2$. In most problems, not much is known about g other than its existence and differentiability and other general properties. In some problems g is badly behaved, but we shall ignore those here. Thus, we shall formulate a sufficient condition involving $\partial F/\partial z$, alone.

Let us proceed in a way that uncovers a "natural" set of sufficient conditions for the global asymptotic stability of (4.1) on bounded trajectories. Assume the solution $z_2 = 0_2$ is locally asymptotically stable for (4.3). (Here $F_2[g(0_2), 0_2] = 0_2$.) This means that there is an open neighborhood $N_2(0_2) \subseteq R^n$ such that $z_{20} \in N_2$ implies $\phi_t[g(z_{20}), z_{20}] \to 0$, $t \to \infty$. Let $A_2(0_2) = \{z_{20} \mid \phi_t[g(z_{20}), z_{20}] \to 0, t \to \infty\}$. If $A_2(0_2)$ is the whole of R^n, we have global asymptotic stability. So suppose that z_{20} is in the boundary of $A_2(0_2)$. Let $u_2 \in R^n$ have unit norm. Consider the vector $z_0(p, u_2) = [g(z_{20} + pu_2), z_{20} + pu_2]$, $p \in [0, \beta]$. Put $y(t, p) = \phi_t[z_0(p, u_2)]$. Let $y_p(t, p) = \partial\phi_t[z_0(p, u_2)]/\partial p$. Consider the following differential equation (let us drop transpose notation except that needed for clarity).

$$T_p = \frac{dT}{dp} = -\frac{y_{1p}F_2[y(T, p)] + y_{2p}F_1[y(T, p)]}{2F_1[y(T, p)]\,F_2[y(T, p)]} = h(T, p),$$
$$T(0, q) = q. \tag{4.4}$$

[6] The function g is the derivative of the value function. If the value is concave, then g will be differentiable almost everywhere [24, p. 405].

Let $y[T(p, q), p] \equiv x(q, p)$ be a solution (if a solution exists), and denote $(\partial/\partial p)\, x(q, p)$ by x_p. Note that

$$x_p = \frac{\partial y}{\partial T}\, T_p + y_p = F[x(q, p)]\, T_p + y_p[T(p, q), p]. \tag{4.5}$$

Also note that

$$x_{1p} F_2[x(q, p)] + x_{2p} F_1[x(q, p)] = 0. \tag{4.6}$$

The latter follows from

$$\begin{aligned}(F_1 T_p + y_{1p})\, F_2 + (F_2 T_p + y_{2p})\, F_1 \\ = 2T_p F_1 F_2 + y_{1p} F_2 + y_{2p} F_1 = 0.\end{aligned} \tag{4.7}$$

But (4.7) is identical to (4.4).

We will call the system of trajectories satisfying (4.6) *Hamiltonian orthogonal* trajectories. They are not the same kind of orthogonal trajectories as in the original Hartman–Olech result. We shall see, however, that it is natural to construct trajectories of type (4.6) for our type of problem.

Consider

$$w(q, p) \equiv x_{1p}(q, p) \cdot x_{2p}(q, p). \tag{4.8}$$

The object w is natural to look at in light of our previous results and the Hartman–Olech technique.

$$\begin{aligned}\frac{\partial w}{\partial q} &= T_{pq}[F_1 x_{2p} + F_2 x_{1p}] + T_q x_p Q x_p \\ &= T_q x_p Q x_p .\end{aligned} \tag{4.9}$$

The last follows from the definition of Hamiltonian orthogonal trajectories, Eq. (4.6). The reader will recall

$$\begin{aligned}Q(x) &\equiv \begin{bmatrix} H_{11}(x) & \rho/2\, I_n \\ \rho/2\, I_n & -H_{22}(x) \end{bmatrix}, \qquad x = x(q, p), \\ x_q &= F(x)\, T_q, \qquad x_{pq} = T_{pq} F(x) + T_q J(x)\, x_p, \qquad T_q > 0.\end{aligned} \tag{4.10}$$

Notice here that both the "Lyapunov" function (4.8) and the method of constructing the "transverse" trajectories (4.4) are different from Hartman–Olech. The method of proof is also different.

A theorem may now be stated.

THEOREM 4.1. *Assume that* (a) $F \in C^1$, (b) $F(0) = 0$, $F(z) \neq 0$ *for* $z \neq 0$, (c) $z_2 = 0_2$ *is a locally asymptotically stable solution of*

$\dot{z}_2 = F_2[g(z_2), z_2]$, *and* (d) *for all bounded trajectories* $z(t)$, *for all vectors* $c \in R^{2n}$ *such that* $\| c \| = 1$, $c = (c_1, c_2)$, $c_1 = J_g c_2$ *where* $J_g = g'(z_2)$, $c_1 F_2[z(t)] + c_2 F_1[z(t)] = 0$ *hold, we have*

$$c^T Q[z(t)]\, c > 0. \tag{4.11}$$

Then $z_2 = 0_2$ *is GAS for* $\dot{z}_2 = F_2[g(z_2), z_2]$. I.e., *all bounded trajectories of* $\dot{z} = f(z)$ *converge to* 0 *as* $t \to \infty$.

The reader should notice that although in the hypothesis (d) of the theorem we used information on the Jacobian matrix of g, we could have stated the stronger hypothesis that for all $z \in R^{2n}$, for all $c \in R^{2n}$ with $\| c \| = 1$, $c_1^T F_2[z(t)] + c_2^T F_1[z(t)] = 0$ implies $c^T Q[z(t)]\, c > 0$, which does not use any properties of J_g. The reason Theorem 4.1 is stated in the form above is that one can obtain the following corollary.

COROLLARY 4.1. *Given an MHDS like* (3b.1) *under hypotheses* (a)–(c) *of Theorem* 4.1, *suppose* J_g *is symmetric and that the Hamiltonian function satisfies for all* $z = (g(z_2), z_2)$

(i) $c_1^T H_{11}(z)\, c_1 < -\alpha \mid c_1 \mid^2$ *for all* $c_1 \neq 0$ *such that* $c_1^T H_1(z) = 0$,

(ii) $c_2^T[-H_{22}(z)]\, c_2 > \beta \mid c_2 \mid^2$ *for all* $c_2 \neq 0$ *such that* $c_2^T[H_2(z) - \rho(z_1 + \bar{q})] = 0$ *for some* $(\alpha, \beta) \in R^2$ *with* $\alpha\beta \geqslant \rho^2/4$.

Then global asymptotic stability of bounded trajectories holds.

Proof. Since in Theorem 4.1, $z(t)$ is a bounded trajectory and (4.2) holds, we have $F_1(z) \equiv \rho(z_1 + \bar{q}) - H_2(z + (\bar{q}, \bar{k})) = J_g F_2(z) = J_g H_1(z + (\bar{q}, \bar{k}))$. Since $c_1 = J_g c_2$, $c_1^T F_2[z(t)] + c_2^T F_1[z(t)] = 0$ iff $(J_g c_2)^T F_2 + c_2^T J_g F_2 = 0$. Hence, (d) of Theorem 4.1 holds iff $c_2^T F_1 = c_1^T F_2 = 0$. By (i) and (ii), $c^T Q[z(t)]\, c > 0$ for all c satisfying (d) of Theorem 4.1. Hence, GAS must hold.

Remark 3. In Corollary 4.1, only the fact that J_g is symmetric was used. As in Remark 1 of Section 2, if a value function R exists and is C^2, $J_g = R''$, and, hence, symmetric.

Remark 4. For $\rho = 0$, (i) and (ii) can be interpreted as quasiconcavity of the Hamiltonian function H. (The Hamiltonian is in fact, always a convex function of q.) Therefore, the equivalent bordered matrix conditions so beloved by economists may be written down in place of (i) and (ii). This generalizes the result of Rockafellar [36] on the GAS of convexo-concave Hamiltonians to the quasi-convex-quasi-concave case.

Remark 5. Given a function F: $R^n \to R$ we will say that F is α-quasi-convex at x if for any $c \neq 0$, $c^T Df(x) = 0$ implies $c^T D^2 f(x)\, c > \alpha \mid c \mid^2$. Define α-quasi-concavity in the obvious way. Inequality (i) of Corollary 4.1

simply says that the Hamiltonian is α-quasi-convex in z_1. Let $\tilde{H}(z) = H(z + (\bar{q}, \bar{k})) - \rho(z_1 + \bar{q})^T z_2$. Then (ii) says that $\tilde{H}$ is β-quasi-concave in z_2. Note that ρ introduces a "distortion" that vanishes at $\rho = 0$.

The general result of [8] may now be stated. Let G: $R^{2n} \to M[R^{2n}, R^{2n}]$ (the set of all $2n \times 2n$ real matrices) be such that $G(z)$ is positive definite for all z. In what follows, $G(z)$ will be assumed to be continuously differentiable. Let

$$j \equiv \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix. Put $\alpha(z) \equiv jG + (jG)^T$. Let $D(z) \equiv [(\alpha(z)\, J(z))^T + (\alpha(z)\, J(z))] + \alpha'(z)$ where $\alpha'(z) \equiv \sum_{r=1}^{2n} (\partial\alpha/\partial z_r) F_r$, the "trajectory" derivative of the matrix $\alpha(z)$. Now consider the system

$$\dot{z} = F(z). \tag{4.12}$$

We may now state the following theorem.

THEOREM 4.2. *Suppose that* (4.12) *obeys assumptions*

(a) 0 *is the unique rest point of* (4.12),

(b) 0 *is LAS in the sense that the linearization of* $\dot{z}_2 = F_2[g(z_2), z_2]$ *at* $z_2 = 0$ *has all eigenvalues with negative real parts,*

(c) *Let* K *be a compact subset of the bounded manifold. Then,* $\bigcup_{z \in K} \bigcup_{t \geq 0} \phi_t(z)$ *is bounded, and, furthermore, the following basic property holds.*

ASSUMPTION 1. *For all* $w \in R^{2n}$, $w \neq 0$ *for all* $z \neq 0$, *we have*

$$w^T \alpha(z) F(z) = 0 \tag{4.13}$$

implies

$$w^T D(z)\, w > 0. \tag{4.14}$$

Then all trajectories in the bounded manifold converge to 0 *as* $t \to \infty$.

The proof of this theorem is long and involved, and is done in a sequence of lemmas. See [8] for the proof.

Hartman and Olech's basic result [21, Theorem 14.1] is closely related to our theorem. To see this, put $\alpha(z) = -2G(z)$. Assumption 1 then becomes Hartman and Olech's (H–0) assumption.

Our theorem would not be interesting if all it did was restate Hartman and Olech. It's interest lies in applicability to systems where the stable manifold $W(W \equiv \{z_0 \mid \phi_t(z_0) \to 0, t \to \infty\})$ *is not all of* R^{2n}. In particular, the important special case of MHDS that we are interested in generates systems where W is n-dimensional.

For example, let $G(z) \equiv I_{2n}$, the $2n \times 2n$ identity matrix. Assumption 1 becomes

$$w^T jF(z) = 0 \text{ implies } w^T[jJ(z) + j^T(z)j]\, w > 0. \tag{4.15}$$

Note that

$$w^T[jJ(z) + J^T(z)j]\, w = 2w^T Q(z)\, w \tag{4.16}$$

where

$$Q(z) = \begin{bmatrix} H_{11}(z) & \rho/2\, I_n \\ \rho/2\, I_n & -H_{22}(z) \end{bmatrix}. \tag{4.17}$$

5. Suggestions For Future Research

We have made in this paper only small progress toward providing a comprehensive analysis of GAS of optimal controls generated by MHDS's. Several topics for future research follow.

1. A more general notion of long-run behavior needs to be formulated. There is really no economic reason to rule out limit cycles, for example. It is, therefore, necessary to build a theory that allows more general limit sets than rest points, and find sufficient conditions on preferences and technology for a "minimal" limit set to be stable in some sense.

2. Both our results and the CSR results are "small ρ" theorems. I.e., the sufficient conditions for GAS are most likely to hold when ρ is small. But the Cass–Koopmans' one good model is GAS independently of the size of ρ. Intuition suggests that if we perturb such an economy slightly that it will still be GAS. This suggests development of a notion of block dominant diagonal for MHDS's to parallel the development of dominant diagonal notions in the study of the Walrasian tâtonnement [34]. An initial rather unsuccessful attempt to do this is reported in [9]. Perhaps a more fruitful approach will build on Pearce's notion of block dominance.

3. It is natural to extend our results and the CSR results to uncertainty. Some results in this area are reported in [7], but much more remains to be done. There is nothing done in the area of extending this work to continuous time stochastic processes,[7] in the nonlinear case.

[7] Magill [45], of Indiana University's Department of Economics, has obtained results for linear quadratic stochastic optimal control problems.

4. One of the original motivations for introducing positive definite matrices, $G(x)$, into the Hartman–Olech framework was that this is needed to obtain their result. Consider the differential equations

$$\dot{x} = f(x), f(0) = 0.$$

Let

$$\Gamma(x) = \underset{1 \leqslant i < j \leqslant n}{\text{maximum}} \{\lambda_i(x) + \lambda_j(x)\}$$

where $\lambda_r(x)$ is an eigenvalue of the symmetric matrix $(J^T(x) + J(x))/2$.

THEOREM (Hartman and Olech [21, p. 549]). *If $\Gamma(x) \leqslant 0$ for all $x \neq 0$ and 0 is LAS, then 0 is GAS.*

See page 549 of Hartman [20] in order to be convinced that matrices of the form $G(x) = p(x)\, I_n$, $p(x) > 0$ for all x, p: $R^n \to R$ must be introduced into the basic Hartman and Olech method in order to obtain the above theorem. The above theorem is important because Γ is an easy quantity to interpret.

Some analog of the above theorem may exist for MHDS, but we have been unable to obtain it.

5. Burmeister and Graham [12] have exhibited a class of models where GAS obtains under conditions not sensitive to the size of ρ. Furthermore, their GAS models do not satisfy either the CSR hypotheses or our hypotheses. Therefore, a general stability hypothesis that covers the one good model, the Burmeister–Graham models and the CSR–Brock–Scheinkman models, remains to be developed. Further evidence to support this proposition is the Ryder–Heal [39] experience. Some of their GAS results are not dependent on the size of ρ. Recent results reported in [6] on adjustment cost models (see [10] for references to the adjustment cost literature) indicate the existence of a large class of models where the CSR–Brock–Scheinkman "small ρ" conditions do not hold, but GAS does hold. This class is basically the class where the Lyapunov function $V = k^T H_{qq}^{-1} k$ together with the concavity of the value function yields $\dot{V} < 0$. The reader is referred to [6] for details.

6. Araujo and Scheinkman [1] obtain conditions for GAS for discrete time models that are independent of the discount rate, except around the steady state.

These results seem to indicate that there exist many other interesting stability theorems to be developed.

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A Growth Property in Concave–Convex Hamiltonian Systems

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The analysis of the stability of Hamiltonian dynamical systems in various economic models depends on the "curvature" of the Hamiltonian function at a rest point of the system, or equivalently, on growth properties involving gradients or subgradients. The purpose of this note is to establish a general property pointing in particular to a simplification of conditions assumed by Cass and Shell [1].

In this context, a Hamiltonian H: $R^n \times R^n \to [-\infty, +\infty]$ is an extended-real-valued function such that $H(k, Q)$ is concave in k and convex in Q. The dynamical system of interest in the case of a constant discount rate ρ is

$$\dot{k} \in \partial_Q H(k, Q), \quad -\dot{Q} + \rho Q \in \partial_k H(k, Q) \tag{1}$$

where $\partial_Q H$ and $\partial_k H$ are the subdifferentials [2] with respect to Q and k (or if differentiability is present, the gradients), and a rest point is a pair, (k^*, Q^*) satisfying

$$0 \in \partial_Q H(k^*, Q^*), \quad \rho Q^* \in \partial_k H(k^*, Q^*). \tag{2}$$

Particular attention is directed to determining the existence of solutions to (1) satisfying

$$k(0) = \bar{k} \qquad \text{and} \qquad \lim_{t \to \infty} e^{-\rho t}(k(t) - k^*) \cdot (Q(t) - Q^*) = 0, \tag{3}$$

and whether such a solution is stable in the sense of converging to (k^*, Q^*) as $t \to +\infty$.

The author in [3] and [4] developed results on existence and stability on the basis of assuming H was strictly or strongly convex–concave in a neighborhood of (2). Cass and Shell [1] showed under different assumptions, to be considered below, that (1) and (3) imply convergence of

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$k(t)$ to k^*. Brock and Scheinkman [5] obtained convergence of $(k(t), Q(t))$ to (k^*, Q^*) under more differentiability but less convexity. Gaines [6] proved the existence of solutions to (1), (3), using a different approach and different growth assumptions on H than the author.

In the paper of Cass and Shell, a certain global growth property of H is implicit in the conditions they impose on the underlying technology. For instance, it is a consequence of their model that solutions to (1) satisfy a universal bound $\| k(t)\| \leq B$. We wish to discuss not this aspect of their work, but their conditions involving the function

$$\Phi(k, Q) = \inf\{-R \cdot (k - k^*) + z \cdot (Q - Q^*) + \rho Q^* \cdot (k - k^*) | \times R \in \partial_k H(k, Q), z \in \partial_Q H(k, 0)\}. \tag{4}$$

(where $\inf \varnothing = +\infty$). Clearly $\Phi(k^*, Q^*) = 0$. It is known that Φ is everywhere nonnegative by virtue of the concavity–convexity of H (see below).

The crucial condition invoked by Cass and Shell can be stated as follows:

(S) For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\| k - k^* \| \geq \epsilon \Rightarrow \Phi(k, Q) + \rho(k - k^*) \cdot (Q - Q^*) > \delta.$$

We shall establish that (S) can be expressed equivalently in the apparently weaker forms in Theorem 1 below. For this we need the harmless technical assumption that H is *closed* in the sense of [2, Section 34]. (This is always true for H arising from an economic model. A direct *sufficient* condition is that $H < +\infty$ everywhere and $H(k, Q)$ is upper semicontinuous in k.)

THEOREM 1. *Suppose H is finite on a neighborhood of (k^*, Q^*) and closed.*

(a) *If $\rho = 0$,* (S) *holds if merely $\Phi(k, Q) > 0$ for all $(k, Q) \neq (k^*, Q^*)$.*

(b) *If $\rho \neq 0$,* (S) *holds if for every (k', Q') with $k' \cdot Q' \leq 0$ the function*

$$\varphi(t) = \Phi(k^* + tk', Q^* + tQ') + \rho t^2 k' \cdot Q'$$

satisfies $\varphi(t) > 0$ for all $t > 0$ and

$$\liminf_{t \to \infty} \varphi(t) > 0.$$

Theorem 1 will be derived from a more general result. Consider an arbitrary multifunction $A: R^N \to R^N$ and any pair x^*, y^* such that $y^* \in A(x^*)$. Define

$$\Phi(x) = \inf\{(x - x^*) \cdot (y - y^*) | y \in A(x)\}. \tag{5}$$

The case above corresponds to $x = (k, Q) \in R^n \times R^n$,

$$A(k, Q) = \{(-R, z) \mid R \in \partial_k H(k, Q),\ z \in \partial_Q H(k, Q)\}, \tag{6}$$

$$x^* = (k^*, Q^*),\ y^* = (\rho Q^*, 0). \tag{7}$$

One says in general that A is *monotone* if

$$y_i \in A(x_i) \qquad \text{for } i = 0, 1, \text{ implies } (x_1 - x_0) \cdot (y_1 - y_0) \geqslant 0. \tag{8}$$

It is *maximal monotone* if it is monotone and its graph

$$G(A) = \{(x, y) \in R^N \times R^N \mid y \in A(x)\}$$

is not properly contained in the graph of any other monotone $A' : R^N \to R^N$. The *effective domain* of A is

$$D(A) = \{x \mid A(x) \neq \varnothing\}.$$

The connection with monotonicity and the present context is the following.

THEOREM 2 [7]. *If A is given by* (6) *for a closed concave–convex function H which is finite on a neighborhood of a point (k^*, Q^*), then A is maximal monotone and $(k^*, Q^*) \in \operatorname{int} D(A)$.*

THEOREM 3. *Let $A : R^N \to R^N$ be an arbitrary maximal monotone multifunction and let Φ be defined by* (5) *for any x^* and y^* satisfying $x^* \in \operatorname{int} D(A)$ and $y^* \in A(x^*)$. Then the* inf *in* (5) *is always attained, and Φ is an everywhere lower semicontinuous function with*

$$\Phi(x) \geqslant \Phi(x^*) = 0 \qquad \textit{for all } x.$$

Moreover, the expression $\Phi(x^* + sx')/s$ is for any x' nondecreasing as a function of $s > 0$.

In particular, the function

$$\theta(s) = (1/s) \min\{\Phi(x) \mid \| x - x^* \| = s\}, \qquad s > 0.$$

is nonnegative, lower semicontinuous, nondecreasing, and

$$\Phi(x) \geqslant \theta(\| x - x^* \|) \| x - x^* \| \qquad \textit{for all } x \neq x^*. \tag{10}$$

Proof. Consider two values $s_1 > s_2 > 0$ and any $y_i \in A(x^* + s_i x')$, $i = 1, 2$. The monotonicity of A implies

$$\begin{aligned} 0 &\leqslant [(x^* + s_1 x') - (x^* + s_2 x')] \cdot (y_1 - y_2) \\ &= (s_1 - s_2)\, x' \cdot (y_1 - y_2) \\ &= (s_1 - s_2)[s_1^{-1}((x^* + s_1 x') - x^*) \cdot (y_1 - y^*) \\ &\quad - s_2^{-1}((x^* + s_2 x') - x^*) \cdot (y_2 - y^*), \end{aligned}$$

and consequently, since the y_i are arbitrary,

$$0 \leqslant (s_1 - s_2)[s_1^{-1}\Phi(x^* + s_1x') - s_2^{-1}\Phi(x^* + s_2x')].$$

Thus $s^{-1}\Phi(x^* + sx')$ is nondecreasing in $s > 0$ as claimed.

The maximal monotonicity of A is known [8] to imply that int $D(A)$ is convex and (since int $D(A)$ is nonempty by hypothesis),

$$D(A) \subset \text{cl int } D(A). \tag{11}$$

Furthermore A is compact-valued and upper semicontinuous on int $D(A)$ (cf. [8]). Therefore, Φ is lower semicontinuous on int $D(A)$ and the inf in (5) is attained there.

Consider now a noninterior point $\bar{x}$ of $D(A)$. By (11) and the convexity of int $D(A)$ we have $x^* + s(\bar{x} - x^*) \in \text{int } D(A)$ for $0 \leqslant s < 1$. In view of this and the monotonicity of $\Phi(x^* + sx')/s$ in $s > 0$ for all x', the global lower semicontinuity of Φ will follow if we show that the restriction of Φ to the line

$$L = \{x^* + \lambda(\bar{x} - x^*) \mid -\infty < \lambda < +\infty\}$$

is lower semicontinuous at $\bar{x}$. Let

$$M = \{u \mid u(\bar{x} - x^*) = 0\},$$

i.e., M is the $N - 1$ dimensional subspace of R^N orthogonal to L. Define

$$\begin{aligned} A_0(x) &= M \qquad \text{if } x \in L \\ &= \varnothing \qquad \text{if } x \notin L \\ A_1(x) = (A + A_0)(x) &= A_0(x) + M \qquad \text{if } x \in L \\ &= \varnothing \qquad \text{if } x \notin L. \end{aligned}$$

The multifunction A_0 is trivially maximal monotone. Therefore, A_1 is maximal monotone, because $A_1 = A + A_0$ and $D(A_0) \cap \text{int } D(A) \neq \varnothing$ [9]. In particular $A_1(x)$ is closed for each $x \in L$. Note that since M is $N - 1$ dimensional, $A_1(x)$ actually has a very simple structure:

$$A_1(x) = (A(x) \cap L) + M \qquad \text{for } x \in L$$

(equivalent to a *one-dimensional* maximal monotone multifunction). Also $y^* \in A_1(x^*)$, and the function

$$\Phi_1(x) = \inf\{(x - x^*) \cdot (y - y^*) \mid y \in A_1(x)\} \tag{12}$$

coincides on L with $\Phi(x)$. One sees easily from the "one-dimensional" nature of A_1 that, relative to L, Φ_1 is lower semicontinuous and the inf in (12) is always attained. Therefore, Φ is lower semicontinuous relative to L and (inasmuch as $x - x^* \perp M$ for $x \in L$) the inf in (5) is attained for all $x \in L$. This finishes the argument that Φ is globally lower semicontinuous.

Proof of Theorem 1. (a) is obvious from (10) and the lower semicontinuity of Φ: if $\Phi(x) > 0$ for all $x \neq x^*$, then $\theta(s) > 0$ for all $s > 0$. For (b), we make use merely of the lower semicontinuity of

$$\Psi(k', Q') = \Phi(k^* + k', Q^* + Q') + \rho k' \cdot Q'.$$

We have $\Psi(k', Q')$ positive by hypothesis except at the origin, where it vanishes. The hypothesis that

$$\liminf_{t \to \infty} \Psi(tk', tQ') > 0$$

implies by a simple compactness argument that actually for some $r > 0$ and $\delta > 0$

$$\Psi(k', Q') \leqslant \delta \Rightarrow \|(k', Q')\| \leqslant r.$$

Therefore, for any $\epsilon > 0$,

$$\inf\{\Psi(k', Q') | \, \| k' \| \geqslant \epsilon\} \geqslant \min\{\delta, \delta_\epsilon\} > 0$$

where

$$\delta_\epsilon = \min\{\Psi(k', Q') | \, \| k', Q' \| \leqslant r, \| k' \| \geqslant \epsilon\}.$$

This is the desired conclusion.

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Homothetic Preferences and Community Excess Demand Functions*

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It has recently been shown that microeconomic theory imposes almost no restriction on community excess demand functions besides Walras' Law, if the economy contains no more commodities than consumers.

It is shown that the same result is true if the preferences of the consumers are further restricted by the seemingly strong requirement that they be homothetic, even when the distribution of initial endowments is a set of independent points in commodity space fixed in advance except for a scaling factor which is common to all consumers.

In a series of recent papers [2–5] it has been shown that microeconomic theory imposes almost no restriction on community excess demand functions besides Walras' Law, if the economy contains no more commodities than consumers.

In the present note it will be shown that the same result is true if the preferences of the consumers are further restricted by the seemingly strong requirement that they be homothetic, even when the distribution of initial endowments is a set of independent points in commodity space fixed in advance except for a scaling factor which is common to all consumers.

The economy will be described by l commodities; R^l is the commodity space. The excess demand function f: $P \to R^l$ will be assumed to be given, where P is a compact convex subset of $H = \{p \in R^l : s \cdot p = 1\}$, the set of normalized price vectors. The vector s_h represents the hth unit vector,

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for $h = 1,\ldots, l$, and $s = \sum_h s_h$. There are l consumers, indexed by i; each consumer has homothetic preferences described by a log-homogeneous, strictly concave utility function $u^i: X \to R$ where X is his consumption set, so that $u^i(\lambda x) = u^i(x) + \log \lambda$ for all $x \in X$ and real $\lambda > 0$. The ith consumer has initial endowments kw_i, where w_i is known in advance, whereas the positive constant k, common to all consumers, will be determined later on.

Let $P^* = \{x \in R^l: p \cdot x \geq 0 \text{ for all } p \in P\}$ be the (positive) polar of the set P. If P is compact and convex, and if $0 \notin P$, then its convex cone hull is pointed, so that P^* has a nonempty interior. It is easy to see that also int $P^* \cap \Omega$, where $\Omega = R_+^l$, is nonempty, since $s \in P^*$.

The following proposition will be shown to be true:

THEOREM. *Let $P \subset H$ be compact, convex. Let $f: P \to R^l$ have second-order partial derivatives which are uniformly bounded on P, and let $p \cdot f(p) = 0$ for all $p \in P$. Let $w_i \in P^*$, $i = 1,\ldots, l$ be independent vectors. Then there exists a real $k > 0$, a convex cone $X \subset P^*$, and l unsatiated consumers with strictly concave, homogeneous utility functions $u^i: X \to R$ and initial endowments kw_i whose excess demand functions add up to f on P.*

Remark 1. Normally one would require that also all $w_i \in \Omega$. In particular, it is possible to specify that $|w_i - w_j| \leq \epsilon$ for all i, j, for any preassigned $\epsilon > 0$.

Proof. Take any l independent vectors $a_i \in H \cap \text{int } \Omega$. Define the $l \times l$ matrices $W = (w_1,\ldots, w_l)$; $A = (a_1,\ldots, a_l)$; $B = AW'$; and $M = (W')^{-1}$, where the prime indicates transposition, and all vectors are assumed to be columns. Note that $A's = s$. Let $X = \{x \in R^l \mid x = B'y \text{ for some } y \in \Omega\}$.

Define the ith indirect utility function $v^i: P \to R$ by

$$v^i(p) = (1/k)\, m_i \cdot f(p) - a_i \cdot \log(Bp), \tag{1}$$

where the log of a vector means the vector of the logs of its coordinates, and m_i is the ith column of M.

For $p \in P$, since $w_i \in P^*$, one has $p \cdot w_i \geq 0$. Furthermore, $W'p \neq 0$ since W is regular, so that $Bp = AW'p > 0$. Hence, the second term in (1) is strictly convex on P. The second-order partial derivatives of f are uniformly bounded, so that it is possible to choose $k > 0$ and large enough for v^i to be strictly convex on P.

The demand function of the ith consumer is, when his income equals $p \cdot kw_i$,

$$x_i(p, p \cdot kw_i) = -kp \cdot w_i v_p{}^i(p),$$

where v_p^i represents the gradient of v^i. From (1) one obtains

$$x_i(p, p \cdot kw_i) = kp \cdot w_i[-(1/k) f_p'(p)\, m_i + B'\widehat{Bp}^{-1}a_i], \tag{2}$$

where f_p is the Jacobian of f, and the hat ($\widehat{\ }$) transforms an l-vector into an $l \times l$ diagonal matrix with the coordinates of the vector along its main diagonal.

Since P is compact, so are the sets $\{x \in R^l \mid x = B'\widehat{Bp}^{-1}a_i$ for some $p \in P\}$. Their union lies in the interior of X. It is therefore possible to assume k to be sufficiently large so that $x_i(p, 1) \in X$ for all $p \in P$. Note that all that is needed for a meaningful result is that $x_i(p, 1) \in \Omega$ for $p \in P$.

Consider now the aggregate demand, obtained from summing (2) over i,

$$\begin{aligned}
\sum_i x_i(p, p \cdot kw_i) &= -f_p'(p)\, MW'p + kB'\widehat{Bp}^{-1}AW'p \\
&= -f_p'(p)p + kB'\widehat{Bp}^{-1}Bp \\
&= f(p) + kB's \\
&= f(p) + kWA's \\
&= f(p) + kWs.
\end{aligned}$$

All steps follow from the construction, except for the identity $f_p'p + f = 0$ which can be obtained by differentiating Walras' Law.

It is easy to see that the demand functions defined by (2) satisfy the strong axiom of revealed preference. Because of the unitary income elasticity of demand it is sufficient to verify the axiom in the case in which income is fixed at unity.

In that case, for any $p, q \in P$, if

$$q \cdot x_i(p, 1) \leqslant q \cdot x_i(q, 1) = 1,$$

one has, whenever $x_i(p, 1) \neq x_i(q, 1)$ so that $p \neq q$,

$$v^i(q) - v^i(p) > v_p^i(p) \cdot (q - p) = 1 - q \cdot x_i(p, 1) \geqslant 0$$

because of the strict convexity of v^i. Therefore, the revealed preference relation is acyclic.

The ith consumer's direct utility function can be derived from the formula

$$u^i(x) = \min_p \{v^i(p) \mid px \leqslant 1, p \in P\},$$

and can be seen to be defined on P^*, being strictly concave and log homogeneous, i.e.,

$$u^i(\lambda x) = u^i(x) + \log \lambda \quad \text{for all} \quad x \in P^* \quad \text{and} \quad \lambda > 0.$$

It will be a perturbation of the function defined by $u^i(x) = a_i \cdot \log[(B^t)^{-1}x]$.

Remark 2. It is remarkable that the present result shows that even a slight deviation from the proportionality of initial endowments is sufficient to decompose any excess demand function into individual functions stemming from homothetic preferences. It is known [1] that if endowments are proportional, homotheticity implies that the economy behaves as if a single consumer were maximizing his preferences. But of course, the theorem stated above allows adjusting preferences to the given endowments. No such decomposition would be attainable if the parameters of the utility functions were uniformly bounded.

Remark 3. It should be noted that the present result is based on the existence of strictly convex consumers whose excess demand functions add up identically to the zero vector, so that any price vector is an equilibrium price vector. The solution is by no means unique. The reader may verify that the indirect utility functions on int Ω

$$v^i(p) = f^i(p) - \sum_j [\alpha p^j/p^i + \beta \log(p^j/p^i)] + 1/p^i$$

are strictly quasiconvex for all positive price–income pairs $(p, m) \in K$, a convex, compact subset of R_+^{l+1}, if the domain of f is Ω, and $\beta > 0$ is chosen sufficiently large. Furthermore, the corresponding demand functions have range in Ω if $\alpha > \beta$ is chosen large enough, and each consumer is endowed with initial holdings $(\alpha - \beta)s$. They also add up to the given excess demand function. In the present case, it should be noted, all endowments are proportional. The unperturbed utility functions in this case are

$$u^i(x) = x^i + \beta \sum_{j \neq i} \log(x^j - \alpha).$$

Remark 4. It is evident that the indirect utility functions obtained by the present construction inherit the differentiability properties of the given excess demand function f. In other words, if f is of class C^k, where $k \geq 1$, the same will be true of each v^i. It would be interesting to know under what conditions the direct utility functions will also be of class C^k. It can be conjectured that this will be true without further assumptions, since the underlying unperturbed preferences are of class C^∞.

Concluding remark. The importance of having ways to rationalize excess demand functions with the strong assumption of bounded second-order derivatives was pointed out to the author by Andreu Mas-Colell, since it provides an excellent tool for constructing counterexamples. The question of whether the construction could be carried out using homothetic preferences and almost proportional endowments has been posed by Hugo Sonnenschein.

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A Law of Large Numbers in the Theory of Consumer's Choice under Uncertainty

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1. INTRODUCTION

Consider a consumer unit with an n-period planning horizon. Suppose that the unit expects with certainty to earn an amount y_1 in the first period, y_2 in the second, and so on, up to the nth period, in which earnings will be y_n. Finally, assume that the consumer unit can borrow and lend freely at a zero rate of interest and that its rate of subjective discount is also zero. Then, under a lifetime wealth constraint, it is optimal to set the consumption level in each period equal to $(y_1 + y_2 + \cdots + y_n)/n$. Now consider a sequence of consumer allocation problems of this sort, with the number of planning periods, n, becoming very large. If, as $n \to \infty$, the quantity $(y_1 + y_2 + \cdots + y_n)/n$, i.e., average earnings, tends to some real number, say μ, then optimal consumption will also tend to μ as $n \to \infty$. Now, let us leave the world of certainty and consider the case where future earnings are random. Specifically, let us assume that $y_1, y_2, y_3, \ldots$ is a sequence of independent, identically distributed random variables, whose common distribution has mean μ. Then, by the law of large numbers, the sample mean $(y_1 + y_2 + \cdots + y_n)/n$ tends to μ almost surely. This means, roughly speaking, that the consumer knows with virtual certainty that, no matter how future earnings might evolve, they will, in the long run, average out to a rate of μ per period. Therefore, it seems reasonable to expect that, as $n \to \infty$, the optimal consumption level will tend to μ. The main result in this paper asserts that this is, in fact, the case.

As a possible motivation for this study, we might turn to Friedman's permanent income hypothesis [2]. In a world of certainty, this hypothesis amounts to the assertion that consumption depends on income only to the extent that income affects lifetime wealth. (The reference here is to

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the general theoretical statement of the permanent income hypothesis, and not to the specific form of the hypothesis proposed by Friedman.) When future earnings are uncertain (and Friedman obviously has this case in mind), statement of a permanent income hypothesis is a much more difficult affair. In the first place, what is "permanent income" in such a setting? Friedman's own account is deliberately vague on this issue. He draws a diagram [2, p. 24] in which three possible "permanent incomes" are depicted. Then, the true permanent income is described as "something of an average" of these three. Now, if we suppose that the consumer believes future earnings to be independent, identically distributed random variables with mean μ, and if both the rate of interest and the rate of subjective discount are zero, then the three "possible permanent incomes" in Friedman's diagram coincide and are equal to μ in each period. Thus, even Friedman's vague definition leads, in this simple-minded setting, to mean earnings μ as permanent income. Given this interpretation, what is meant by a permanent income *hypothesis* is still far from clear. Friedman seems to lean toward a hypothesis that would assert the following: The actual consumption level in the next period will depend, to some extent, on what earnings will be this period. (Some fraction of a windfall is consumed immediately.) But the *expected value* of consumption in the next period (i.e., the so-called permanent consumption) will depend only on permanent income. Given our simple-minded setting, with μ playing the role of permanent income, this hypothesis is, in general, false. The expected value of consumption in the next period depends, in general, on the entire distribution of earnings and on the exact shape of the utility function, with quantities like the third derivative playing a crucial role. But, if, instead of looking at expected consumption with a small number of planning periods, we look at actual optimal consumption with a large number of periods, we arrive at a permanent income hypothesis that turns out to be valid. Specifically, as the number of planning periods becomes very large, optimal consumption tends to permanent income, regardless of the distribution of earnings or the shape of the utility function.

2. A Dynamic Programming Problem

We turn now to a formal statement of a decision problem for a consumer whose future earnings are not known with certainty. Since our aim is to study asymptotic properties of optimal behavior, it seems advisable at this stage to keep everything else as simple as possible. To this end, we make the highly unrealistic assumptions that future earnings are inde-

pendent, identically distributed random variables and that the rate of interest and the rate of subjective discount are both zero. Section 5 will contain some remarks to the effect that the methods used here are also applicable when these assumptions are relaxed.

Two building blocks are needed to start the analysis: a utility function u, and a random variable y. u is a real function, which we will assume is defined for *all* real numbers and is twice continuously differentiable. It is assumed, further, that u is increasing and strictly concave, i.e., that $u'(x) > 0$ and $u''(x) < 0$ for all real x. Without loss of generality, we normalize u to satisfy $u(0) = 0$. y is a real nonnegative random variable, whose range is assumed to be finite. (I have not been able to prove the results in the sequel for the case where y has an infinite range. In addition, the arguments in Section 3 assume implicitly that the distribution of y is continuous. This is a simplifying assumption, and dropping it causes the notation to become very cumbersome.) Let the mean of y be denoted μ. The symbols $y_1, y_2, y_3, \ldots$ will be used to describe a sequence of independent random variables that are all distributed like y. The symbol E_y will denote expectation with respect to the distribution of y. The symbol E alone will be used to denote expectation with respect to an appropriate joint distribution, such as that of $\langle y_1, y_2, \ldots, y_n \rangle$.

The consumer's decision problem, for T planning periods, may now be stated as follows. Find T real functions $c_1, c_2, \ldots, c_T$ so as to

$$\text{Maximize } E \sum_{t=1}^{T} u(c_t(s_t)) \tag{1}$$

Subject to s_1 given;

$$s_{t+1} = s_t + y_t - c_t(s_t), \qquad t = 1, 2, \ldots, T - 1;$$

$$c_T(s_T) = s_T.$$

Thus, s_t is the consumer's *assets* at the beginning of period t. Assets play the role of a state variable in the analysis.

The foregoing statement of the decision problem has the consumption level for period t being decided upon *before* earnings in period t are realized. However, if we interpret y_t to be the earnings realized in period $t + 1$, and if we assume that first-period earnings are subsumed under s_1, we convert the problem into one where decisions take place *after* earnings are realized.

The last constraint in (1), $c_T(s_T) = s_T$, represents a simple way to take account of the restriction imposed by the consumer's lifetime wealth by forcing the last period's consumption to equal the last period's assets. (Note, in this connection, that we have neglected the constraint that

consumption must be non-negative. For the asymptotic results that we shall be interested in, this is of no consequence, because consumption will be shown to converge to a positive limit.) It turns out that the analysis is virtually unchanged if, instead of doing this, we use a bequest function to evaluate terminal wealth.

It will be argued that the maximum in (1) exists. So, let us denote this maximum by $U_T(s_1)$. In general, the symbol $U_n(s)$ will be used to denote the maximum expected utility of an n-period process that starts from an initial asset level s. The maximization problem may now be stated recursively in the following manner:

$$\begin{aligned} &U_n(s) = \max_c\{u(c) + E_y U_{n-1}(s + y - c)\}, \qquad \text{for} \quad n > 1; \\ &U_1(s) = u(s), \end{aligned} \tag{2}$$

where, in general, the maximizing c in (2) will be different for different values of s.

Before proceeding to analyze asymptotic behavior, we turn to a proposition in which the basic properties of the functions U_n are enumerated.

PROPOSITION 1. *Let the value of c at which the maximum in* (2) *is attained be denoted $c(s)$. Then,*

(i) *The function $c(\ )$ exists. (Consequently, U_n exists also.)*

(ii) *$c(\ )$ is continuously differentiable.*

(iii) *$0 < c'(s) < 1$ for all real numbers s.*

(iv) *Except for the normalization $u(0) = 0$, the function U_n has the same properties (differentiability, monotonicity, and concavity) that u is assumed to have.*

(v) *The range of the derivative U_n' is contained in the range of the derivative u'.*

(vi) *The following inequality holds:*

$$U_n(s) \leqslant nu\left(\frac{s + (n-1)\mu}{n}\right).$$

Proof. By induction on n. Assume that U_{n-1} exists and satisfies (iv) and (v). We know this to be true for U_1. For fixed s, consider the equation

$$u'(c) = E_y U_{n-1}'(s + y - c). \tag{3}$$

(This expectation exists, in view of the fact that y has finite range.) The left-hand side of (3) is strictly decreasing in c, and the right-hand side is

strictly increasing in c. Furthermore, the range of U'_{n-1} is an interval that is contained in the range of u', and therefore, so is the range of the right-hand side of (3) as a function of c. Hence, Eq. (3) has a unique solution to be denoted $c(s)$. By strict concavity, $c(s)$ is the solution of the maximization problem. Thus, $U_n(s)$ exists. If we now let s vary, we see, upon looking at Eq. (3), that the implicit function theorem is applicable (since both u' and U'_{n-1} are continuously differentiable) and this leads immediately to the continuous differentiability of the optimal policy $c(\ )$, i.e., to our assertion (ii). Differentiating Eq. (3), we get

$$c'(s)\,u''(c(s)) = (1 - c'(s))\,E_y U''_{n-1}(s + y - c(s))$$

and, by strict concavity, we have that $c'(s)$ and $1 - c'(s)$ must be of the same sign, which establishes (iii).

To obtain a proof of (iv) and (v), we write

$$U_n(s) = u(c(s)) + E_y U_{n-1}(s + y - c(s)).$$

It is clearly legitimate to differentiate once, which leads, in view of (3), to

$$U_n'(s) = u'(c(s)),$$

thus proving (v). But now we see that it is legitimate to differentiate the last equation once more, which leads to

$$U''_n(s) = c'(s)\,u''(c(s)) < 0,$$

proving (iv).

Finally, assume that U_{n-1} satisfies (vi). (Once again, we note that U_1 satisfies it.) Using this assumption in (2), we proceed to write

$$U_n(s) \leqslant \max_c \left\{u(c) + (n-1)\,E_y u\left(\frac{s + y - c + (n-2)\mu}{n-1}\right)\right\}.$$

From the concavity of u, we now get

$$\begin{aligned} U_n(s) &\leqslant \max_c \left\{u(c) + (n-1)\,u\left(\frac{s + (n-1)\,\mu - c}{n-1}\right)\right\} \\ &= nu\left(\frac{s + (n-1)\,\mu}{n}\right), \end{aligned}$$

which proves (vi). This completes the proof of the proposition.

3. Convergence of Average Expected Utility

In this section, we study the asymptotic behavior of the quantity U_n/n, i.e., of maximum expected utility per period.

Proposition 2. *As $n \to \infty$, we have*

$$U_n(s)/n \to u(\mu),$$

where the convergence is uniform in s on finite intervals.

Proof. According to Proposition 1(vi), we may write

$$\frac{U_n(s)}{n} \leqslant u\left(\frac{s + (n-1)\,\mu}{n}\right)$$

where, as $n \to \infty$, the right-hand side tends to $u(\mu)$ uniformly on finite intervals. We now proceed to construct a lower bound for U_n/n that converges to the same limit. To this end, consider the following consumption policy:

$$c_t(s_t) = \min(s_t, \mu), \qquad t = 1, 2, \ldots, n-1. \tag{4}$$

By maximality, $U_n(s)$ is at least as great as the expected utility associated with the policy (4). Let $k(s)$ be the number of periods until the first arrival of the process $\langle s_t \rangle$ in the interval $[\mu, \infty)$, starting from initial assets s and following the policy (4). Also, let T_n be the number of periods t in $\{1, 2, \ldots, n\}$, for which $s_t < \mu$, calculated on the assumption that initial assets are equal to μ. Since $s_t \geqslant 0$ for $t > 1$, and since $u(0) = 0$, we may write

$$U_n(s) \geqslant u(\min(s, \mu)) + [n - 1 - Ek(s) - ET_{n-1-k(s)}]\, u(\mu).$$

But $Ek(s) = 0$ or a positive constant (given by q^{-1}, where q is the probability that $y \geqslant \mu$) according to whether $s \geqslant \mu$ or $s < \mu$. Hence, to show that a lower bound on U_n/n converges to $u(\mu)$ uniformly on finite intervals, it suffices to show that

$$\frac{ET_{n-1-k(s)}}{n} \to 0 \qquad \text{as} \quad n \to \infty. \tag{5}$$

Now, the quantity $(1/n)\, ET_{n-1-k(s)}$ can also be written as follows:

$$E_{k(s)}\left[E_T \frac{T_{n-1-t}}{n} \,\middle|\, k(s) = t\right],$$

where the expectation $E_T(T_{n-1-t}/n)$ is bounded between 0 and 1. Therefore, if we can show that

$$E_T \frac{T_{n-1-t}}{n} \to 0 \qquad \text{as } n \to \infty \tag{6}$$

for each *fixed* t, then, by Lebesgue's Bounded Convergence Theorem, it will follow that (5) also holds. But, for fixed t, (6) is clearly equivalent to the statement that

$$ET_n/n \to 0 \qquad \text{as} \quad n \to \infty.$$

To establish this, let us transform the process $\langle s_t \rangle$ by defining

$$w_t = s_{t+2} - y_{t+1}, \qquad \text{for} \quad t = 0, 1, 2, \ldots .$$

Since T_n is calculated on the assumption that $s_1 = \mu$, we have $w_0 = 0$. The process $\langle w_t \rangle$ is governed by the rule

$$w_{t+1} = \max\{w_t + y_{t+1} - \mu, 0\}.$$

Thus, $\langle w_t \rangle$ is a queuing process [1, p. 194], and T_n is the number of periods t in $\{1, 2, \ldots, n-2\}$ for which $w_t = 0$. To avoid the introduction of still further notation, we will think of T_n as the number of periods t in $\{1, 2, \ldots, n\}$, rather than in $\{1, 2, \ldots, n-2\}$, for which $w_t = 0$. Let $t_1, t_2, \ldots, t_{T_n}$ be the successive indices in $\{1, 2, \ldots, n\}$ for which $w_t = 0$, and define $k_1, k_2, \ldots, k_{T_n}$ by

$$\begin{aligned} k_1 &= t_1 \\ k_2 &= t_2 - t_1 \\ &\vdots \\ k_{T_n} &= t_{T_n} - t_{T_n - 1}. \end{aligned}$$

Then, $k_1, k_2, \ldots, k_{T_n}$ are independent, identically distributed random variables, with *infinite* mean [1, p. 380]. Now, as $n \to \infty$, $T_n \to \infty$ almost surely. (Actually, the fact $P[T_n \to \infty] = 1$ is not needed in the argument because, in the event $T_n < M$ for all n, the desired result on T_n/n certainly holds.) Therefore, on a set of probability 1, we have, as $n \to \infty$,

$$(k_1 + k_2 + \cdots + k_{T_n})/T_n \to \infty \text{ almost surely.}$$

This is a version of the law of large numbers for the case of an infinite mean. Now, we also know that $k_1 + k_2 + \cdots + k_{T_n} \leqslant n$, by definition.

Hence,

$$n/T_n \to \infty \text{ almost surely} \qquad \text{or} \qquad T_n/n \to 0 \text{ almost surely.}$$

But T_n/n is bounded by $0 \leqslant T_n/n \leqslant 1$, so that convergence almost surely implies convergence in the moments. In particular, we have

$$(E\,T_n)/n \to 0,$$

as was to be shown.

4. Convergence of the Optimal Policy

Consider once again the dynamic programming problem (2). It is well-known that the result of Section 3, i.e., the convergence of maximum expected utility per period, does not imply convergence of the optimal policy. To show that the optimal policy converges, one usually has to show not only that U_n/n tends to $u(\mu)$, but also that $U_n(s) - nu(\mu)$ tends to some fixed function of s. This is the case in many dynamic programming problems (see [3]), including inventory theory [4] and Ramsey's optimal growth problem [5]. Unfortunately, it turns out that in the present framework, the quantity $U_n(s) - nu(\mu)$ in general fails to converge. So, instead of examining this quantity, we will examine a quantity of the form $U_n(s) - K_n$, where, among other things, the constants $K_1, K_2, \ldots$ satisfy the condition $K_n/n \to u(\mu)$ but not the condition $K_n/n = u(\mu)$. As it turns out, this is what is needed for the convergence of the optimal consumption policy.

Proposition 3. *In the dynamic programming problem* (2), *let* $c^n(s)$ *be the optimal consumption level in the first period of an n-period process that starts with assets s. Assume that the random variable y has its range given by the interval* $[0, Y]$, *for some* $Y \geqslant 0$. *Then, there exists a real number* c^*, *satisfying* $0 \leqslant c^* \leqslant Y$, *such that, as* $n \to \infty$,

$$c^n(s) \to c^*,$$

uniformly on finite intervals.

Proof. We begin by constructing a sequence of functions, $V_1, V_2, \ldots$ in the following manner. For every real number s, let

$$V_n(s) = \max_c \{u(c) + E_y V_{n-1}(s + y - c) - E_y V_{n-1}(y)\}, \qquad \text{for } n = 2, 3, \ldots, \tag{7}$$

$$V_1(s) = u(s).$$

Comparing (7) with (2), we see that V_n differs from U_n by a constant. So, $c^n(s)$ is the value of c at which the maximum in (7) is attained. Note also that all the assertions about U_n in Proposition 1 are valid also for V_n. The optimal policy, $c^n(s)$, is defined uniquely by

$$u'(c^n(s)) = E_y V'_{n-1}(s + y - c^n(s)). \tag{8}$$

We will now show that the sequence $\langle V_n \rangle$ and the sequence of derivatives $\langle V_n' \rangle$ both converge uniformly on finite intervals. (The fact that the sequence $\langle V_n \rangle$ converges will be used in the following theorem.) In particular, it will be shown that, as $n \to \infty$,

$$V_n'(s) \to \alpha,$$

where α is some positive constant. It will then follow, in view of (8), that $c^n(s) \to c^*$, where $c^* = u'^{-1}(\alpha)$.

Since the range of V'_{n-1} is contained in the range of u' (see Proposition 1), there exists a real number s_n^* such that

$$u'(s_n^*) = E_y V'_{n-1}(y). \tag{9}$$

The number s_n^* has the following four properties: (a) s_n^* is unique; (b) $c^n(s_n^*) = s_n^*$; (c) $V_n(s_n^*) = u(s_n^*)$; and (d) $V_n'(s_n^*) = u'(s_n^*)$. These properties follow at once from the various definitions involved in conjunction with Proposition 1. Somewhat less immediate is the next property,

$$0 \leqslant s_n^* \leqslant Y, \qquad \text{for all } n. \tag{10}$$

In fact, a more general inequality is true, namely,

$$s/n \leqslant c^n(s) \leqslant (s + (n - 1)\, Y)/n, \tag{11}$$

for all n and for all s. Note that (10) follows from (11) upon setting $c^n(s) = s = s_n^*$.

The inequality (11) is proved by induction on n. The induction basis is given by $c^1(s) = s$. Now, for the induction step, suppose (11) is true for $n - 1$. That is, suppose that

$$s/(n - 1) \leqslant c^{n-1}(s) \leqslant (s + (n - 2)\, Y)/(n - 1). \tag{12}$$

Since $V'_{n-1}(s) = u'(c^{n-1}(s))$ and u is concave, we may write

$$u'\left(\frac{s + (n-2)Y}{n-1}\right) \leqslant V'_{n-1}(s) \leqslant u'\left(\frac{s}{n-1}\right). \tag{13}$$

Using the concavity of V_{n-1} and the fact that $0 \leqslant y \leqslant Y$ in Eq. (8) leads to

$$V'_{n-1}(s + Y - c^n(s)) \leqslant u'(c^n(s)) \leqslant V'_{n-1}(s - c^n(s)). \tag{14}$$

Combining (13) and (14), we get

$$u'\left(\frac{s + (n-1)\,Y - c^n(s)}{n-1}\right) \leqslant u'(c^n(s)) \leqslant u'\left(\frac{s - c^n(s)}{n-1}\right),$$

from which (11) follows immediately.

Note also that, by setting $c = s$ in (7), we obtain the inequality $V_n(s) \geqslant u(s)$ for all s and n.

Collecting all the information we have so far about the functions V_n, we may say, in summary, that each of the functions V_n is increasing, concave, and twice continuously differentiable; that it is tangent to the function u from above, and that all the points of tangency are within the interval $[0, Y]$. Also, it follows from (11) that the *derivatives*, both of V_n and of V_n', are uniformly bounded on finite s-intervals. These facts lead immediately to the conclusion that the sequences $\langle V_n \rangle$ and $\langle V_n' \rangle$ are both *equicontinuous* on any finite interval. It now follows, by the Ascoli–Arzela theorem (see, e.g., [6, p. 144]) that the sequences $\langle V_n \rangle$ and $\langle V_n' \rangle$ both have subsequences that converge uniformly on any finite interval.

To proceed further, let us revert to the use of subscripts on period-by-period asset levels. Specifically, let $s_1, s_2, s_3, \ldots$ be defined by

$$\begin{aligned} s_1 &= s \\ s_{t+1} &= s_t + y_t - c^{n-t+1}(s_t), \qquad \text{for} \quad t = 1, \ldots, n-1. \end{aligned} \tag{15}$$

Differentiating V_n twice and making use of (8), we get

$$\begin{aligned} V_n'(s_1) &= EV'_{n-1}(s_2) \\ V_n''(s_1) &= \left[1 - \frac{dc^n(s_1)}{ds_1}\right] E\, V''_{n-1}(s_2). \end{aligned} \tag{16}$$

(Here, and through (19), the symbol E stands for expectation with respect to y_1.) But, by differentiating (8), we have

$$1 - \frac{dc^n(s_1)}{ds_1} = \frac{u''(c^n(s_1))}{u''(c^n(s_1)) + E\, V''_{n-1}(s_2)}. \tag{17}$$

Hence, by combining (16) and (17), we arrive at

$$V_n''(s_1) = \frac{1}{(1/u''(c^n(s_1))) + (1/E\, V''_{n-1}(s_2))}, \tag{18}$$

which leads immediately to the inequality

$$V_n''(s_1) \geqslant u''(c^n(s_1)).$$

By the same token, however,

$$V_{n-1}''(s_2) \geqslant u''(c^{n-1}(s_2)).$$

Hence, substituting in (18), we get

$$V_n''(s_1) \geqslant \frac{1}{(1/u''(c^n(s_1))) + (1/E\,u''(c^{n-1}(s_2)))}. \tag{19}$$

Now, if there exists some constant, say m, such that $u''(c^n(s_1)) \geqslant m$ and $u''(c^{n-1}(s_2)) \geqslant m$, the latter holding regardless of the value taken on by y_1, then (19) implies that

$$V_n''(s_1) \geqslant m/2.$$

But, this can be said about $V_{n-1}''(s_2)$ as well, which leads, on substitution in (18), to the following conclusion: If $u''(c^n(s_1)) \geqslant m$ and if $u''(c^{n-1}(s_2)) \geqslant m$ (regardless of the value taken on by y_1) and if $u''(c^{n-2}(s_3)) \geqslant m$ (regardless of the values taken on by y_1 and y_2), then

$$V_n''(s_1) \geqslant m/3.$$

Thus, it is clear that the following general statement holds: If, regardless of the values taken on by the random variables $y_1, \ldots, y_k$ (with $k < n$), the inequalities

$$u''(c^{n-i+1}(s_i)) \geqslant m, \qquad \text{for} \quad i = 1, \ldots, k \tag{20}$$

hold, then

$$V_n''(s_1) \geqslant m/(k+1). \tag{21}$$

Select some $\beta > 0$, and define m by

$$m = \min\{u''(s) \mid -\beta \leqslant s \leqslant Y + \beta\}. \tag{22}$$

We will show that, given any positive integer k, there exists an integer N, such that, for $n > N$, the inequality (20) holds, for $i = 1, \ldots, k$ and for all

values of s_1 in an arbitrary finite interval (and regardless of the values taken on by $y_1, \ldots, y_k$). In view of (21), this would imply that

$$V_n''(s_1) \to 0 \tag{23}$$

uniformly on finite intervals.

To establish this last claim, fix the integer k. We now assert that, for any such fixed k, there exist two sequences of real numbers, $\langle \theta_n{}^k \rangle$ and $\langle \delta_n{}^k \rangle$, such that, as $n \to \infty$,

$$\theta_n{}^k \to 0 \qquad \text{and} \qquad \delta_n{}^k \to 1, \tag{24}$$

and such that, regardless of the values taken on by the random variables $y_1, \ldots, y_k$, we have

$$\frac{s_1}{n} - \theta_n{}^k Y \leqslant c^{n-k+1}(s_k) \leqslant \frac{s_1}{n} + \delta_n{}^k Y. \tag{25}$$

This result, if true, clearly suffices. For, given that s_1 ranges in some finite interval, it is possible to select N in such a way that, for $n > N$, both bounds in (25) fall inside the interval $[-\beta, Y + \beta]$.

The existence of two double sequences, $\langle \theta_n{}^k \rangle$ and $\langle \delta_n{}^k \rangle$, such that (24) and (25) hold, is proved by induction on k. Unfortunately, we must introduce two additional double sequences, $\langle \sigma_n{}^k \rangle$ and $\langle \nu_n{}^k \rangle$, to carry out the induction. All four double sequences are defined recursively, in the following manner:

$$\sigma_n{}^1 = \nu_n{}^1 = \theta_n{}^1 = 0, \qquad \delta_n{}^1 = \frac{n-1}{n},$$

$$\sigma_n^{k+1} = \sigma_n{}^k + \theta_n{}^k + 1, \qquad \nu_n^{k+1} = \nu_n{}^k + \delta_n{}^k,$$

$$\theta_n^{k+1} = \frac{\nu_n^{k+1}}{n-k}, \qquad \delta_n^{k+1} = \frac{\sigma_n^{k+1} + (n-k-1)}{n-k}, \tag{26}$$

for all positive integers k and for all integers n satisfying $n > k$. It is immediately verified that, with these definitions, we have, as $n \to \infty$,

$$\sigma_n{}^k \to k-1, \qquad \nu_n{}^k \to k-1, \qquad \theta_n{}^k \to 0, \qquad \delta_n{}^k \to 1. \tag{27}$$

We shall now prove, by induction on k, that (25) holds, together with the relationship

$$\frac{n-k+1}{n} s_1 - \nu_n{}^k Y \leqslant s_k \leqslant \frac{n-k+1}{n} s_1 + \sigma_n{}^k Y. \tag{28}$$

For $k = 1$, (25) reduces to (11), and (28) is trivial. Therefore, assume that (25) and (28) have been shown to hold for some k. We must show that they hold also for $k + 1$. By definition, $s_{k+1} = s_k + y_k - c^{n-k+1}(s_k)$. So, from (25), (26), and (28), together with the inequality $0 \leqslant y_k \leqslant Y$, it follows that

$$\frac{n-k}{n} s_1 - \nu_n^{k+1} Y \leqslant s_{k+1} \leqslant \frac{n-k}{n} s_1 + \sigma_n^{k+1} Y, \tag{29}$$

i.e., (28) holds for $k + 1$. However, from (11), we have

$$\frac{s_{k+1}}{n-k} \leqslant c^{n-k}(s_{k+1}) \leqslant \frac{s_{k+1} + (n-k-1)Y}{n-k}, \tag{30}$$

and combining (29) and (30) leads to (25) being satisfied for $k + 1$. This completes the proof of the assertion that $V_n''(s_1) \to 0$, uniformly on finite intervals.

Let us return now to the sequence $\langle V_n' \rangle$ of first derivatives. Since the second derivatives tend to zero, every convergent subsequence of $\langle V_n' \rangle$ must converge to a constant. We must still show that it is not possible for two subsequences to converge to different constants. To see this, we rewrite the first derivatives in (16) in full, while omitting the time subscripts on assets:

$$V_n'(s) = E_y V_{n-1}'(s + y - c^n(s)).$$

Now, if V_{n-1}' is uniformly within, say, ϵ of some constant (on intervals of length Y), then so must V_n' be. In other words, if V_{n-1}' is selected from a subsequence converging to some constant α, then V_n' must be in the same subsequence. Thus, we now have the result that there exists a real number α such that, as $n \to \infty$,

$$V_n'(s) \to \alpha,$$

the convergence being uniform on finite intervals. Recall, however, that the functions V_n are all tangent to u, with the points of tangency in the interval $[0, Y]$. This means that the sequence $\langle V_n \rangle$ also converges and, furthermore, that

$$\alpha = u'(c^*)$$

for some c^* in $[0, Y]$. (Indeed, the sequence $\langle V_n \rangle$ converges to the straight line that is tangent to u at c^*.) Checking Eq. (8), we see that the proof of Proposition 3 is complete.

Propositions 2 and 3 together lead to the following result.

THEOREM. *Under the assumptions of Propositions* 2 *and* 3, *and with* $c^n(s)$ *defined as in Proposition* 3, *we have*

$$c^n(s) \to \mu, \tag{31}$$

as $n \to \infty$, *the convergence being uniform on finite intervals.*

Proof. Let U_n be defined as in (2) and let V_n be defined as in (7). Then,

$$U_n(s) = V_n(s) + \sum_{k=1}^{n-1} E_y V_k(y). \tag{32}$$

We have seen, in Proposition 3, that the sequence $\langle V_n \rangle$ converges. For the moment, let its limit be denoted V_∞. Note also that $V_n/n \to 0$ as $n \to \infty$, since the sequence $\langle V_n \rangle$ is uniformly bounded on any finite interval. Finally, recall that $U_n/n \to u(\mu)$, by Proposition 2. Thus, if we divide (32) by n and take the limit, as $n \to \infty$, we arrive at

$$u(\mu) = E_y V_\infty(y). \tag{33}$$

But, $V_\infty(y) = u(c^*) + (y - c^*)\,u'(c^*)$, where c^* is the limit of $c^n(s)$ in Proposition 3. Therefore, (33) reduces to

$$u(\mu) = u(c^*) - (c^* - \mu)\,u'(c^*). \tag{34}$$

By strict concavity, (34) has the unique solution $c^* = \mu$.

5. REMARKS

A. *The Non-negativity Constraint*

Even though we have not taken account explicitly of the constraint $c_t \geqslant 0$, our results show that with a large number of planning periods, the constraint is satisfied. To incorporate non-negativity explicitly, we must change the way in which the lifetime wealth constraint is treated. For, if we impose $c_t \geqslant 0$ for all t, together with $c_T = s_T$, we find that consumption in every period must obey the additional constraint $0 \leqslant c_t \leqslant s_t$. In other words, no borrowing is possible. (See Remark B, below.) One way to get around this problem is to introduce a bequest function, say φ, and work with the maximand

$$E\left[\sum_{t=1}^{T} u(c_t) + \varphi(s_{T+1})\right], \tag{35}$$

instead of the maximand in (1). (The function φ must satisfy the condition $\sup \varphi' > \inf u'$, or else the problem would have no solution.) This changes the initial conditions of our dynamic programming problem from $U_1 = V_1 = u$ to $U_1 = V_1 = \varphi$, which does not affect any of the asymptotic results, as may readily be verified. While φ must be defined for all real numbers, u may now be defined only for non-negative values.

B. *Lifetime Wealth Constraint*

If, instead of using a penalty function for the evaluation of terminal assets, one wishes to use an explicit endpoint constraint on terminal assets, one finds that the problem changes into one where borrowing is restricted throughout the process. Suppose, for instance, that we were to introduce the constraint $P(s_{T+1} \geqslant 0) = 1$. By monotonicity, this can be changed to $P(s_{T+} = 0) = 1$. Now, without loss of generality, we may assume that the distribution of the random variable y, which describes future earnings, has positive mass in the neighborhood of 0. This is so because any future earnings that are anticipated with certainty may be subsumed under initial assets. Thus, if $s_t < 0$ for some t, then $s_{T+1} < 0$ with positive probability. In other words, to ensure that $s_{T+1} = 0$ with probability 1, we must impose the constraint that requires $c_t \leqslant s_t$ for all t. Consider, therefore, the maximization problem (1), with the added constraint $0 \leqslant c_t \leqslant s_t$ for all t. Proposition 2 is still valid without change. However, Proposition 3 is no longer true. For an analysis of this problem, the reader is referred to [7].

C. *Nonindependent Earnings*

There is some empirical evidence [8] to the effect that a Markov chain fits the distribution of earnings fairly well. If earnings constitute a Markov chain that possesses an asymptotic distribution, then the techniques used above can be applied, without far-reaching modifications, to establish the convergence of the optimal consumption policy to the long-run mean.

D. *The Rate of Interest and the Rate of Subjective Discount*

Both of these rates are assumed in the foregoing analysis to be zero. While this assumption is very restrictive, it does force us to avoid the use of any arguments that depend on convergence of total expected utility. Whether or not these techniques might be applicable, when one relaxes the assumption on the rate of interest and the rate of subjective discount, is an open question.

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An Income Fluctuation Problem

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A consumer has a t period planning horizon problem, at each period, he gets additionals incomes that are i.i.d. random variables, and he must decide how much of his income will be spent in consumption, yielding some utility, and how much will be saved to maximize the total expected utility. No borrowing is allowed, and for the amounts saved no interest is paid. Using the concepts of competitive prices it is shown that as $t \to \infty$ the corresponding limit of the consumption policy is strictly bounded above by the expected value of the random income.

INTRODUCTION

A consumer is faced with the following situation. He has a t period planning horizon and at each period he must decide how much to consume and save, to maximize the total expected utility accumulated in t periods. If y is the available income and $0 \leqslant x \leqslant y$ is the amount saved at period k, then the available income in the next period is $x + w^k$, where w^k are i.i.d. random variables. If $c = y - x$ units are consumed in period k, then this produces $u(c)$ units of satisfaction or utility to the consumer in the period k.

When y is the initial income, the earnings $\omega^1,\ldots,\omega^t$ are known with certainty and $u(c)$ is strictly concave, it is known that it is optimal to set the consumption level $c_t(y)$ at

$$c_t(y) = \min\left(y, \frac{\omega^1 + \cdots + \omega^{t-1} + y}{t}\right).$$

If $(\omega^1 + \cdots + \omega^t)/t$ converges to an average ω as $t \to \infty$, then the optimal consumption level will converge to $\min(y, \omega)$. In the stochastic case, in which ω^t are i.i.d. random variables, it is natural to expect that $c_t(y)$ will converge to $c(y) = \min(y, \omega)$, since the expected average income $(\omega^1 + \cdots + \omega^t)/t$ will converge to $\omega = E\omega^t$ by the law of large numbers.

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In this paper, we will show that this is not true whenever $u'(0) < \infty$ and $\omega^t \equiv \omega$ are nondegenerate random variables. In fact, we will obtain the surprising result that $0 < c(y) < \omega$ for $0 < y < \infty$. Yaari [16] considers the case in which borrowing is allowed at zero interest rate, no constraints are imposed on $c_t(y)$, i.e., $c_t(y) \in (-\infty, +\infty)$ and obtains the result that $c(y) = \lim_{t\to\infty} c_t(y) = \omega$ for all y.

In this paper we use the concepts of competitive prcies and policies developed in [17] in a more general context. For the sake of completeness some proofs that are resteatments of more general ones of [17] are also presented.

Lemmas, theorems and corollaries are numbered by section in the order in which they appear and the end of a proof is represented by ∎.

1. The Model

A consumer is faced with the following situation. He has a t period planning horizon and at each period he must decide how much to consume and save, to maximize the total expected utility accumulated in t periods. If y is the available income and $0 \leqslant x \leqslant y$ is the amount saved at period k, then the available income in the next period is $x + \omega^k$, where k are random variables. If $c = y - x$ units are consumed in period k, then this produces $u(c)$ units of satisfaction or utility. We will assume that the random variables ω^k are i.i.d. and will be simply denoted by ω. The range of ω is an interval $[0, A]$, $0 < A < +\infty$, and if $v(0, \epsilon) \equiv \text{Prob}[\omega \leqslant \epsilon]$, $\epsilon \geqslant 0$, we will assume that $1 > v(0, \epsilon) > 0$ for $A > \epsilon > 0$. The utility function is assumed to be increasing, strictly concave, and differentiable. We also assume that $u'(0) < +\infty$ and without loss of generality that $u(0) = 0$. The problem can be stated as

$$\max E \sum_{j=1}^{t} u(c_j).$$

Subject to:

$$c_j + x_j = y_j, \qquad y_{j+1} = x_j + w_j, \qquad x_j \geqslant 0,\ c_j \geqslant 0.$$

If $V_t(y)$ is the maximum expected utility that we can get when y is the stock available and we have t periods to go, then, the usual dynamic programming formulation leads us to the following functional equations:

$$\begin{aligned} V_t(y) &= \max_{\substack{c+x=y \\ c\geqslant 0, x\geqslant 0}} \{u(c) + EV_{t-1}(x + \omega)\} \qquad \text{for} \quad t > 1 \\ V_1(y) &= \max_{\substack{c+x=y \\ c\geqslant 0, x\geqslant 0}} \{u(c)\}. \end{aligned} \tag{1.1}$$

A solution of (1.1) will be denoted by $(x_t(y), c_t(y))$. Studying the basic properties of $(x_t(y), c_t(y))$, using the concepts of prices that will be introduced later, we will be able to obtain our main results by taking the limit as $t \to \infty$.

First we will show that the function $V_t(y)$ inherits the basic properties of $u(c)$.

LEMMA 1.1. *The function $V_t(y)$ is increasing and strictly concave. Furthermore, $V_t(y)$ is differentiable for $y > 0$ and for all t.*

Proof. The first two parts are immediate and have as a consequence the fact that $x_t(y)$ and $c_t(y)$ are uniquely defined. The differentiability will be shown by induction.

(I) For $t = 1$ we have, $V_1(y) = u(y)$. Thus, $V_1(y)$ is differentiable for $y > 0$, since by assumption $u(c)$ is differentiable for $c > 0$.

(II) Assume that $V_{t-1}(y)$ is differentiable for $y > 0$. To prove that $V_t(y)$ is differentiable for $y > 0$, we must show that $V_t(y)$ has a unique support for all $y > 0$.

Let p be a support of $V_t(y)$ at y. Then,

$$V_t(y') - V_t(y) \leqslant p(y' - y), \qquad \text{for all} \quad y' \geqslant 0$$

or

$$V_t(y') - py' \leqslant V_t(y) - py. \tag{1.2}$$

Now, choose any c', $x' \geqslant 0$ and let $y' = c' + x'$. Then,

$$u(c') + \phi_{t-1}(x') \leqslant V_t(y') \tag{1.3}$$

from the definition of $V_t(y)$, where $\phi_{t-1}(x) \equiv EV_{t-1}(x + \omega)$. Hence, from (1.2) and (1.3) we get:

$$\begin{aligned} u(c') + \phi_{t-1}(x') - p(c' + x') \\ \leqslant V_t(y') - py' \leqslant V_t(y) - py \\ = u(c_t(y)) + \phi_{t-1}(x_t(y)) - p(x_t(y) + c_t(y)). \end{aligned} \tag{1.4}$$

Thus,

$$\begin{aligned} u(c') - pc' + \phi_{t-1}(x') - px' \\ \leqslant u(c_t(y)) - pc_t(y) + \phi_{t-1}(x_t(y)) - px_t(y). \end{aligned} \tag{1.5}$$

Now, either $x_t(y) > 0$ or $c_t(y) > 0$ since $y = c_t(y) + x_t(y) > 0$.

Suppose $c_t(y) > 0$; then for $x' = x_t(y)$ in (1.5), we get:

$$u(c') - pc' \leqslant u(c_t(y)) - pc_t(y) \qquad \text{for all} \quad c' \geqslant 0 \tag{1.6}$$

or, $u(c) - pc$ has a maximum at $c_t(y) > 0$. Then,

$$u'(c_t(y)) = p. \tag{1.7}$$

In an analagous way, for $x_t(y) > 0$ we get

$$\phi_{t-1}(x') - px' \leqslant \phi_{t-1}(x_t(y)) - px_t(y), \qquad \text{for all} \quad x' \geqslant 0 \tag{1.8}$$

or, $\phi_{t-1}(x) - px$ has a maximum at $x_t(y) > 0$. Thus,

$$p = \phi'_{t-1}(x_t(y)). \tag{1.9}$$

Despite the fact that $V_t(y)$ is differentiable for $y > 0$, the function $c_t(y)$ need not be. Consider the following case:

$$f(x; \omega) = x + \omega, \qquad u(c) = \log c$$

and, ω assumes the values $a_1 = 1$, $a_2 = 2$ with corresponding probabilities $\pi_1 = \frac{1}{2}$ and $\pi_2 = \frac{1}{2}$. Now,

$$V_2(y) = \max_{\substack{c+x=y \\ c\geqslant 0, x\geqslant 0}} \{\log c + E \log(x + \omega)\},$$

which is equivalent to maximizing

$$\log c + \sum_{i=1}^{2} \pi_i \log(y - c + a_i) \qquad \text{for} \quad c \geqslant 0. \tag{1.10}$$

Taking the derivations in (1.10), we get:

$$1/c = \sum_{i=1}^{2} \pi_i/(y - c + a_i). \tag{1.11}$$

Making $c = y$ in (1.11) we get a critical value

$$\mathbf{y} = \frac{1}{\sum\limits_{i=1,2} \pi_i/a_i} = \frac{4}{3},$$

below which, we consume all the stock of goods available, i.e.,

$$c_2(y) = y \qquad \text{for} \quad 0 \leqslant y \leqslant \mathbf{y}.$$

For $y \geqslant \mathbf{y}$ the solution of (1.10) is given by

$$c_2(y) = \frac{6y + 9 - (4y^2 + 12y + 17)^{1/2}}{8}.$$

Now, taking the right and left-hand side derivatives of $c_2(y)$, at the point $\mathbf{y}$ we get:

$$\frac{dc_2(y)}{dy}\Big|_{\mathbf{y}}^{\cdot} = \frac{1280}{2432} \quad \text{and} \quad \frac{dc_2(y)}{dy}\Big|_{\mathbf{y}}^{-} = 1,$$

which shows $c_2(y)$ is nondifferentiable at the point $\mathbf{y}$. This example could be generalized for the case in which ω is any discrete random variable with a finite range.

LEMMA 1.2. *The function $V_t(y)$ is differentiable from the right at the point $y = 0$. Furthermore $V_t'(0) = u'(0)$.*

Proof. First we immediately have that

$$V_t(y) - V_t(0) \geqslant u(y). \tag{1.12}$$

Now we will show by induction that

$$V_t(y) - V_t(0) \leqslant tu(y/t). \tag{1.13}$$

For $t = 1$ we have $V_1(y) = u(y)$. Hence, $V_1(y) \leqslant u(y)$. Assume that it is true for $t - 1$, i.e.,

$$V_{t-1}(y) - V_{t-1}(0) \leqslant (t-1)\, u(y/(t-1))$$

Now,

$$\begin{aligned} V_t(y) - V_t(0) &\leqslant u(c_t(y)) + EV_{t-1}(X_t(y) + \omega) - EV_{t-1}(\omega) \\ &\leqslant u(c_t(y)) + V_{t-1}(X_t(y)) - V_{t-1}(0), \end{aligned}$$

where in the last inequality we used the fact that $V_{t-1}(y)$ is concave. (If $\phi(c)$ is concave then $\phi(c + h) - \phi(h)$ is a decreasing function of h.) Hence, using the induction hypothesis we get

$$\begin{aligned} V_t(y) - V_t(0) &\leqslant u(c_t(y)) + (t-1)\, u\left(\frac{1}{t-1}\, x_t(y)\right) \\ &= t\left(\frac{1}{t}\, u(c_t(y)) + \frac{t-1}{t}\, u\left(\frac{1}{t-1}\, x_t(y)\right)\right) \\ &\leqslant tu\,\frac{1}{t}\, c_t(y) + \frac{1}{t}\, x_t(y) \\ &= tu\left(\frac{1}{t}\, y\right). \quad \blacksquare \end{aligned}$$

Now, from (1.12) and (1.13) we get

$$u(y) \leqslant V_t(y) - V_t(0) \leqslant tu((1/t)\, y)$$

Dividing by $y > 0$ we have

$$\frac{u(y)}{y} \leqslant \frac{V_t(y) - V_t(0)}{y} \leqslant \frac{t}{y}\, u\left(\frac{1}{t}\, y\right).$$

Now, taking the limit as $y \to 0^+$ we finally get

$$V_t'(0) = u'(0). \quad \blacksquare$$

From Lemma 1.1 and Lemma 1.2 we can state the following theorem:

THEOREM 1.3. *The function $V_t(y)$ is differentiable for $y \geqslant 0$, and for all t. Furthermore $V_t'(0) = u'(0)$.*

The derivatives of $V_t(y)$ will be denoted by $p_t(y)$, and will be called prices.

The prices $p_t(y)$ are continous decreasing functions since they are the derivatives of strictly concave functions. The continuity is defined on $[0, +\infty)$.

The following corollary is an immediate consequence of Theorem 1.3.

COROLLARY 1.4. *The pair $(x_t(y), c_t(y))$ satisfies the following optimality conditions*:

$$p_t(y) \geqslant Ep_{t-1}(x_t(y) + \omega) \tag{1.14}$$

with equality if $x_t(y) > 0$, and

$$p_t(y) \geqslant u'(c_t(y)) \tag{1.15}$$

with equality if $c_t(y) > 0$.

Now we will study the basic properties ot the functions $x_t(y)$, $c_t(y)$, and $p_t(y)$.

THEOREM 1.5. *The functions $x_t(y)$, $c_t(y)$ are nondecreasing functions. Furthermore, $x_t(y)(c_t(y))$ is increasing whenever $x_t(y) > O(c_t(y) > 0)$.*

Proof. Let $y' > y \geqslant 0$. First we will show that $c_t(y') \geqslant c_t(y)$. The proof depends on the values that $c_t(y')$ assumes; for the sake of simplicity, we will consider only one case; the other cases have similar proves.

Let $c_t(y') > 0$ and $c_t(y) > 0$. Then from the optimality conditions (1.15), it follows that

$$p_t(y') = u'(c_t(y')) \qquad \text{and} \qquad p_t(y) = u'(c_t(y)).$$

Thus,

$$u'(c_t(y')) < u'(c_t(y))$$

by monotonicity of $p_t(y)$. Therefore,

$$c_t(y') > c_t(y)$$

by monotonicity of $u'(c)$.

An analagous proof holds for $x_t(y)$, except that we use the optimality condition (1.14). ∎

A proof of the following theorem can be seen in [17, p. 18].

THEOREM 1.6. *The functions $x_t(y)$ and $c_t(y)$ are continuous functions.*

THEOREM 1.7. $p_t(y) \geq p_{t-1}(y)$ *for* $t \geq 2$.

Proof. The proof will be done by induction.

(1) For $t = 2$, we have from the optimality conditions (1.15) that

$$p_2(y) \geq u'(c_2(y)) \qquad \text{and} \qquad p_1(y) = u'(y).$$

Hence,

$$p_2(y) \geq p_1(y)$$

because $c_2(y) \leq y$ and $u'(c)$ is a decreasing function.

(2) Assume that it is true for $t - 1$, i.e., $p_{t-1}(y) \geq p_{t-2}(y)$. We need to consider two cases depending on the values that $x_t(y)$ and $x_{t-1}(y)$ assume. We will consider only the case, $0 \leq x_t(y) \leq y$ and $0 < x_{t-1}(y) \leq y$; a similar proof holds for the other case. From the optimality condition (1.14) we get:

$$p_t(y) \geq Ep_{t-1}(x_t(y) + \omega) \tag{1.16}$$

and

$$p_{t-1}(y) = Ep_{t-2}(x_{t-1}(y) + \omega). \tag{1.17}$$

Now, suppose that $p_t(y) < p_{t-1}(y)$. Hence,

$$c_t(y) \geq c_{t-1}(y) \tag{1.18}$$

and consequently

$$x_t(y) \leqslant x_{t-1}(y). \tag{1.19}$$

From (1.14) and the induction hypothesis we get

$$\begin{aligned} p_t(y) &\geqslant E\,p_{t-1}(x_t(y) + \omega) \\ &\geqslant E\,p_{t-2}(x_t(y) + \omega). \end{aligned} \tag{1.20}$$

So, from (1.20), (1.19) and the monotonicity of $p_t(y)$, we have

$$\begin{aligned} p_t(y) &\geqslant Ep_{t-2}(x_{t-1}(y) + \omega) \\ &= p_{t-1}(y) \quad \text{by (1.17),} \end{aligned}$$

which is a contradiction. ∎

COROLLARY 1.8. *$c_t(y) \leqslant c_{t-1}(y)$ and $x_t(y) \geqslant x_{t-1}(y)$ for all $t \geqslant 2$.*

We can now prove a result that will be needed later when we consider infinite horizon problems.

THEOREM 1.9. *The limiting policy*

$$x(y) = \lim_{t\to\infty} x_t(y) \qquad \textit{and} \qquad c(y) = \lim_{t\to\infty} c_t(y)$$

exist and are nondecreasing continuous functions. Furthermore, the sequence of functions $x_t(y)$ and $c_t(y)$ converge uniformly in any finite closed interval.

Proof. From Corollary 1.8 we have

$$x_{t+1}(y) \geqslant x_t(y) \qquad \text{for all } t, y \geqslant 0$$

and

$$c_{t+1}(y) \leqslant c_t(y) \qquad \text{for all } t, y \geqslant 0.$$

But $c_t(y) > 0$ and $x_t(y) < y$ for all t. Hence,

$$x(y) = \lim_{t\to\infty} x_t(y) \qquad \text{and} \qquad c(y) = \lim_{t\to\infty} c_t(y)$$

are well-defined functions. Now, since $x_t(y)$, $c_t(y)$ are nondecreasing continuous functions, it follows that $x(y)$ and $c(y)$ are also nondecreasing functions. The continuity of $x(y)$ and $c(y)$ will follow by the same arguments that are used in the proof of Theorem 1.6. From the fact that a sequence of monotonic functions converging pointwise to a continuous function on a closed interval converges uniformly the theorem is finally proved. ∎

The derivatives of the utility functions $u(c) = \log c$, $u(c) = c^\gamma, 0 < \gamma < 1$, $u(c) = ke^{-c}$ usually found in the literature are convex functions. For these cases, we can state the following theorem.

THEOREM 1.10. *If $u'(c)$ is a convex function, then $p_t(y)$ is also convex for all t.*

Proof. For $t = 1$ it is immediate, since $p_1(y) = u'(c)$.

Now, assume that $p_{t-1}(y)$ is convex, and suppose that $p_t(y)$ is not convex, i.e., there exist y_1 and $y_2 \in (0, +\infty)$, and $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$ such that for $\mathbf{y} = \lambda_1 y_1 + \lambda_2 y_2$ we have

$$p_t(\mathbf{y}) > \lambda_1 p_t(y_1) + \lambda_2 p_t(y_2). \tag{1.21}$$

We need to consider three cases depending on the values that $x_t(y)$ assumes at y_1, y_2, and $\mathbf{y}$. We will consider only the case

$$0 \leqslant x_t(y_1) \leqslant y_1, \qquad 0 \leqslant x_t(y_2) \leqslant y_2, \qquad 0 < x_t(\mathbf{y}) < \mathbf{y}$$

and whenever the three inequalities hold. From (1.15) and (1.14) it follows that

$$\begin{aligned} p_t(y_1) &= u'(c_t(y_1)) \\ p_t(y_2) &\geqslant u'(c_t(y_2)) \\ p_t(\mathbf{y}) &= u'(c_t(\mathbf{y})) \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} p_t(y_1) &\geqslant Ep_{t-1}(x_t(y_1) + \omega) \\ p_t(y_2) &\geqslant Ep_{t-1}(x_t(y_2) + \omega) \\ p_t(\mathbf{y}) &= Ep_{t-1}(x_t(\mathbf{y}) + \omega). \end{aligned} \tag{1.23}$$

Now, from (1.21) and (1.22) we have

$$\begin{aligned} u'(c_t(\mathbf{y})) &> \lambda_1 u'(c_t(y_1)) + \lambda_2 u'(c_t(y_2)) \\ &\geqslant u'(\lambda_1 c_t(y_1) + \lambda_2 c_t(y_2)) \end{aligned}$$

by convexity of $u'(c)$. Thus,

$$c_t(\mathbf{y}) < \lambda_1 c_t(y_1) + \lambda_2 c_t(y_2)$$

by monotonicity of $u'(c)$. And consequently

$$x_t(\mathbf{y}) > \lambda_1 x_t(y_1) + \lambda_2 x_t(y_2). \tag{1.23}$$

Now, from (1.21) and (1.23) we get

$$\begin{aligned} p_t(\mathbf{y}) &> \lambda_1 E p_{t-1}(x_t(y_1) + \omega) \\ &\quad + \lambda_2 E p_{t-1}(x_t(y_2) + \omega) \\ &\geqslant E p_{t-1}(\lambda_1 x_t(y_1) + \lambda_2 x_t(y_2) + \omega), \end{aligned} \tag{1.24}$$

where in the last inequality we used the convexity of $p_{t-1}(y)$. Finally from (1.24) and (1.23) we have

$$p_t(\mathbf{y}) > E p_{t-1}(x_t(\mathbf{y}) + \omega) = p_t(\mathbf{y})$$

by (1.22), which is a contradiction.

2. Bounds for the Limiting Policy

If $\omega \equiv v$, i.e., ω is a degenerated random variable, then the corresponding prices and policies will be denoted by $p_t^v(y)$ and $(x_t^v(y), c_t^v(y))$; the optimality conditions (1.12) and (1.13) are

$$p_t^v(y) \geqslant p_{t-1}^v(x_{t-1}^v(y) + v) \tag{2.1}$$

with equality if $x_t^v(y) > 0$, and

$$p_t^v(y) \geqslant u'(c_t^v(y)) \tag{2.2}$$

with equality if $c_t^v(y) > 0$.

Now we will prove a theorem that will give a lower bound for $c_t(y)$, i.e., we will compare $c_t(y)$ with the optimal consumption policy $c_t^0(y)$ of a "pessimistic consumer" who assumes $\omega \equiv 0$.

Theorem 2.1. *$p_t(y) \leqslant p_t^0(y)$ for all t, and consequently $c_t(y) \geqslant c_t^0(y)$ for all t.*

Proof. The proof will be done by induction.

(1) For $t = 1$ it is true since $p_1(y) = p_1^0(y) = u'(y)$.

(2) Assume it is true for $t - 1$, i.e., $p_{t-1}(y) \leqslant p_{t-1}^0(y)$. We need to consider two cases, depending on the values that $x_t(y)$ assumes.

Case i. $x_t(y) = 0$. Then $c_t(y) = y$ and $p_t(y) = u'(y)$. So, $p_t(y) \leqslant p_t^0(y)$ since $c_t^0(y) \leqslant y$ and $u'(c)$ is decreasing function.

Case ii. $x_t(y) > 0$. By contradiction suppose that $p_t(y) > p_t^0(y)$. Then,

$$c_t(y) \leqslant c_t^0(y) \tag{2.3}$$

and

$$x_t(y) \geqslant x_t^0(y). \tag{2.4}$$

Now, from the optimality condition (2.1) we get

$$p_t(y) = Ep_{t-1}(x_t(y) + \omega). \tag{2.5}$$

Thus,

$$p_t(y) \leqslant Ep_{t-1}(x_t^0(y) + \omega) \tag{2.6}$$

since from (2.4) $x_t(y) \geqslant x_t^0(y)$ and $p_{t-1}(y)$ is a decreasing function

Now, from (2.6) using the fact that $p_{t-1}(y)$ is decreasing, and the induction hypothesis we get

$$\begin{aligned} p_t(y) &\leqslant p_{t-1}(x_t^0(y)) \\ &\leqslant p_{t-1}^0(x_t^0(y)) \\ &\leqslant p_t^0(y) \text{ by (4.1),} \end{aligned}$$

which is a contradiction. ∎

An immediate consequence is the following corollary.

COROLLARY 2.2. *If* $c(y) = \lim_{t\to\infty} c_t(y)$ *and* $c^0(y) = \lim_{t\to\infty} c_t^0(y)$ *then* $c^0(y) \leqslant c(y)$.

In a similar way we can prove the following theorem and corollary that compare $c_t(y)$ with the consumption $c_t^A(y)$ of an "optimistic consumer" who assumes $\omega = A$.

THEOREM 2.3. $p_t(y) \geqslant p_t^A(y)$ *for all* t *and consequently* $c_t(y) \leqslant c_t^A(y)$ *for all* t.

COROLLARY 2.4. *If* $c(y) = \lim_{t\to\infty} c_t(y)$ *and* $c^A(y) = \lim_{t\to\infty} c_t^A(y)$ *then* $c(y) \leqslant c^A(y)$.

If we consider the cases in which $u'(c)$ is a convex function, we will be able to compare the optimal policy $c_t(y)$ with the corresponding solution $c_t^{\omega}(y)$ of the deterministic problem in which ω is replaced by $\omega = E\omega$, i.e., we will show that $c_t(y) \leqslant c_t^{\omega}(y)$.

(1) It is true for $t = 1$, since $p_1(y) = p_1^{\omega}(y) = u'(y)$.

(2) Assume that it is true for $t - 1$. We need to consider two cases depending on the values that $x_t^{\omega}(y)$ assumes.

Case i. $x_t^{\omega}(y) > 0$. By contradiction suppose that $p_t(y) < p_t^{\omega}(y)$. Thus,

$$c_t(y) \geqslant c_t^{\omega}(y), \tag{2.7}$$

and consequently

$$x_t(y) \leqslant x_t^{\omega}(y). \tag{2.8}$$

Now, from (2.2) and (2.8) and the fact that $p_{t-1}(y)$ is a decreasing function we get:

$$\begin{aligned} p_t(y) &\geqslant E\, p_{t-1}(x_t(y) + \omega) \\ &\geqslant p_{t-1}(x_t^{\omega}(y) + \omega). \end{aligned} \tag{2.9}$$

But, from Theorem 1.9 we have that $p_t(y)$ is a convex function. Hence from (2.9), using the convexity and the induction hypothesis we get

$$\begin{aligned} p_t(y) &\geqslant E\, p_{t-1}(x_t^{\omega}(y) + \omega) \\ &= p_{t\ 1}(x_t^{\omega}(y) + \omega) \\ &\geqslant p_{t-1}^{\omega}(x_t^{\omega}(y) + \omega) \\ &= p_t^{\omega}(y) \end{aligned}$$

by (2.2), which is a contradiction.

Case ii. $x_t^{\omega}(y) = 0$. Then, $c_t^{\omega}(y) = y$ and $p_t^{\omega}(y) = u'(y)$. Now, $p_t(y) \geqslant u'(c_t(y))$. So, $p_t(y) \geqslant p_t^{\omega}(y)$, since $c_t(y) \leqslant y$ and $u'(c)$ is a decreasing function. ∎

The following corollary is an immediate consequence.

COROLLARY 2.6. *If* $c^{\omega}(y) = \lim_{t\to\infty} c_t^{\omega}(y)$ *then* $c(y) \leqslant c^{\omega}(y)$.

THEOREM 2.7. *The limiting policy satisfies the following inequalities*:

$$c^0(y) \leqslant c(y) \leqslant c^A(y) \tag{2.10}$$

Moreover if $u'(c)$ *is a convex function then*

$$c^0(y) \leqslant c(y) \leqslant c^{\omega}(y). \tag{2.11}$$

Proof. It is an immediate consequence of Corollary 2.4 and Corollary 2.6. ∎

COROLLARY 2.8. *The policy $c_t(y)$ satisfies the following inequalities*

$$\frac{y}{t} \leqslant c_t(y) \leqslant \frac{y}{t} + \frac{t-1}{t} A \qquad \text{for all } t. \tag{2.12}$$

Moreover if $u'(c)$ is convex then

$$\frac{y}{t} \leqslant c_t(y) \leqslant \frac{y}{t} + \frac{t-1}{t} \omega \qquad \text{for all } t.$$

Proof. From Theorem 2.1 and Theorem 2.3 we have without any assumption about $u'(c)$ that

$$c_t^0(y) \leqslant c_t(y) \leqslant c_t^A(y) \qquad \text{for all } t.$$

But, it is know that $c_t^0(y) = y/t$ and $c_t^A(y) = \min(y, (y/t) + ((t-1)/t) A)$. So, (2.12) holds. Now, for $u'(c)$ convex, using Theorem 2.5 we get

$$c_t(y) \leqslant c_t^\omega(y) \qquad \text{for all } t.$$

Hence,

$$c_t(y) \leqslant \frac{y}{t} + \frac{t-1}{t} \omega$$

since it is known that $c_t^\omega(y) = \min(y, (y/t) + ((t-1)/t)\,\omega)$. Thus, (2.13) holds. ∎

Now we will obtain the surprising fact that $c(y)$ is uniformly bounded.

COROLLARY 2.9. *The limiting policy $c(y)$ satisfies the following inequalities*:

$$0 \leqslant c(y) \leqslant A.$$

Moreover if $u'(c)$ is convex then

$$0 \leqslant c(y) \leqslant \omega.$$

Proof. It follows immediately from Theorem 2.8 by taking limits as $t \to \infty$ in (2.12) and (2.13).

We should mention at this point that the result $0 \leqslant c(y) \leqslant \omega$, for the case in which $u'(c)$ is convex, has been obtained with very simple arguments. Unfortunately for the more general case, i.e., we do not necessarily assume that $u'(c)$ is convex, and to obtain the surprising fact

that $0 < c(y) < \omega$ (observe the strict inequality) it will be necessary to use other results, some of them obtained in [16], with a "little bit" of work, as it will be done in the following paragraphs.

LEMMA 2.10. *The sequence of functions* $\{p_t(y)\}$ *converges uniformly in any finite and closed interval to a continuous nonincreasing function* $p(y)$.

Proof. It converges pointwise, since

$$p_t(y) \geqslant p_{t-1}(y)$$

and

$$\begin{aligned} p_t(y) &= V_t'(y) \\ &\leqslant V_t'(0) \\ &= u'(0) \end{aligned}$$

by concavity and Theorem 1.3. Now, from the fact that $c_t(y) \geqslant y/t > 0$ for $y > 0$ (Theorem 2.8), and the optimality condition (1.15) we have that

$$p_t(y) = u'(c_t(y)) \qquad \text{for all } y.$$

Taking the limit and using the fact that $u'(c)$ is a continuous function and $c_t(y)$ converges by Theorem 1.9, we will have that

$$p(y) = u'(c(y)).$$

So, $p(y)$ is continuous, since $c(y)$ is continuous from Theorem 1.9. Now, the theorem follows from the fact that $\{p_t(y)\}$ is a sequence of monotonic functions converging pointwise to a continuous function; consequently, it converges uniformly in any finite closed interval. ∎

LEMMA 2.11. *The limiting policy* $(x(y), c(y))$ *satisfies the following relations*:

$$p(y) = u'(c(y)) \tag{2.14}$$

and

$$p(y) \geqslant Ep(x(y) + \omega) \tag{2.15}$$

with equality if $x(y) > 0$.

Proof. From the optimality conditions (1.15) and (1.4) we get

$$p_t(y) = u'(c_t(y))$$

since $c_t(y) > 0$ for $y > 0$, $p_t(0) = u'(0)$, and

$$p_t(y) \geqslant Ep_{t-1}(x_t(y) + \omega)$$

with equality if $x_t(y) > 0$. Now, taking the limit and using the fact that $\{p_t(y)\}$ converges uniformly in any finite closed interval we get (2.14) and (2.15). ∎

LEMMA 2.12. *If $c(y) = 0$ for some $y > 0$ then $c(y) \equiv 0$.*

Proof. Let $\mathbf{y} = \sup\{y \mid c(y) = 0\}$. The supremum is actually achieved from the continuity of $c(y)$. If $\mathbf{y} = +\infty$ we have done. Now suppose that $\mathbf{y} < +\infty$. Then,

$$\begin{aligned} p(\mathbf{y}) &= u'(c(\mathbf{y})) \\ &= u'(0). \end{aligned}$$

Now, for $\omega > 0$, $c(\mathbf{y} + \omega) > 0$ and $p(\mathbf{y} + \omega) < u'(0)$. Let ϵ be any number such that $0 < \epsilon < A$. Then,

$$\begin{aligned} p(\mathbf{y}) &= E\, p(\mathbf{y} + \omega) \\ &= \int_0^{\epsilon} p(\mathbf{y} + \omega)\, \nu(d\omega) + \int_{\epsilon}^{A} p(\mathbf{y} + \omega)\, \nu(d\omega) \\ &\leqslant u'(0)\, \nu(0, \epsilon) + p(\mathbf{y} + \epsilon)(1 - \nu(0, \epsilon)) \\ &< u'(0)\, \nu(0, \epsilon) + u'(0)(1 - \nu(0, \epsilon)) \\ &< u'(0), \end{aligned}$$

which is a contradiction. ∎

THEOREM 2.13. *If $c(y) > 0$ for $y > 0$ then $p(y)$ is strictly decreasing.*

Proof. By contradiction suppose not, i.e., there exists $y' > y \geqslant 0$ such that

$$p(y') = p(y) = \mathbf{p}. \tag{2.16}$$

Let,

$$\mathbf{y} = \min\{y \mid p(y) = \mathbf{p},\ 0 \leqslant y \leqslant y'\}.$$

Now, $\mathbf{y} > 0$ since $c(y') > 0$ and

$$\begin{aligned} p(\mathbf{y}) &= p(y') \\ &= u'(c(y')) \\ &< u'(0). \end{aligned}$$

Hence

$$p(y) > p(\mathbf{y}) \qquad \text{for} \quad y < \mathbf{y}. \tag{2.17}$$

From (2.15) we have that

$$p(\mathbf{y}) \geqslant Ep(x(\mathbf{y}) + \omega) \tag{2.18}$$

and

$$p(y') = Ep(x(y') + \omega). \tag{2.19}$$

Hence, from (2.16), (2.18), and (2.19) we have that

$$Ep(x(y') + \omega) \geqslant Ep(x(\mathbf{y}) + \omega).$$

Therefore,

$$E[p(x(y') + \omega) - p(x(\mathbf{y}) + \omega)] \geqslant 0. \tag{2.20}$$

But

$$E[p(x(y') + \omega) - p(x(\mathbf{y}) + \omega)] \leqslant 0 \tag{2.21}$$

since $x(y') > x(\mathbf{y})$ and $p(y)$ is nondecreasing. From (2.20) and (2.21) we have

$$E[p(x(y') + \omega) - p(x(\mathbf{y}) + \omega)] = 0.$$

Hence,

$$p(x(y') + \omega) = p(x(\mathbf{y}) + \omega) \qquad \text{for} \quad \omega = 0, \tag{2.22}$$

since $\phi(\omega) \equiv p(x(y') + \omega) - p(x(\mathbf{y}) + \omega)$ is nonpositive, continuous and $\nu(0, \omega) > 0$ for $\omega > 0$. From the fact that $p(\mathbf{y}) = p(y')$ it follows that

$$x(y') = x(\mathbf{y}) + y' - \mathbf{y}. \tag{2.23}$$

Thus from (2.23) and (2.22) we have

$$p(x(\mathbf{y})) = p(x(\mathbf{y}) + y' - \mathbf{y}),$$

and consequently for $y \in [x(\mathbf{y}), x(\mathbf{y}) + y' - \mathbf{y}]$ we have

$$p(y) = p(x(\mathbf{y})).$$

If $x(\mathbf{y}) = 0$, then $c(y) = 0$ for $y \in [x(\mathbf{y}), x(\mathbf{y}) + y' - \mathbf{y}]$ which contradicts the fact that $c(y) > 0$ for $y > 0$. For $x(\mathbf{y}) > 0$ consider the following cases:

Case i. $x(\mathbf{y}) + y' - \mathbf{y} \geqslant \mathbf{y}$. Then it follows that $\mathbf{y} \in [x(\mathbf{y}), x(\mathbf{y}) + y' - \mathbf{y}]$ and hence, $p(x(\mathbf{y})) = p(\mathbf{y})$, which contradicts (2.17) since $x(\mathbf{y}) < \mathbf{y}$.

Case ii. $x(\mathbf{y}) + y' - \mathbf{y} < \mathbf{y}$. Now repeat the proof starting with the new interval $[x(\mathbf{y}), x(\mathbf{y}) + y' - \mathbf{y}]$ and computing the new

$$\mathbf{y} = \min\{y \mid p(y) = p(x(\mathbf{y})), y \geqslant 0, y \leqslant x(\mathbf{y})\}$$

The repetition of Case ii will generate a sequence of intervals that are nonoverlapping, have nondecreasing lengths, and are all contained in the interval $[0, y']$. So, Case ii can be repeated only a finite number of times. ∎

Now we show with a proof similar to the one given in [16] that

$$\lim_{T\to\infty} \frac{V_T(y)}{T} = u(\omega).$$

LEMMA 2.14.

$$\lim_{T\to\infty} \frac{V_T(y)}{T} \geqslant u(\omega), \qquad \textit{where} \quad \omega = E\omega.$$

Proof. It suffices to show that $\lim_{T\to\infty} (V_T(0)/T) \geqslant u(\omega)$, since $V_T(y) \geqslant V_T(0)$. For $V_T(0)$, consider the policy $c_t(y) = \min(y, \omega)$ for all $1 \leqslant t \leqslant T$. If y^t (observe the superscript) is the stock level at the beginning of period t when we follow the policy $c_t(y) = \min(y, \omega)$, then

$$y^t = \max(y^{t-1} + z^t, 0) \qquad t \geqslant 2$$
$$y^1 = 0$$

where $z^t = \omega - \omega$, and hence $Ez^t = 0$.

When $y^k = 0$, we have at the kth period (trial is the usual terminology in Renewal Theory) a renewal point, since at this point the process will start all over again. If $N(T)$ is the number of renewals in T periods, then the expected return from the policy $c_t(y) = \min(y, \omega)$ for a T period problem is greater or equal to $u(\omega)(T - E(N(T)) - 1)$. Hence

$$V_T(0) \geqslant u(\omega)(T - E(N(T)) - 1).$$

Thus, to show that

$$\lim_{T\to\infty} \frac{V_T(0)}{T} \geqslant u(\omega)$$

it suffices to show that

$$\lim_{T\to\infty} \frac{EN(T)}{T} = 0.$$

Let τ_i be the number of trials between the ith and $(i-1)$th renewals. From Renewal Theory we have

$$\lim_{T\to\infty} \frac{EN(T)}{T} = \frac{1}{E\tau_1} \quad \text{if} \quad E\tau_1 < \infty$$

and

$$\lim_{T\to\infty} \frac{EN(T)}{T} = 0 \quad \text{if} \quad E\tau_1 = \infty.$$

(For a proof, see [13].) But, from the Random Walk Theory it can be shown that $E\tau_1 = \infty$ if $Ez^t = 0$ [3, p. 397]. Thus, the lemma holds. ∎

LEMMA 2.15.

$$\lim_{T\to\infty} \frac{V_T(y)}{T} \leqslant u(\omega).$$

Proof. It suffices to show that

$$V_T(y) \leqslant Tu\left(\frac{y}{T} + \frac{(T-1)}{T}\,\omega\right).$$

It is true for $T = 1$, since $V_1(y) \leqslant u(y)$. Assume it is true for t, i.e.,

$$V_t(y) \leqslant tu\left(\frac{y}{t} + \frac{t-1}{t}\,\omega\right) \quad \text{for all} \quad y. \tag{2.24}$$

Now,

$$\begin{aligned} V_{t+1}(y) &= u(c_t(y)) + EV_t(x_t(y) + \omega) \\ &\leqslant u(c_t(y)) + V_t(x_t(y) + \omega) \end{aligned} \tag{2.25}$$

by concavity of $V_t(y)$. Thus, from (2.24) and (2.25) we get:

$$\begin{aligned} V_{t+1}(y) &\leqslant u(c_t(y)) + tu\left(\frac{1}{t}\,(x_t(y) + \omega) + \frac{t-1}{t}\,\omega\right) \\ &= (t+1)\left(\frac{1}{t+1}\,u(c_t(y))\right) + \frac{t}{t+1}\,u\left(\frac{1}{t}\,(x_t(y) + t\omega)\right) \\ &\leqslant (t+1)\,u\left(\frac{1}{t+1}\,(y + t\omega)\right) \end{aligned} \tag{2.26}$$

by concavity of $u(c)$. ∎

From the previous two lemmas we can state the following Theorem.

THEOREM 2.16.

$$\lim_{T\to\infty} \frac{V_T(y)}{T} = u(\omega) \qquad \textit{for all} \quad y.$$

The sequence $V_t(y)$ diverges as $t \to \infty$. To study the behavior of the limiting policy $c(y)$ we will define a special function, $h_t(y)$, that has the same properties of $V_t(y)$, but which sequence converges.

DEFINITION. Let $h_t(y) = V_t(y) - V_t(0)$. Using the fact that

$$\begin{aligned} V_t(0) &= u(0) + EV_{t-1}(\omega) \\ &= EV_{t-1}(\omega), \end{aligned}$$

since by assumption $u(0) = 0$, it can be shown that the function $h_t(y)$ satisfies the following relations

$$\begin{aligned} h_t(y) + V_t(0) - V_{t-1}(0) &= \max_{\substack{c+x=y \\ c\geqslant 0, x\geqslant 0}} \{u(c) + Eh_{t-1}(x+\omega)\} \\ h_1(y) = V_1(y) &= \max_{\substack{c+x=y \\ c\geqslant 0, x\geqslant 0}} \{u(c)\} \end{aligned} \tag{2.27}$$

and

$$V_t(y) = h_t(y) + E\sum_{k=1}^{t-1} h_k(\omega). \tag{2.28}$$

THEOREM 2.17. *The function $h_t(y)$ is strictly concave, increasing, and differentiable for all t. Furthermore,*

$$h_t'(y) = p_t(y)$$

and if $(x_t(y), c_t(y))$ is the solution of (1.1), *it is also the solution of* (2.27).

Proof. It is an immediate consequence of the fact that $h_t(y)$ and $V_t(y)$ differ by a constant. ∎

Now, we will state without proof the following lemma (for a proof see [14]):

LEMMA 2.18. *Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$ and such that $\{f_n(x)\}$ converges for some x_0 on $[a, b]$. If the sequence of derivatives $\{f_n'\}$ converges uniformly on $[a, b]$ then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad a \leqslant x \leqslant b.$$

THEOREM 2.19. *The sequence of functions $h_t(y)$ converges uniformly in any finite interval* $[0, b]$. *Furthermore if* $h(y) = \lim_{t\to\infty} h_t(y)$, *then*

$$h'(y) = \lim_{t\to\infty} h_t(y) = p(y).$$

Proof. (I) $h_t(0) = 0$ for all t. Hence, $\lim_{t\to\infty} h_t(0) = 0$.

(II) $h_t(y)$ is differentiable for all t, $y \geqslant 0$.

(III) $h_t'(y) = p_t(y)$ converges uniformly on any finite interval $[0, b]$ by Lemma 5.4.

Hence, the theorem follows from the previous lemma. ∎

COROLLARY 2.20.

$$\lim_{t\to\infty} Eh_t(\omega) = Eh(\omega).$$

LEMMA 2.21.

$$Eh(\omega) = u(\omega).$$

Proof. From (2.28) we have

$$V_t(y) = h_t(y) + \sum_{k=1}^{t-1} Eh_k(\omega).$$

Now, dividing by t, and taking the limit we get

$$u(\omega) = Eh(\omega)$$

where we have used the following facts:

(I) $\lim_{t\to\infty} \dfrac{V_t(y)}{t} = u(\omega)$ from Theorem 2.16.

(II) $\lim_{t\to\infty} \dfrac{\sum_{k=1}^{t-1} Eh_k(\omega)}{t} = Eh(\omega)$ from Corollary 2.20.

(III) $\lim_{t\to\infty} (h_t(y)/t) = 0$, since $h_t(y)$ is uniformly bounded in any finite interval. ∎

LEMMA 2.22.

$$\lim_{t\to\infty} V_t(0) - V_{t-1}(0) = u(\omega).$$

Proof. From the definitions of $V_t(0)$, $V_{t-1}(0)$ and $h_{t-1}(y)$ we have

$$V_t(0) - V_{t-1}(0) = u(0) + EV_{t-1}(\omega) - V_{t-1}(0)$$
$$= Eh_{t-1}(\omega).$$

Now, taking the limit and using Corollary 2.20 we get that

$$\lim_{t\to\infty} V_t(0) - V_{t-1}(0) = u(\omega). \quad \blacksquare$$

LEMMA 2.23.

$$h(y) + u(\omega) = u(c(y)) + Eh(x(y) + \omega).$$

Proof. From (2.27) we have

$$h_t(y) + V_t(0) - V_{t-1}(0) = u(c_t(y)) + Eh_{t-1}(x_t(y) + \omega).$$

Now, taking the limit, using Lemma 2.22 and the uniform convergence of $h_t(y)$ in any finite interval we get

$$h(y) + u(\omega) = u(c(y)) + Eh(x(y) + \omega). \quad \blacksquare$$

THEOREM 2.24. *The limiting policy $c(y)$ is always positive for $y > 0$.*

Proof. By contradiction suppose not, i.e., $c(y) = 0$ for some $y > 0$. Hence, from Lemma 2.12 $c(y) = 0$. Now, from Lemma 2.21 we have

$$u(\omega) = Eh(\omega). \tag{2.29}$$

But $p(y) = u'(0)$ for all y since $c(y) = 0$. Hence,

$$h(y) = u'(0)\, y \tag{2.30}$$

Finally, from (2.29) and (2.30) we get

$$u(\omega) = u'(0)\, \omega,$$

which is a contradiction, since $u(c)$ is strictly concave. $\blacksquare$

LEMMA 2.25. *The function $h(y)$ is strictly concave.*

Proof. From Theorem 2.24 $c(y) > 0$ for $y > 0$; hence, from Theorem 2.13 $p(y) = h'(y)$ is strictly decreasing and consequently $h(y)$ is strictly concave. $\blacksquare$

Now, we are able to show the surprising fact that $c(y) < \omega$ for $0 < y < +\infty$.

THEOREM 2.26. *The limiting policy satisfies the following inequalities*

$$0 < c(y) < \omega \qquad \textit{for} \quad 0 < y < +\infty.$$

Proof. From Lemma 2.23 we have

$$h(y) + u(\omega) = u(c(y)) + Eh(x(y) + \omega).$$

Now, suppose by contradiction that for some $y < +\infty$, $c(y) = \omega$. Hence,

$$h(y) + u(\omega) = u(\omega) + Eh(y - \omega + \omega).$$

Thus,

$$h(y) = Eh(y - \omega + \omega) < h(y)$$

by strict concavity of $h(y)$, which is a contradiction. ∎

THEOREM 2.27. $\lim_{y\to\infty} c(y) = \omega$.

Proof. Suppose not, i.e., $\lim_{y\to\infty} c(y) = \delta < \omega$. Hence,

$$\lim_{y\to\infty} h'(y) = \lim_{y\to\infty} u'(c(y)) = u'(\delta).$$

From Lemma 2.23 we have

$$u(\omega) - u(c(y)) = Eh(y - c(y) + \omega) - h(y). \tag{2.31}$$

Now, from the mean value theorem we get

$$h(y - c(y) + \omega) - h(y) = h'(y_\omega(y))(\omega - c(y)), \tag{2.32}$$

where

$$y_\omega(y) \in [y - c(y) + \omega, y] \qquad \text{if} \quad c(y) \geqslant \omega$$

and

$$y_\omega(y) \in [y, y - c(y) + \omega] \qquad \text{if} \quad c(y) \geqslant \omega.$$

Replacing (2.32) in (2.31) we get

$$u(\omega) - u(c(y)) = Eh'(y_\omega(y))(\omega - c(y)).$$

Now, taking the limit as $y \to \infty$ and using the fact that $y_\omega(y) \to \infty$ when $y \to \infty$ ($-\omega \leqslant -c(y) + \omega \leqslant A$ since $0 \leqslant c(y) \leqslant \omega$ and $\omega \leqslant A$), we get

$$\begin{aligned} u(\omega) - u(\delta) &= Eu'(\delta)(\omega - \delta) \\ &= u'(\delta)(\omega - \delta) \end{aligned}$$

which is a contradiction since $u(c)$ is strictly concave and $\omega < \delta$. ∎

Now we will obtain some properties of the capital y^t accumulated in t periods.

THEOREM. *The sequence of capitals* $\{y^t\}$ *converges almost surely to* $+\infty$.[1]

Proof. It suffices to show that $\{y^t\}$ converges almost surely, since either $\{y^t\}$ does not converge or it converges almost surely to $+\infty$. From the optimality condition (2.15), we have that

$$p(y^t) \geqslant E[p(y^{t+1}) \mid y^t]. \tag{2.33}$$

Now, if we define $p^t \equiv p(y^t)$ we will have from (2.23) that

$$p^t \geqslant E[p^{t+1} \mid p^t] \tag{2.34}$$

where we have used the fact that $p(y)$ is a strictly decreasing function, and hence, for a given p^t, y^t is uniquely defined. From (2.34) we have that $\{p^t\}$ is a positive supermartingale, and hence, it converges almost surely to a random variable $\mathbf{p}$. Since $p(y)$ is a continuous strictly decreasing function it is possible to define a continuous function $g(p)$ such that

$$y^t = g(p^t).$$

The Theorem now follows from the fact that $\{p^t\}$ converges almost surely and $g(p)$ is a continuous function. ∎

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Public Finance in a Keynesian Temporary Equilibrium

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1. INTRODUCTION

The theory of public finance has always been separated into two compartments. In the first of these we find general equilibrium models of allocative and distributive effects of government activity, and in the second we have income–expenditure models for analyzing the effects of such activity on employment and output. While each approach contributes important insights, each in isolation has serious limitations. Differential incidence models, to be useful, must rely on rapid adjustment to equilibrium, and there are well-known difficulties concerning that. Income–expenditure models are notorious for their neglect of relative prices. In examining the effects of various taxes and government expenditures on the economy, it therefore has been necessary to use these two models separately, and supplement them by ad hoc judgements.

Policy discussions in practice usually rely on crude income–expenditure models alone. The key considerations are the size and the balance of the budget. Except for some attention paid to differences in saving propensities, important distinctions among different fiscal measures are thus blurred. For example, it is recognized that indirect taxes can have price-inflationary effects in addition to demand-deflationary ones, but we have no systematic framework for analyzing the two together.

The means for providing a unified approach are now available. Following the work of Clower [6], it is possible to construct models of short-run equilibrium with interrelated markets, where some of the markets clear by quantity rationing and others by price adjustment. The dual decision framework of demand in these models, where the realized transactions in rationed markets can influence effective demand in other markets, provides a way to combine a Keynesian consumption function concept with the usual price-theoretic analysis of consumer behavior. Such models, therefore, seem well suited for joint analyses of price and employment effects of government activity.

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It might be said that the conventional Keynesian consumption function, by introducing actual income as a determinant of effective consumption demand, already incorporates the Clower hypothesis. However, this is an ad hoc modification, not based on any rigorous formulation of the dual decision problem. When other arguments such as the price level, interest rates, or real cash balances are to be considered, they are introduced again in an ad hoc manner. It is therefore difficult to interpret the results in terms of relations between price and income effects. A more rigorous formulation is clearly necessary.

Models that are better in this regard have been studied at an abstract level, with emphasis on testing their overall consistency and leading to results consisting largely of existence theorems.[1] In this paper I shall put one such model to use in studying the effects of government policy in conditions of unemployment. I hope this will prove to be a useful first step in integrating the two approaches to public finance, and also in providing economists the same kind of intuition about the neo-Keynesian models as they now have for the general equilibrium and income–expenditure models.

I shall derive and compare the effects of taxation, government purchases, and fiat money supply on various price and quantities. Concerning tax incidence, I shall show that the case of more than 100% incidence on producers is more likely in a Keynesian context than it would be in general equilibrium. Government purchases will have multiplier effects, and the expressions for these will involve price elasticities and expectations as well as the traditional marginal propensity to consume.

I should emphasize one point at the outset. While I shall obtain some new and interesting results using this approach, the model itself is too simplified to be directly applicable. This paper therefore should be seen as an indication of the potentialities of the approach, and should not be contrasted unfavorably with the sophisticated level that the general equilibrium and income–expenditure models have reached after years of research.[2] In the concluding section, I shall comment on the shortcomings of the model used here, and also suggest lines for future development.

[1] See Grandmont and Laroque [8] and Benassy [3]. One exception is the model of a supply-constrained full employment situation developed by Barro and Grossman [2].

[2] Blinder and Solow [4] and Barro and Grossman [1], in different ways, provide important examples of the way in which the conventional consumption function can be modified in ad hoc ways to allow arguments besides income. These yield some results similar to mine, but their interpretation cannot be soundly based on consumer theory.

2. Equilibrium with Rationing

The model I shall use is that of Grandmont and Laroque [8], with minor modifications and different notation. The transactors in this model are concerned with two periods, but prevailing markets only provide a Hicksian temporary equilibrium for goods and services for the first period. Agents therefore must form expectations concerning the state of markets that will prevail when the second period arrives. For simplicity, it is assumed that such expectations are held with subjective certainty, but of course there is no presumption that they will turn out to be correct. The expectations are functions of the current market experience.

Current markets may clear either by quantity rationing or by price adjustment, according to specified rules. In particular, it is interesting to consider the case where the labor market clears by rationing and all other markets by price adjustment.[3] Much of the difficulty of proving existence arises from the problems of proper piecing together of possible regimes, but I shall circumvent the difficulty here by assuming that the rationing constraint in the labor market is binding in the first period and expected to remain binding in the second. The quantity of labor services traded is then set by the short side of the market, i.e., by the firms' demand. This is rationed among the sellers by some rule that need not be specified. Commodity markets clear by price adjustment; in fact I shall make matters simple by assuming that at each date there is only one commodity. This suffices to illustrate some of the important new features of the model, and complications arising from cross-substitution among several commodities are best left for future elaboration. There is no possibility of borrowing or lending, but each agent enters the economy with a fixed endowment of money which he can use in the first period or carry over to the second, and I shall assume that the endowments of money are sufficient to ensure that no one finds the constraint of carrying over a nonnegative amount binding, i.e., there is no desire to borrow at a zero rate of interest. I shall also assume that all consumers and all firms are identical within their respective groups, and thus, in effect, that there is only one of each.

Consider the consumer first. He comes to the first period markets with an endowment M_1 of money. This includes any amount received from the government, and his expectations of distribution of profit by the firm, but it is assumed that the current actions of firms do not affect such expectations. This seems reasonable in a short-run model. The consumer faces prices q_1 for commodities and v_1 for labor. His employment is constrained at e_1, and he demands x_1 of commodities and plans to carry m_2 of money

[3] See Bliss [5] for a defense of this interpretation of Keynesian unemployment.

to the second period. He then expects to face prices q_2 and v_2, to be constrained to sell e_2 of labor and purchase x_2 of commodities. Thus, his budget constraints are

$$q_1x_1 - v_1e_1 + m_2 = M_1,$$
$$q_2x_2 - v_2e_2 = m_2,$$

and he wishes to maximize utility

$$u(x_1, x_2, e_1, e_2).$$

Since utility does not depend explicitly on m_2, and since I am assuming that the constraint $m_2 \geqslant 0$ is not binding, the constraints can be collapsed into one,

$$q_1x_1 + q_2x_2 = M_1 + v_1e_1 + v_2e_2. \tag{1}$$

With e_1 and e_2 fixed, the choice variables are x_1 and x_2. It is then convenient to describe the consumer's choice by means of the "partial" expenditure function

$$E(q_1, q_2, e_1, e_2, u) = \min \{q_1x_1 + q_2x_2 \mid u(x_1, x_2, e_1, e_2) \geqslant u\}. \tag{2}$$

This will be increasing in all arguments, concave and homogeneous of degree one in q_1 and q_2, and convex in e_1 and e_2. Then, given the quantities which are taken as parameters by the consumer, his utility level and his first period demand for commodities[4] is defined by

$$E(q_1, q_2, e_1, e_2, u) = M_1 + v_1e_1 + v_2e_2, \tag{3}$$
$$x_1 = E_1(q_1, q_2, e_1, e_2, u). \tag{4}$$

Subscripts on functions, as usual, indicate partial derivatives with respect to the appropriate argument.

As mentioned before, q_2, v_2, and e_2 will be functions of the current market observations. I shall assume for simplicity that each is a function only of the corresponding variable for the first period. Grandmont and Laroque assume that e_2 is constant. This has to do with ensuring continuity for existence proofs where it is not specified in advance whether rationing will occur, and it is therefore important to have continuity across possible regimes. I shall neglect this problem.

As in the first model of Grandmont and Laroque, I shall assume that the producer is not rationed on any market, and that he wishes to maxi-

[4] This is the rigorously formulated compensated demand function under rationing that we have been striving for. A related approach can be found in Pollak [9].

mize profit given his price expectations. He enters Period 1 with a given amount Y_1 of commodities and enough money to finance his activities. He can combine commodities y_1 and labor l_1 to produce commodities y_2 available in Period 2. As I shall consider the economy with taxation, the consumer and the producer can face different prices. Suppose the producer faces a price p_1 for commodities in Period 1 and a wage rate w_1 for labor in Period 1, and expects the price of commodities in Period 2 to be p_2, where p_2 is a function of p_1. He will determine his own demand for commodities y_1 (and hence his supply $Y_1 - y_1$ to the rest of the economy) and his demand for labor l_1 to maximize

$$p_1(Y_1 - y_1) + p_2 y_2 - w_1 l_1 = p_1 Y_1 + (p_2 y_2 - p_1 y_1 - w_1 l_1).$$

This choice is best described by means of a cost function. While a more general treatment is no more difficult, I shall be able to interpret the results somewhat more easily by using a cost function of the form

$$y_2^n C(p_1, w_1)$$

where C is increasing, concave, and homogenous of degree one and $n > 1$. Profit maximization will then require

$$p_2 = n y_2^{(n-1)} C(p_1, w_1), \tag{5}$$

and the input demands will be given by

$$y_1 = y_2^n C_1(p_1, w_1), \tag{6}$$

$$l_1 = y_2^n C_2(p_1, w_1). \tag{7}$$

Finally, consider the government. I shall allow five types of policies. The government can levy taxes in Period 1 markets, at specific rates t_1 on commodities and r_1 on labor. It can purchase amounts x_g of commodities and l_g of labor in the first period markets. Finally, it can hand out m_1 units of money to consumers. Any budget deficit is financed by creation of money, and any surplus is destroyed. Thus, the deficit, defined as the rate of net money creation is given by

$$D = q_1 x_g + w_1 l_g - t_1(x_1 + x_g) - r_1(l_1 + l_g) + m_1.$$

It is simpler to net out the tax paid on the government's own purchases, and write

$$D = p_1 x_g + v_1 l_g - t_1 x_1 - r_1 l_1 + m_1. \tag{8}$$

Suppose that the market wage w_1 cannot change within the period. We can then write down the conditions for an equilibrium with rationing:

$$p_1 = q_1 - t_1, \tag{9}$$

$$v_1 = w_1 - r_1, \tag{10}$$

$$x_1 + x_g + y_1 = Y_1, \tag{11}$$

$$e_1 = l_1 + l_g. \tag{12}$$

As a check, note that in (3)–(7) and (9)–(12) we have nine equations, and nine unknowns: u, x_1, y_1, y_2, l_1, e_1, q_1, p_1, and v_1. The second period (expected) values of the last four are uniquely related to the first period values, and therefore need not be considered as separate variables.

Note the assumption that it is the market wage before taxation that is "sticky," producing an equilibrium in which the sellers of labor are rationed. This need not always be the case, and stickiness may appear in v_1, the wage net of tax. This is a possibility worth separate treatment.

3. Comparative Statics

To study the effects of government policy, it will be necessary to find out how the equilibrium shifts in response to various policy changes. The calculations become quite complex, and it is convenient to begin with the comparative statics of each agent considered in isolation.

First the consumer. For him the endogenous variables are x_1 and u, and the exogenous ones are q_1, v_1, e_1, and M_1. Total differentiation of (3) and (4) yields

$$\begin{aligned} E_1\,dq_1 + E_2q_2'(q_1)\,dq_1 + E_3\,de_1 + E_4e_2'(e_1)\,de_1 + E_5\,du \\ = dM_1 + v_1\,de_1 + v_2e_2'(e_1)\,de_1 + e_1\,dv_1 + e_2v_2'(v_1)\,dv_1 \\ dx_1 = E_{11}\,dq_1 + E_{12}q_2'(q_1)\,dq_1 + E_{13}\,de_1 + E_{14}e_2'(e_1)\,de_1 + E_{15}\,du \end{aligned}$$

Substituting from the first of these into the second, we can write

$$dx_1 = -A\,dq_1 + B\,de_1 + \Gamma\,dv_1 + \Delta\,dM_1 \tag{13}$$

where

$$-A = (E_{11} - E_1E_{15}/E_5) + (E_{12} - E_2E_{15}/E_5)\,q_2'(q_1), \tag{14}$$

$$B = [E_{13} + (v_1 - E_3)\,E_{15}/E_5] + [E_{14} + (v_2 - E_4)\,E_{15}/E_5]\,e_2'(e_1), \tag{15}$$

$$\Gamma = (E_{15}/E_5)[e_1 + e_2v_2'(v_1)], \tag{16}$$

$$\Delta = E_{15}/E_5 = \partial x_1/\partial M_1. \tag{17}$$

We see at once that provided x_1 is a normal good, Δ will be positive. Then, provided $v_2'(v_1)$ is nonnegative, i.e., provided an increase in wage in Period 1 does not generate expectations of a decrease in wage in Period 2, Γ will be positive. Both these assumptions seem sensible. I shall similarly assume $q_2'(q_1)$ and $e_2'(e_1)$ (and later $p_2'(p_1)$) to be nonnegative, but the signs of A and B are still not so clear.

Consider A first. We have assumed that E_{15} and $q_2'(q_1)$ are nonnegative. We know from standard consumer theory that E_{11} is nonpositive. Thus only one sign is doubtful, namely that of E_{12}. If commodities at the two dates are complements, this is negative. In fact from the Slutsky–Hicks equation, we recognize the combination in which it enters as the uncompensated derivative $\partial x_1/\partial q_2$. Therefore, it suffices to have the weaker requirement of gross complementarity. If price expectations are inelastic, $q_2'(q_1) = 0$ and E_{12} does not matter. Finally, if they are unit-elastic $q_2'(q_1) = q_2/q_1$, and we have

$$E_{11} + E_{12}q_2/q_1 = (q_1E_{11} + q_2E_{12})/q_2 = 0$$

by the homogeneity of degree zero of E_1 in (q_1, q_2). This last result says simply that if q_2 is expected to change in the same proportion as q_1, then this latter change has no pure substitution effect. Then what is left of A is positive. In view of all these results, it seems safe to assume that A will be positive.

The magnitude of A ultimately must be a matter for empirical study. However, a qualitative remark can be made. Note that E_{11} is the compensated derivative of x_1 with respect to q_1 when employment is fixed. We know from the Le Chatelier–Samuelson principle[6] that it has a smaller numerical value than the corresponding derivative when the wage rate is fixed and labor supply can be chosen optimally, both derivatives being evaluated at the point where the constraint on employment just begins to bind. This leads to the general presumption that demands under rationing are relatively inelastic, and therefore we have reason to think that A will be relatively small.

Finally, consider B. Observe that E_3 is the marginal rate of substitution between e_1 and M_1, i.e., the supply price of labor in Period 1. Under conditions of excess supply and rationing, i.e., involuntary unemployment, in the labor market, we must have $(v_1 - E_3)$ positive. As the same state is expected to prevail in Period 2, $(v_2 - E_4)$ will similarly be positive. Now E_{13} will be positive if an increase in e_1, i.e., a decrease in Period 1 leisure, increases the compensated demand for x_1, i.e., if x_1 is a substitute for

[6] For a recent exposition, see Samuelson [10].

Period 1 leisure. Similarly, E_{14} will be positive if it is a substitute for Period 2 leisure. Thus, if we rule out strong complementarity between x_1 and leisure in either period, we can assume B to be positive. This seems a reasonable assumption at this level of aggregation; in a disaggregated model, we would wish to allow some particular commodities to be complementary to leisure.

The essence of Clower's dual decision hypothesis is that the amount of actual employment affects the demand for commodities. This effect appears in B, but it must be sorted out from other relationships. Examining the terms in B, we first have E_{13}, which gives the substitution effect of e_1 on x_1. Next, the term $-E_3E_{15}/E_5$ gives the general equilibrium income effect. These two can be combined into an overall general equilibrium effect. The term v_1E_{15}/E_5 is the Keynesian income effect, where each additional unit of employment provides v_1 more units of money income, and each unit of money income means E_{15}/E_5 more units of x_1 demanded as in (17). Expectations concerning e_2 contribute further terms of each type. Thus, equal changes in money income arising from money transfers and additional employment have different effects on commodity demand, since the latter has general equilibrium repercussions as well. We must accordingly distinguish between the corresponding marginal propensities to consume.

The implications of this will be seen shortly.

The comparative statics of the producer's decisions is easier except that the formation of expectations introduces a link between p_1 and p_2. Define $\epsilon = d \log p_2/d \log p_1$, the elasticity of expectations. Then, taking small proportional changes in (5)–(7), we find

$$\epsilon\, dp_1/p_1 = dp_2/p_2 = (n-1)\, dy_2/y_2 + \theta\, dp_1/p_1,$$
$$dy_1/y_1 = n\, dy_2/y_2 - (1-\theta)\,\sigma\, dp_1/p_1,$$
$$dl_1/l_1 = n\, dy_2/y_2 + \theta\sigma\, dp_1/p_1,$$

where θ is the imputed share of Period 1 commodity inputs in factor cost, and σ is the elasticity of substitution between commodities and labour in production. Thus, the responses of derived demands to parameter changes are given by

$$dy_1/y_1 = [(\epsilon - \theta)\, n/(n-1) - (1-\theta)\,\sigma]\, dp_1/p_1, \tag{18'}$$

$$dl_1/l_1 = [(\epsilon - \theta)\, n/(n-1) + \theta\sigma]\, dp_1/p_1. \tag{19'}$$

In the 'normal' case, we would like the coefficient in (18') to be negative and that in (19') to be positive. The first would give a downward sloping demand curve for commodities as inputs to production, i.e., an upward

sloping supply curve of Period 1 commodities to the rest of the economy, while the second would mean that a fall in the price of Period 1 commodities received by producers would cause them to reduce labor demand. However, it is quite easy for one of these to be false. The first can fail since sufficiently elastic expectations can lead producers to expand production as p_1 increases, while the second can fail since the fall in the marginal cost curve as p_1 falls can cause increased production. The two require opposing circumstances, and it is easy to see that one of the signs must be normal. However, if n is close to one, the first term in each coefficient is large in absolute value, and this increases the possibility of one of the two ceasing to be normal. This is because returns to scale are then nearly constant, the marginal cost curve is very flat, and any vertical shifts in it or in the expected price produce large output effects.

In this paper I shall assume that both effects are "normal," as I would like to emphasize some other anomalies that remain as an essential feature of an equilibrium with rationing. However, the other cases also deserve attention.

Introducing some more abbreviations, I shall accordingly write

$$dy_1 = -\Phi\, dp_1 \tag{18}$$

$$dl_1 = \Psi\, dp_1 \tag{19}$$

where Φ and Ψ are both positive.

One more condition can be imposed on these expressions, by considering the total demand for Period 1 commodities as a function of q_1, assuming the appropriate quantity adjustment in the labor market, and holding all other variables fixed. Now, corresponding to a change dq_1, we have the same change in p_1, and a change of $\Psi\, dq_1$ in e_1. Using (13) and (18), we see that the change in the total demand for Period 1 commodities is

$$dx_1 + dy_1 = -A\, dq_1 + B\Psi\, dq_1 - \Phi\, dq_1 .$$

If we assume a downward sloping demand curve for stability in this market, we have

$$A + \Phi - B\Psi > 0. \tag{20}$$

I shall write Σ for the expression on the left-hand side.

We can now put the pieces together to study the comparative statics of the whole equilibrium. Suppose the tax rates change by dt_1 and dr_1, the government purchases by dx_g and dl_g, and the money transfers to consumers by dm_1 (and of course $dM_1 = dm_1$). From (9)–(12) we have

$$dp_1 = dq_1 - dt_1 , \qquad dv_1 = -dr_1 ,$$
$$de_1 = dl_1 + dl_g ,$$

and then

$$\begin{aligned} 0 &= dx_1 + dx_g + dy_1 \\ &= [-A(dp_1 + dt_1) + B(dl_1 + dl_g) - \Gamma\, dr_1 + \Delta\, dm_1] \\ &\quad - \Phi\, dp_1 + dx_g\,. \end{aligned}$$

Therefore,

$$dp_1 = \Sigma^{-1}[-A\, dt_1 - \Gamma\, dr_1 + dx_g + B\, dl_g + \Delta\, dm_1], \tag{21}$$

$$dq_1 = \Sigma^{-1}[(\Phi - B\Psi)\, dt_1 - \Gamma\, dr_1 + dx_g + B\, dl_g + \Delta\, dm_1], \tag{22}$$

$$de_1 = \Sigma^{-1}[-A\Psi\, dt_1 - \Gamma\Psi\, dr_1 + \Psi\, dx_g + (A + \Phi)\, dl_g + \Delta\Psi\, dm_1] \tag{23}$$

$$dx_1 = \Sigma^{-1}[-A\Phi\, dt_1 - \Gamma\Phi\, dr_1 - (A - B\Psi)\, dx_g + B\Phi\, dl_g + \Delta\Phi\, dm_1] \tag{24}$$

Finally, we can calculate the effect of various policies on the budget deficit. After some substitution and simplification, we find

$$\begin{aligned} dD = (q_1 + \Omega)\, dx_g + (v_1 + B\Omega)\, dl_g + (1 + \Delta\Omega)\, dm_1 \\ - (x_1 + A\Omega)\, dt_1 - (e_1 + \Gamma\Omega)\, dr_1 \end{aligned} \tag{25}$$

where

$$\Omega = \Sigma^{-1}(x_g - t_1\Phi - r_1\Psi). \tag{26}$$

The sign of Ω is ambiguous. However, if we assume that increases in government purchases increase the deficit and increases in tax rates reduce it, then all the coefficients in (25) must be positive.

In the two sections that follow, these formulas will be put to use in examining and comparing various policy measures.

4. Prices and Quantities

Consider first the simplest (and the most reassuring) result. If the government increases its purchases of labor slightly, we have

$$de_1 = \Sigma^{-1}(A + \Phi)\, dl_g > dl_g\,. \tag{27}$$

This is the conventional employment multiplier. However, the conventional output multiplier does not arise quite as clearly. If the government increases its purchases of commodities slightly, we have

$$\begin{aligned} d(x_1 + x_g) &= [1 - \Sigma^{-1}(A - B\Psi)]\, dx_g \\ &= \Sigma^{-1}\Phi\, dx_g > 0, \end{aligned} \tag{28}$$

but we cannot say in general that the output multiplier will exceed one. The problem is that the increased government demand raises the price of commodities, and this decreases the quantity demanded by the consumers. If this demand is price-elastic, i.e., if A is large, we can have

$$dx_1 = -\Sigma^{-1}(A - B\Psi)\, dx_g < 0.$$

This is the first example of a phenomenon that the conventional fixed-price income–expenditure models miss.

The output multiplier would be stronger if the supply of commodities could be increased within the first period very elastically in response to the price increase. This would be the case in a model that allowed unused capacity and assumed a zero production lag.

Having mentioned price changes, let us examine them in greater detail. Begin with the question of the incidence of the commodity tax. We have

$$dp_1 = -A\Sigma^{-1}\, dt_1\,, \quad dq_1 = \Sigma^{-1}(\Phi - B\Psi)\, dt_1\,. \tag{29}$$

It is now possible that an increase in the tax rate will lower the price to consumers. The price to producers will then fall by more than the amount of the tax increase, i.e., the incidence will be more than 100% on the producers. This happens if $\Phi < B\Psi$. i.e., $B > \Phi/\Psi$.

To understand this result, compare it with conventional microeconomic incidence results. In a partial equilibrium setting, more than 100% incidence on either party cannot occur with a downward sloping demand curve and an upward sloping supply curve. In general equilibrium, such curves for each commodity shift as the prices of related commodities change. Extreme incidence then becomes possible, although it can be ruled out using natural sufficient conditions for the uniqueness of equilibrium, as shown in [7]. In a temporary equilibrium, however, the demand curve for commodities shifts as the amount of employment changes on account of the Keynesian income effect. Thus, an initial fall in the price received by the producer leads him to reduce the amount fo labor demanded. This shifts the demand curve for commodities to the left, causing a further fall in the price to the producer. The cumulative outcome of this process can be to lower the price to the consumer. To see this more precisely, examine the expression (15) for B. The Keynesian income effect adds the terms $\{v_1 + v_2 e_2'(e_1)\}\, \Delta$, thus making B larger, and therefore, a tax incidence of more than 100% on the producer more likely.

Conversely, it is possible that a reduction in indirect taxes, by stimulating demand, will actually increase prices. This is another example of a possibility that the unified approach to public finance allows us to model.

This leads us naturally to a comparison of the price and employment effects of various policies. Three of the policies—the wage tax, commodity purchases, and money supply changes—are identical in this regard. For each of them, we have

$$dq_1/de_1 = 1/\Psi. \tag{30}$$

Labor purchases cause a smaller change in commodity prices per unit of employment change; for this policy we have

$$dq_1/de_1 = B/(A + \Phi), \tag{31}$$

and it is easy to verify that the right-hand side of (31) is less than $1/\Psi$. For the commodity tax we have

$$dq_1/de_1 = (B\Psi - \Phi)/(A\Psi). \tag{32}$$

This may be of either sign, but will always be less than $1/\Psi$.

These comparisons are important not only for their policy implications, but also because they have an indirect bearing on the possibility of finding stable relations of the Philips curve type in an economy in which discretionary macropolicy is pursued. Of course a model which will allow us to investigate such a curve must be much more sophisticated, in particular we must allow a flexible market wage and compare temporary equilibria in successive periods, taking into account changes in expectations, but it is unlikely that the necessary complications will eliminate the differences among policies. Thus we have reason to suspect that the *kind* of discretionary policy being pursued affects the slope of a Phillips curve. This might be an interesting empirical problem.

5. Budget Deficits and Employment

The standard question in general equilibrium models of public finance is the policy choice which, subject to the appropriate constraints, maximizes utility. Given a revenue constraint, for example, we look for a tax structure which in an interior optimum equates the marginal dead-weight loss per unit of revenue. In a temporary equilibrium framework, however, utility maximization loses much of its welfare significance. This is the case since consumers maximize utility given their expectations, and these expectations are in general inconsistent and incorrect. In fact the welfare economics of temporary equilibria has hardly been studied, and therefore it seems more appropriate to catalog various comparisons of policies rather than to try to reduce them all to a scalar measure of welfare. One

comparison of particular interest is the expansionary effect of policies in relation to the effect on the budget deficit. Since the constraint on employment is the main new feature of this model, it seems natural to consider expansionary effects on employment. In considering the budget deficit, we must of course include both direct and indirect effects of each policy, and expression (25) does this.

If we could be assured of the existence of an interior solution to the problem of finding an optimal policy mix for employment, we would write down first-order condition equating their effects on employment per unit of added budget deficit at the margin. However, interior solutions do not always exist, and I shall pose the problem as one of the various binary comparisons of employment expansions per unit of marginal budget deficit.

The first such comparison will be between the government's commodity purchases and transfers of money to consumers. For the former, we see from (23) and (25) that

$$de_1/dD = \Sigma^{-1}\Psi/(q_1 + \Omega).$$

For the latter, the corresponding expression is

$$de_1/dD = \Sigma^{-1}\,\Delta\Psi/(1 + \Delta\Omega).$$

Therefore, commodity purchases are more expansionary per unit of deficit at the margin if

$$\Sigma^{-1}\Psi/(q_1 + \Omega) > \Sigma^{-1}\,\Delta\Psi/(1 + \Delta\Omega).$$

This reduces to

$$q_1\Delta < 1. \tag{33}$$

Now $q_1\,\partial x_1/\partial M_1$ is the marginal propensity to consume first period commodities, and therefore (33) is true under the standard assumption.

This is the familiar result about the balanced budget multiplier. It should not be surprising that the appropriate comparison is with transfers of money, since the simple income-expenditure analysis treats taxes as lump sum transfers. Comparisons of commodity purchases with other tax reductions yield related results, but in the new framework the price effects of these taxes must be considered.

Proceeding in a similar way, we find that commodity purchases are more expansionary per unit deficit at the margin than a reduction in the wage tax if

$$(q_1\,\partial x_1/\partial M)[1 + (e_2/e_1)\,v_2'(v_1)] < 1. \tag{34}$$

This is equivalent to (33), and therefore true, if wage expectations are totally inelastic. Otherwise it is less likely to be true. The point is simply that a unit reduction in the wage tax increases the wage net of tax by a unit, which is exactly like a transfer of e_1 units of money except in that it can induce expectations of further transfers in Period 2.

Comparing commodity purchases with a commodity tax reduction, we find the former more expansionary if

$$-\left(\frac{q_1}{x_1}\frac{\partial x_1}{\partial q_1}\right)-\left(\frac{q_2}{x_1}\frac{\partial x_1}{\partial q_2}\right)\cdot\left(\frac{q_1}{q_2}q_2'(q_1)\right)<1. \tag{35}$$

This amounts to requiring a price-inelastic demand, in an overall sense including the effect on expectations.

The presumption that this price elasticity will be relatively low was discussed before. Of course, its precise magnitude cannot be determined without empirical work.

One interesting case can be discussed further. If expectations are unit-elastic, there is no pure substitution effect, and the comparison depends on income effects alone. We find that (35) reduces to

$$EE_{15}/E_5 < E_1 .$$

This is true if $\partial(E_1/E)/\partial u < 0$, i.e., if Period 1 commodities are necessities, and, correspondingly, Period 2 commodities are luxuries. If preferences are homothetic, therefore, commodity purchases and commodity tax reductions become equivalent with unit-elastic expectations.

Next compare the government's labor purchases with money transfers. The condition for the former to be more expansionary per unit deficit at the margin is

$$\Sigma^{-1}(A+\Phi)/(v_1+B\Omega) > \Sigma^{-1}\Psi\Delta/(1+\Delta\Omega).$$

For this, it is sufficient to have

$$\Sigma^{-1}B\Psi/(v_1+B\Omega) \geqslant \Sigma^{-1}\Psi\Delta/(1+\Delta\Omega),$$

which reduces to

$$B \geqslant v_1\Delta. \tag{36}$$

If the Keynesian income effect is the only impact of employment on commodity demand, i.e., an increase of de_1 in employment is exactly like an additional transfer of $v_1 de_1$ units of money, then $B = v_1\Delta$, and (36) holds. However, there are other effects, namely the general equilibrium interactions and the expectations concerning e_2. The latter adds to the

Keynesian income effect, and thus contributes to making (36) true. The general equilibrium effects must be examined more closely. If commodities at Date 1 are gross substitutes for leisure at either date in the sense that an increase in the quantity of e_1 or e_2 will increase the demand for x_1, then $(E_{13} - E_3E_{15}/E_5)$ and $(E_{14} - E_3E_{15}/E_5)$ will be positive. Then

$$B > v_1 + v_2e_2'(e_1),$$

and (36) is true. Thus, we have a sufficient condition with a ready interpretation, but it is very restrictive.

Using the same method and assuming gross substitutes, we find that labor purchases are more expansionary than a wage tax reduction if

$$e_1e_2'(e_1)/e_2 > v_1v_2'(v_1)/v_2\,, \tag{37}$$

i.e., if employment expectations are more elastic than wage expectations. This seems sensible, and the only surprise is that the condition is only sufficient. This is again a consequence of the general equilibrium effects of employment changes.

The condition for a wage tax cut to win in the comparison with money transfers is

$$\Gamma > e_1\Delta, \tag{38}$$

which is always true. If the market wage showed some flexibility, this effect would be further strengthened through substitution in production.

Finally, a wage tax cut wins against a commodity tax cut if

$$-\left(\frac{q_1}{x_1}\frac{\partial x_1}{\partial q_1}\right) - \left(\frac{q_2}{x_1}\frac{\partial x_1}{\partial q_2}\right)\left(\frac{q_1}{q_2}\,q_2'(q_1)\right) < q_1\frac{\partial x_1}{\partial M_1}\left(1 + \frac{e_2}{e_1}\,v_2'(v_1)\right). \tag{39}$$

Thus a high marginal propensity to consume and elastic wage expectations favor the wage tax cut; price-elastic demand favors a commodity tax cut.

6. Modifications and Extensions

The familiar and new results of the preceding sections are, I hope, sufficient to persuade readers of the fruitfulness of this approach. However, the model has serious limitations that must be removed in order to make it more realistic. The most important modification is the introduction of borrowing and lending, and it will affect the model in several ways.

First, it will enable us to give a better treatment of investment. Commodities as inputs to production are a form of investment in the present

model, but the theory of this investment is a very crude one, with only internal finance and only the opportunity cost of sales to consumers. With financial capital markets we can give a much richer theory of this important component of aggregate demand. Also, the theory of producer behavior itself is weak in absence of such markets. I have assumed profit maximization, but the basis on which the producer compares p_2 and p_1 is at best unclear. In fact any behavior that leads to (18) and (19) will serve for the purpose of the model, but borrowing and lending possibilities will be important if we are to make sense of most types of behavior.

The second point is that financial capital markets will remove the need to assume that the constraint $m_2 \geqslant 0$ is never binding on the consumers. This will be particularly important if we are to introduce heterogeneity among consumers, as must be done if we are to study the structure and not merely the magnitude of involuntary unemployment.

Finally in this connection we have the issue of finance of the government's budget deficit. In the present context, there is little reason for having any concern about the magnitude of the deficit per se. However, if the deficit is financed by borrowing, it affects interest rates and thus has other repercussions on the economy. The exercises of finding fiscal measures which yield most expansion per unit of deficit acquire more meaning in such a setting.

On a related point, there is equally little reason in the present model why expansion should not be carried to a point where the rationing constraint on labor ceases to be binding. In reality, second- or third-best policies with regard to the balance of payments provide the reason for maintaining some involuntary unemployment. It would be desirable to include these considerations directly in the model.

Another modification that may be thought desirable is to allow quantity adjustment in the commodity market as well. This can happen, for example, if firms are rationed as to the quantity of commodities they can sell. This dual rationing of firms and consumers is an attractive way to think about processes of cumulative deflation like the multiplier. This approach is taken, for example, by Barro and Grossman [1] and by Grandmont and Laroque [6] in a second, and in their opinion more properly Keynesian, model. However, we have seen that the aproach adopted here, with workers rationed but firms able to act as price-takers, also leads to a multiplier. Thus, the modification is not essential. Also, for what it may be worth, it is clear that the present approach accords with Keynes' intentions quite explicitly stated in Chapter 2 of the General Theory.

Finally, of course, it is important to extend the horizon and consider a succession of temporary equilibria, allowing for changes in expectations

and speculative action in anticipation of such changes. That, however, seems a rather ambitious task at this stage.

I would like to defend the model against one possible charge. It can be said that the model is too specific, assuming as it does a particular situation in which some constraints are binding and others are not. The model in conditions of excess demand, for example, must be recast and is not symmetric. However, I believe that such specificity is desirable, and that more progress will result from building different models applicable to different situations than can be expected from a very general model that tries to encompass the whole range of the underlying theory.

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Democratic Preference Functions

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One often finds in the literature on axiomatic social choice theory reference to the social preference function, which maps the configuration of individual preference relations into a social preference relation, as the constitution. If one accepts this definition of a constitution it is possible to deduce that no constitution is able to satisfy a few fundamental normative principles, principles that claim to be vital to a democratic and liberal society. Perhaps the two most striking of such impossibility theorems are the Arrow paradox [1] and the Sen paradox [7]. This paper argues that the Arrow and Sen theorems depend on a very narrow definition of a constitution amounting to the identification of the constitution with the social preference function. A constitution is more complex than this and when it is defind properly and when the normative criteria are properly represented by the constitution the force of the impossibility theorems is spent.

In the case of the Arrow theorem, the misplaced condition is Independence of Irrelevant Alternatives. This should be introduced, not as a normative restriction on the mapping of individual into social preference, but as a technical requirement the force of which is to ensure that a social preference function can be implemented by some iterative procedure (like a parliamentary system). With the Independence condition put in its place all of the Arrow criteria, including transitivity, are consistent. Moreover, simple majority rule is used to define a social choice algorithm that always selects an alternative that is socially best according to *any* social preference function satisfying all of the restrictions employed.

Sections 1 to 5 describe a constitution and the way in which its elements will be represented here. Section 6 presents a social preference function that satisfies all of the conditions imposed. Section 7 shows how the best alternative, specified by a democratic preference function, may be found by an iterative procedure. Section 8 concludes the paper.

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1. A Constitution

Basic to the notion of a constitution is the recognition that certain features of community life must be determined in an explicitly collective fashion. In such cases a single decision will be imposed on all by the coercive power of the state. The decision may affect different individuals in different ways; it may merely set out guidelines or parameters (like income tax schedules) which influence, but do not determine, individual decision making. But no individual may escape the implications of the decision except by leaving the community (which act itself may be regarded as an implication of the decision).

An apparatus for making collective decisions having been set up, its scope must be limited, if only slightly. The domain of collective decision making must be determined. This is the first element of a constitution. Can the community collectively decide upon an individual's attendance at church, use of a gun, choice of newspapers, smoking in an auditorium? Where the constitution leaves initiative to the individual, it must provide for an organization that allows the individual to exercise his preference to a degree. For example, it may provide for a market system to serve individual wants. Interestingly, neither private property nor the market mechanism are mentioned in the constitutions of Canada or the United States. The constitutions of China and the U.S.S.R. do specify collective ownership and socialist planning but leave details up to "the plan" [9]. This is not anomalous. A constitution may be written or unwritten and the difference is almost a matter of ink and paper. If a function is supported by an unwritten part of the constitution it is not subject to the usual rules of amendment and so may be modified in a different way than are parts of the written constitution. But this difference is ephemeral. The constitution defined here will follow the above examples and not specify explicitly the organization of decentralized activity. The parameters governing such activity are, however, part of the domain of collective choice.

The second feature of a constitution is the social choice function, which is defined on the domain of collective choice and not on the set of feasible states of society. The social choice function depends on individual preferences and its purpose is to specify which of the feasible collective actions are "best" for society. If the choice function satisfies some simple conditions it can be represented by a social preference function, which fixes a binary relation (called social preference) for every configuration of individual preference relations. This is where Arrow began. The choice function summarizes society's normative criteria regarding collective choice.

The third important feature of a constitution is the social choice algorithm, which is used to find a socially best state (according to the choice function) in the collective domain. Most constitutions are devoted primarily to outlining the algorithm, which is given concrete form by a legislature, committee structure, election rules, and so on. The algorithm is a practical computational device while the choice function is normative.

2. The Domain of Collective Choice

The first step in determining the scope of collective choice is to specify the set of possible states of society. Let X be that set. For example, assuming that citizens are ultimately interested in consumption broadly defined and not in production or the organization of economic activity per se, a typical state x in X can be written $x = (x_0, x_1, ..., x_n)$, where x_0 is a vector of public activities and x_i is the ith individual's private consumption plan. An item of public output is not necessarily a collective good in the sense of Samuelson [5]. The distinction between public and private goods in the constitutional setting is one of function. The plan x_0 is determined by the collective decision-making mechanism, while x_i is determined privately by individual i subject to limitations implied by the action of others including the state. Hence, the mth component of x_0 may specify the amount of asparagus consumed by individual j, or the political party to which j may belong, etc.

The set of states D which is the domain of the collective choice apparatus is not a subset of X. Rather, a typical element d of D specifies a mix x_0 of public output and a set of "parameters" r that guide private activity. Therefore, we can set $d = (x_0, r)$. The control r may specify for each i a set r_i of commodity bundles from which i may select his consumption plan. Or r may be a tax schedule that will determine, given x_0, a market equilibrium allocation, which in turn determines the consumption plan x_i of each individual. The word "determine" is used advisedly. The collective machinery cannot predict the final consumption plan for each choice of parameter r but once r is known individual plans are determined in the sense of being implied by utility maximization subject to the constraints imposed by r and the actions of other economic agents including the public sector.

Individual preference over D must be defined with care. An individual "consumes" points such as x in X but is presented with alternatives in D that by implication determine which states in X will prevail. If the individual knows precisely which alternative x will be established by the

implementation of an arbitrary plan d, then the preference on D can be derived from preference on X with the aid of a utility maximizing axiom and the assumption that simple strategies are used (i.e., the individual does not misrepresent his preferences for any reason). If the link between points in D and X can be perceived only vaguely by individuals, a more complicated method of deriving preference on D will be used. The crucial assumption is that individuals will possess a determinate preference relation on D.

3. Individual and Social Preference

There are n individuals whose preferences are to be consulted. $N = \{1, 2, \ldots, n\}$ is the set of all voters, or citizens. Let $\mathscr{P}$ be the set of all preference n-tuples $(P_1, \ldots, P_n)$ on D; that is, for each $P = (P_1, \ldots, P_n) \in \mathscr{P}$, P_i $(i \in N)$ is a binary relation on D. The only assumption made is that each P_i $(i \in N)$ is asymmetric on D.

For each $P \in \mathscr{P}$ define the function num P on $D \times D$:

$$\text{num } P(a, b) = |\{i \in N; \text{ not } bP_i a\}|, \quad \text{the number of } i \text{ for which } bP_i a \text{ is false.}$$

Let $P(a, b) = \text{num } P(a, b)/[\text{num } P(a, b) + \text{num } P(b, a)]$. Since each P_i is asymmetric, num $P(a, b)$ + num $P(b, a) \neq 0$ and $P(a, b)$ is defined for all $a, b \in D$. Further, $P(b, a) = 1 - P(a, b)$ for all $a, b \in D$.

A *choice function* C on D is a mapping from 2^D into 2^D (the set of all subsets of D) and satisfies

(i) $C(S) \subset S$ for all $S \in 2^D$, and

(ii) $C(S) \neq \varnothing$ if S is a nonempty, finite subset of D.

The elements of $C(S)$ are supposed to be "best" in S according to C. A *social choice function* is a function from $\mathscr{P}$ into the set of all choice functions on D. For each $P \in \mathscr{P}$, CP is a choice function on D, defined by the social choice function C, and $CP(S)$ is the value of CP at S.

If B is a binary relation on D we can define the function max B on 2^D: For $S \in 2^D$

$$\max B(S) = \{a \in S; bBa \text{ implies either } aBb \text{ or } b \notin S\}.$$

Then max B satisfies (i) but not necessarily (ii). It is well-known that max B is a choice function if and only if B is a suborder [6]. (B is a suborder if it is asymmetric and $aBbB \ldots Bc$ implies not cBa.) If B represents social

strict preference, Arrow required that B be a weak order. That is, for all $a, b \in D$,

aBb implies not bBa (asymmetry), and

aBb implies, for all $c \in D$, cBb, or aBc (negative transitivity).

Herzberger [4] and Sen [8] give conditions which, if satisfied by a choice function C, guarantee the existence of a weak order B such that $C = \max B$. We assume throughout that this is possible so that social choice may be represented by a *social preference function* F on $\mathscr{P}$ such that FP is a weak order on D for all $P \in \mathscr{P}$ (Arrow's transitivity axiom). This representation will not be used explicitly until Section 5.

4. Social Choice Algorithms

The social choice algorithm is used to compute a member of the choice set, $CP(S)$. In this context an algorithm is merely a rule, depending on P and S, for transitioning from state a^t (a member of S) to state a^{t+1} (also a member of S). State a^1 is arbitrary and may be regarded as the status quo. If the algorithm reached a point where it alternated between two alternatives a and b in $CP(S)$ it would have reached a stage where a socially best alternative (as determined by C) always prevailed but the alternating would be wasteful. Therefore, we require the algorithm to continue to transition from one member to another in S until it finds a member a^* of $CP(S)$ and then it remains at a^*.

Formally, an algorithm A is defined by means of a set $AP(S \mid a^1,..., a^t)$ for each $P \in \mathscr{P}$, $S \subset D$, and finite sequence $a^1,..., a^t$ of members of S. The members of this set are the only ones that may be selected by the algorithm at the tth state. At the tth state the algorithm transitions from a^t to a member of $AP(S \mid a^1,..., a^t)$. No method of breaking ties specified in case $AP(S \mid a^1,..., a^t)$ contains more than one element. Therefore, A is really an entire class of algorithms, one for each tie-breaking mechanism. In the voting context the different ways of breaking ties correspond to different primary systems, different committee structures, etc.

The algorithm operates until at state T a member a^T of $CP(S)$ is presented. An equilibrium of the algorithm A for feasible set $S \subset D$ and $P \in \mathscr{P}$ is a member a^T of the sequence $\{a^t; i = 1, 2,..., ...\}$ such that

$$a^{t+1} \in AP(S \mid a^1,..., a^t) \qquad \text{for all } t$$

and

$$AP(S \mid a^1,..., a^t) = \{a^T\} \qquad \text{for all } t \geqslant T.$$

Let $AP(S)$ be the set of all equilibria for the algorithm A applied to $P \in \mathscr{P}$ and $S \subset D$. Different initial states $a^1 \in S$ may lead to different equilibria so $AP(S)$ may contain more than one member (or it may be empty). Once an equilibrium is selected by the algorithm it cannot be displaced by any other feasible alternative. Some elementary restrictions are placed on social choice algorithms.

The following hold for arbitrary $P \in \mathscr{P}$ and $S \subset D$.

A_1: If $a^m \in AP(S \mid a^1, \ldots, a^{m-1})$ for $m = 2, \ldots, t$ then $a^m \in AP(S)$ if $a^m = a^{m-1}$ and if the $a^1, \ldots, a^t$ are distinct, $a^t \in S - \{a^1, \ldots, a^{t-1}\}$.

A_2: If S is finite and $a^1 \in S$ there exists a sequence $\{a^1, \ldots, a^t\}$ in S such that a^t is an equilibrium.

A_3: $AP(S) \subset CP(S)$.

To state A_4 some new notation is required. For each $P \in \mathscr{P}$ and $S \subset D$ let

$$[P, S] = \{P' \in \mathscr{P}; \forall a, b \in S, \forall i \in N, aP_i b \text{ if and only if } aP_i' b\}$$

A_4: If $P' \in [P, S]$ then $AP(S) = AP'(S)$.

Condition A_1 prevents the algorithm from cycling; an alternative a^t selected by the algorithm at the tth stage is never selected at a subsequent stage unless a^t is an equilibrium. A_2 requires the algorithm to reach an equilibrium whenever S is finite. An alternative way of stating A_3, and one that closely reflects the development of welfare economics, is that the algorithm must generate equilibria that are optimal, in this case with respect to the choice function CP. The term *optimal* will be used throughout in this sense.

The last condition is obviously related to Arrow's Independence of Irrelevant Alternatives criterion [1, p. 26] and can be defended by borrowing from Arrow's support of his Independence condition. An algorithm cannot be expected to be sensitive to individual preference for alternatives that are not feasible. It must work by inviting participants to reveal their preferences and using the information acquired to effect a new, better, social state. But the agency charged with administering the algorithm cannot look into people's minds, and it must coax individuals to reveal preference by presenting alternatives and observing reactions to them. Obviously, only feasible alternatives can be presented. Hence, it is impossible to design an algorithm to distinguish between two regimes with identical feasible sets and identical individual preferences over those sets. This is exactly what is embodied in A_4. "The essential point . . . is the application of Leibniz's principle of the identity of indiscernibles" [1, p. 109]. The next proposition shows how the restrictions on the algorithm

are related to Independence of Irrelevant Alternatives. First, a weak form of this condition is introduced.

WI. For any $P \in \mathscr{P}$ and any finite $S \subset D$, $\bigcap_{[P,S]} CP'(S) \neq \varnothing$.

The choice sets generated by all preference configurations that agree on S should have at least one member in common. Since the algorithm generates a member of the choice set (A_3) and the set of equilibria are identical on $[P, S]$ the conditions A_2–A_3 imply WI. Since A_1–A_4 will be satisfied by a collective choice algorithm, WI will be satisfied by the social preference function. Arrow's independence condition is

AI. For any $P \in \mathscr{P}$ and any $S \subset D$

$$CP(S) = CP'(S) \qquad \text{for all } P' \in [P, S].$$

AI implies WI but the converse is not true.

One particular algorithm, simple majority rule, will be shown to be optimal for all democratic choice rules (defined below). Simple majority rule (M) is easily defined.

Given $P \in \mathscr{P}$ and a finite, nonempty $S \subset D$:

(i) Choose an arbitrary a^1 in S as the initial state. $MP(S \mid a^1) = \{a^1\}$ if $a \in S$ implies $P(a, a^1) \geqslant \frac{1}{2}$ and otherwise $MP(S \mid a^1) = \{a \in S;\ P(a, a^1) > \frac{1}{2}\}$.

(ii) With $a^1, \ldots, a^t$ and the corresponding $MP(S \mid a^1, \ldots, a^m)$, $m \leqslant t$, already defined, let a^{t+1} be any member of $MP(S \mid a^1, \ldots, a^t)$ and let $MP(S \mid a^1, \ldots, a^{t+1}) = \{a^{t+1}\}$ if $a \in S$ and $P(a, a^{t+1}) > \frac{1}{2}$ implies $a = a^m$ for some $m \leqslant t$ and otherwise $MP(S \mid a^1, \ldots, a^{t+1}) = \{a \in S - \{a^1, \ldots, a^{t+1}\};\ P(a, a^{t+1}) > \frac{1}{2}\}$.

It is easily verified that M satisfies A_1, A_2, and A_4.

5. The Normative Criteria

Two criteria (Anonymity and Neutrality, defined below) will be added to Arrow's [1]. Arrow's first condition, Unrestricted Domain, has been strengthened since any asymmetric individual preference relation on D is admissible. The modified Independence condition is introduced as A_4. Arrow's Nondictatorship condition is implied by Anonymity. The other two Arrow conditions are implied by Monotonicity and Sovereignty. The conditions are imposed on an arbitrary social preference function F.

MONOTONICITY. If $P, P' \in \mathscr{P}$, $a^* \in D$ and for all $i \in N$ and all $a, b \in D$,

(1) $aP_i'b$ if and only if aP_ib for $a^* \notin \{a, b\}$,

(2) a^*P_ib implies $a^*P_i'b$, and

(3) not bP_ia^* implies not $bP_i'a^*$,

then a^*FPb implies $a^*FP'b$ and not $bFPa^*$ implies not $bFP'a^*$, for all $b \in D$.

(CITIZENS') SOVEREIGNTY. If $a, b \in D$ and $a \neq b$ then $aFPb$ for some $P \in \mathscr{P}$.

A permutation mapping π on some $P \in \mathscr{P}$ is a member $\pi(P) \in \mathscr{P}$ such that $\pi(P)$ is obtained from P by relabeling the members of N.

ANONYMITY. If $P \in \mathscr{P}$ and π is a permutation mapping, $FP = F\pi(P)$.

Two neutrality conditions are defined here. It turns out that they are interchangeable in the proofs of the theorems to follow. Some preliminary notation is required. If R is a binary relation on D the inverse R^{-1} of R is given by: $aR^{-1}b$ if and only if bRa, for all $a, b \in D$. If $P = (P_1, \ldots, P_n) \in \mathscr{P}$ $P^{-1} = (P_1^{-1}, \ldots, P_n^{-1})$. Let G be the set of biunique functions from D to D.

NEUTRALITY (I). For all $P \in \mathscr{P}$, $FP^{-1} = (FP)^{-1}$.

NEUTRALITY (II). For all $P, P' \in \mathscr{P}$ and all $g \in G$, if $\forall i \in N$, $aP_i'b$ if and only if $g(a)\, P_i\, g(b)$, for all $a, b \in D$, then

$$aFP'b \quad \text{if and only if} \quad g(a)\, FPg(b), \qquad \text{for all } a, b \in D.$$

First, neither of the two conditions implies the other. Let $P_1 = \{(a, b), (b, c), (a, c)\}$ and $P_2 = \{(b, c), (c, a), (b, a)\}$ and $P = (P_1, P_2)$. There is no $g \in G$ for which

$$eP_i^{-1}d \quad \text{if and only if} \quad g(e)\, P_i\, g(d) \qquad \text{for} \qquad e, d \in D = \{a, b, c\},$$

for $i = 1, 2$. Similarly, for g given by; $g(a) = c$, $g(b) = b$, and $g(c) = a$, and P' defined by

$$eP_i'd \quad \text{if and only if} \quad g(e)\, P_i\, g(d) \qquad \text{for all } e, d \in D$$

it is not the case that $P' = P^{-1}$.

Neutrality (II), employed by Hansson [3], is widely used whereas the other neutrality condition is not. But II is obviously closely related to AI while I stands aloof from Arrow's independence condition (see Guha [2] for an examination of the link between II and AI). As it happens, for the results presented here they are interchangeable; thus, both will be carried.

Anonymity requires the social preference function to be independent of the way in which individuals are labeled while Neutrality imposes the same restriction upon alternatives.

The next section shows that the above four conditions and the weak order requirement are consistent. Section 6 proves that simple majority rule M is optimal for every democratic preference function. A *democratic preference function* is a social preference function F satisfying Monotonicity, Sovereignty, Anonymity, Neutrality (either I or II) and WI, and such that FP is a weak order for all $P \in \mathscr{P}$. We want to impose A_1–A_4 as well but we do this in a roundabout way by making sure that WI, a necessary condition for A_1–A_4, is satisfied. Then the satisfaction of A_1–A_4 is demonstrated directly by showing that simple majority rule is an acceptable algorithm for a democratic choice rule.

6. Democratic Preference Functions

An entire class of democratic preference functions will be defined, one for each real number in (0, 1]. For each $\alpha \in (0, 1]$ the democratic preference function F_α will be the set difference between M_α, α-majority rule, and Q_α, a cycling relation for M_α. That is, for each $P \in \mathscr{P}$,

$$F_\alpha P = M_\alpha P - Q_\alpha P,$$

where binary relations are represented as sets of ordered pairs.

For any $a, b \in D$ and any $P \in \mathscr{P}$,

$$aM_\alpha Pb \quad \text{if and only if} \quad P(a, b) \geqslant \alpha > P(b, a).$$

If t is a positive integer, $\{a^1, \ldots, a^t\} \subset D$; $P \in \mathscr{P}$, and

$$\begin{array}{lll} \text{either} & P(a^h, a^{h+1}) \geqslant \alpha & \\ \text{or} & P(a^h, a^{h+1}) < \alpha \quad \text{and} & P(a^{h+1}, a^h) < \alpha \end{array}$$

for all $h = 1, \ldots, t$, with $a^{h+1} = a^1$, then we say that $a^1a^2 \cdots a^ta^1$ is a cycle (for P) and a^h and a^k cycle for $h, k \in \{1, \ldots, t\}$. Then for all $a, b \in D$ and all $P \in \mathscr{P}$,

$$aQ_\alpha Pb \quad \text{if and only if} \quad a \text{ and } b \text{ cycle} \quad (\text{for } P).$$

Finally, for any $a, b \in D$, and all $P \in \mathscr{P}$,

$$aF_\alpha Pb \quad \text{if and only if} \quad aM_\alpha Pb \text{ and not } aQ_\alpha Pb.$$

Clearly, $Q_\alpha P$ is an equivalence (to establish transitivity, note that if $a \ldots h \ldots a$ and $b \ldots c \ldots b$ are cycles then $a \ldots b \ldots c \ldots b \ldots a$ is a cycle).

Consider the following examples; $D = \{a, b, c\}$ and $P = (P_1, P_2, P_3)$ has the following representation.

$$\begin{array}{llll} aP_1b, & bP_1c, & \text{and} & aP_1c. \\ bP_2c, & cP_2a, & \text{and} & bP_2a. \\ cP_3a, & aP_3b, & \text{and} & cP_3b. \end{array}$$

Then *abca* is a cycle for every value of α so $F_\alpha P$ is empty; $eF_\alpha Pd$ holds for no e, $d \in D$ whatever the value of α. Now add alternative a^* so that $D = \{a^*, a, b, c\}$ and form P_i' $(1 \leqslant i \leqslant 3)$ by adding a^*P_ia, a^*P_ib, and a^*P_ic to P_i above. Then $eF_\alpha Pd$ if and only if $e = a^*$ and $d \in \{a, b, c\}$ for all values of α. We see that F_α collapses cycles by creating indifference but these are the only instances of indifference (note that *aa* is a cycle).

PROPOSITION 1. *For any $a \in (0, 1]$ and any $P \in \mathscr{P}$, $F_\alpha P$ is a weak order on D.*

Proof. Suppose not $bF_\alpha Pa$ and not $cF_\alpha Pb$. If $P(a, c) \geqslant \alpha > P(c, a)$ then certainly $cF_\alpha Pa$ is false. Then assume that $P(a, c) \geqslant \alpha > P(c, a)$ is false. If both

(1) $P(b, a) \geqslant \alpha > P(a, b)$, and

(2) $P(c, b) \geqslant \alpha > P(b, c)$

are true it follows that $aQ_\alpha Pb$ and $bQ_\alpha Pc$. Therefore, $aQ_\alpha Pc$ (and hence, not $cF_\alpha Pa$) by transitivity of $Q_\alpha P$. If both (1) and (2) are false then *cabc* is a cycle and again not $cF_\alpha Pa$. If (1) is true and (2) is false we must have $bQ_\alpha Pa$. Let $b \ldots a \ldots b$ be a cycle containing a and b. Then $a \ldots bca$ is a cycle and so not $cF_\alpha Pa$. If (1) is false and (2) is true let $c \ldots b \ldots c$ be a cycle containing c and b. Then $b \ldots cab$ is a cycle implying not $cF_\alpha Pa$. Therefore, in every logically possible case $cF_\alpha Pa$ is false. Therefore, $F_\alpha P$ is a weak order. Q.E.D.

It is easy to show that $F_\alpha P$ satisfies Sovereignty, Anonymity, and Neutrality (I) and (II).

PROPOSITION 2. *F_α satisfies Monotonicity for all $\alpha \in (0, 1]$.*

Proof. Suppose a^* is related to P and $P' \in \mathscr{P}$ by the hypothesis of Monotonicity.

If $a^*F_\alpha Pb$ then $P(a^*, b) \geqslant \alpha$ and $P(b, a^*) < \alpha$. But clearly $P'(a^*, b) \geqslant P(a^*, b)$ and so $P'(b, a^*) < \alpha$. Therefore, $a^*F_\alpha P'b$ unless $a^*Q_\alpha P'b$. If the latter is true the terms in the cycle between a^* and b may be omitted since we know $P'(a^*, b) \geqslant \alpha > P'(b, a^*)$. Let $a^*ba^1a^2 \ldots a^ta^*$ represent

the cycle. But then $a^*ba^1a^2 \ldots a^ta^*$ is a cycle for P since $P(a^i, a^{i+1}) = P'(a^i, a^{i+1})$ for $i = 1,\ldots, t-1$, $P(a^*, b) \geqslant \alpha > P(b, a^*)$, $P(a^t, a^*) \geqslant P'(a^t, a^*)$, and $P(b, a^1) = P'(b, a^1)$. Therefore, $a^*Q_\alpha P'b$ is false and $a^*F_\alpha P'b$.

Similarly, if not $bF_\alpha Pa^*$ but $bF_\alpha P'a^*$ then there is a cycle for P that may be represented $ba^*a^1a^2 \ldots a^tb$, but one can show that this is also a cycle for P', contradicting $bF_\alpha P'a^*$. Therefore, not $bF_\alpha Pa^*$ implies not $bF_\alpha P'a^*$. Q.E.D.

7. The Optimality of Simple Majority Rule

We show that the algorithm M defined in Section 4 is optimal for every democratic preference function. First, we show that M is optimal for $F_{1/2}$ and then demonstrate that $\max F_{1/2}P(S)$ is a subset of $\max FP(S)$ for all democratic F (for arbitrary P in $\mathscr{P}$ and S in 2^D).

Proposition 3. *If F is a democratic preference function and* num $P(a, b) =$ num $P(b, a)$ *for arbitrary $P \in \mathscr{P}$, $a, b \in D$, then not $aFPb$ and not $bFPa$.*

Proof. Set $S = \{a, b\}$ and suppose num $P(a, b) =$ num $P(b, a)$ with $P \in \mathscr{P}$. Define P^{I} and P^{II} in $\mathscr{P}$; for all $c, d \in D$, $cP_i^{\mathrm{I}}d$ if and only if dP_ic for all $i \in N$, and $cP^{\mathrm{II}}d$ if and only if $g(c)\,P_ig(d)$ for all $i \in N$ where $g: D \to D$ is defined by: $g(a) = b$, $g(b) = a$ and $g(x) = x$ for all $x \in D - S$.

Since num $P(a, b) =$ num $P(b, a)$ there is a $P' \in \mathscr{P}$ such that

$$P' = \pi(P) \in [P^{\mathrm{I}}, S] = [P^{\mathrm{II}}, S]$$

for some permutation π. Suppose $aFPb$. Then $aFP'b$ by Anonymity. But $bFP^{\mathrm{I}}a$ by Neutrality (I) and $bFP^{\mathrm{II}}a$ by Neutrality (II). Therefore,

$$\max FP^{\mathrm{I}}(S) \cap \max FP'(S) = \varnothing$$

and

$$\max FP^{\mathrm{II}}(S) \cap \max FP'(S) = \varnothing,$$

contradicting WI, whichever neutrality condition is used. Therefore not $aFPb$. Similarly, not $bFPa$. Q.E.D.

Now define the *characteristic value* $\alpha(F)$ of an arbitrary social preference function F. First, let

$$A_F = \{P(a, b);\ P \in \mathscr{P} \text{ and } bFPa \text{ for some } a, b \in D\}$$

Sovereignty guarantees that A_F is not empty. A_F is certainly finite so $\max A_F$, the largest number in A_F, exists. Then $\alpha(F)$ is the smallest member of the set

$$\{P(a, b) > \max A_F ; P \in \mathscr{P} \text{ and } a, b \in D\}$$

The next result implies that $\alpha(F) \leqslant \frac{1}{2}$.

PROPOSITION 4. *If F is a democratic preference function, for any $P \in \mathscr{P}$, and any $a, b \in D$,* num $P(a, b) \geqslant$ num $P(b, a)$ *implies not $bFPa$, and as a consequence $P(a, b) \geqslant P(b, a)$ implies not $bFPa$, or equivalently, $P(a, b) \geqslant \frac{1}{2}$ implies not $bFPa$.*

Proof. Everything follows readily from the first inequality.

Suppose num $P(a, b) \geqslant$ num $P(b, a)$. Let

$$h = |\{i \in N; aP_ib\}|,$$
$$j = |\{i \in N; \text{ not } aP_ib \text{ and not } bP_ia\}|$$
$$k = |\{i \in N; bP_ia\}|.$$

Since each P_i is asymmetric $h + j + k = n$ and num $P(a, b) = h + j$ so $h \geqslant k$.

Anonymity permits us to assume: aP_ib for $i = 1,..., h$ not aP_ib and not bP_ia for $i = h + 1,..., n - k$, and bP_ia for $i = n - k + 1,..., n$.

Construct $P' \in \mathscr{P}$: $aP_i'b$ for $i = 1,..., k$, not $aP_i'b$ and not $bP_i'a$ for $i = k + 1,..., n - k$, $bP_i'a$ for $i = n - k + 1,..., n$ and $dP_i'e$ if and only if dP_ie for all $i \in N$ and all $d, e \in D$ such that $\{d, e\} \neq \{a, b\}$.

Not $aFP'b$ and not $bFP'a$ by Proposition 3. Then not $bFPa$ by Monotonicity since P and P' are related by the hypothesis of Monotonicity (with respect to a). Q.E.D.

THEOREM 1. *If F is a democratic preference function with characteristic value α then for all $P \in \mathscr{P}$ and all $S \subset D$,*

$$\max F_{1/2}P(S) \subset \max F_\alpha P(S) \subset \max FP(S).$$

Proof. It suffices to show that $aFPb$ implies $aF_\alpha Pb$ implies $aF_{1/2}Pb$. First, show that $aFPb$ implies $aF_\alpha Pb$. If $aFPb$ then $P(b, a) < \alpha$ since $P(b, a) \in A_F$ and $\max A_F < \alpha$. Since $\alpha \leqslant \frac{1}{2}$ we have $P(a, b) > \frac{1}{2} \geqslant \alpha > P(b, a)$. Therefore, $aF_\alpha Pb$ unless $aQ_\alpha Pb$. Since $\alpha \leqslant \frac{1}{2}$, $P(a^h, a^{h+1}) < \alpha$ and $P(a^{h+1}, a^h) < \alpha$ is impossible so if $aQ_\alpha Pb$ we have a cycle $a^1a^2 \cdots a^ta^1$ in D satisfying:

(i) $P(a^h, a^{h+1}) \geqslant \alpha$ for $h = 1,..., t$ and $a^{t+1} = a^1$, and

(ii) $a^1 = b$ and $a^t = a$.

By definition of the characteristic value (i) implies

$$\text{not } a^h FP a^{h-1} \qquad \text{for} \quad h = t, t-1, \ldots, 3, 2.$$

But then not $a^t FP a^1$ (that is, not $bFPa$) since FP is a weak order. But this we have ruled out so $aF_\alpha Pb$ must hold also.

Suppose now that $aF_\alpha Pb$. Then $P(a, b) > \frac{1}{2} > P(b, a)$. Hence, $aF_{1/2}Pb$ unless $aQ_{1/2}Pb$. But it is easy to show that $aQ_{1/2}Pb$ implies $aQ_\alpha Pb$ for all $\alpha \leqslant \frac{1}{2}$. Therefore $aF_\alpha Pb$ implies $aF_{1/2}Pb$. Q.E.D.

THEOREM 2. *Simple majority rule M is optimal for any democratic preference function; that is, any simple majority rule equilibrium is a member of the choice set defined by any democratic preference function.*

Proof. We need to show that, for an arbitrary $P \in \mathscr{P}$ and $S \subset D$, $MP(S) \subset \max FP(S)$ if F is a democratic preference function. Theorem 1 permits us to show only that $MP(S) \subset \max F_{1/2}P(S)$.

If $a^* \in MP(S)$ there is some path $a^1, \ldots, a^T$ in S with $a^* = a^T$ an equilibrium and for all $t = 2, \ldots, T$,

$$P(a^t, a^{t-1}) > \tfrac{1}{2} \qquad a^t \in S - \{a^1, \ldots, a^{t-1}\}.$$

Suppose $a^* \notin \max F_{1/2}P(S)$. Then $aF_{1/2}Pa^*$ for some $a \in S$. Since $\{a \in S - \{a^1, \ldots, a^T\}; aF_{1/2}Pa^T\} = \varnothing$, $a = a^m$ for some $m \in \{1, \ldots, T-1\}$. But then $a^T a^{T-1} \ldots a^m a^T$ is a cycle that contradicts $aF_{1/2}Pa^*$. Therefore, $a^* \in \max F_{1/2}P(S)$. Q.E.D.

Theorem 2 provides one interpretation of simple majority rule, not as the sort of social preference function Arrow sought, but as an iterative procedure leading to an outcome that embodies the normative criteria that Arrow stressed as being fundamental to democratic decision making.

8. CONCLUSION

With the Independence condition put in its proper place not only is it possible to satisfy all of Arrow's criteria, and two others besides, in addition simple majority rule can be used to generate an alternative that is socially best with respect to any social preference function satisfying all of the conditions.

Another impossibility theorem loses its force when a constitution is properly defined—Sen's "paradox of the Paretian Liberal" [6]. Sen discovered that the Pareto criterion was inconsistent with the liberal doctrine that some choices should be governed entirely by the individual

"concerned." Whereas Sen took this to imply a restriction on the way in which C maps the set of feasible alternatives into a smaller choice set, in fact it implies a restriction on the *domain* of C. With the proper definition of a constitution and the appropriate introduction of the liberal ethic the Sen theorem is vitiated.

What the structure of this paper amounts to is a rationale for simple majority rule. No mention has been made of political parties, legislatures, and so on. These institutions serve to activate the social choice algorithm to determine which of the members of $AP(S \mid a^1,\ldots, a^{t-1})$ will actually be presented at the tth stage. The structure that has been called a democratic preference function can be used to determine which of the various institutional webs of government—parliamentary versus presidential systems, proportional versus direct representation, etc.—are acceptable. This undertaking and the application of the algorithm to specific problems such as the determination of a tax structure would seem to be the chief applications of the results of this paper.

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Oligopoly in Markets with a Continuum of Traders: An Asymptotic Interpretation*

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This paper tries to make relevant for a finite economy the results of Shitovitz on the "pricing-out" of cores of economies with a non-atomless measure space of agents. It is hoped that the comparison of the two sets of results will shed light on the interpretation and modeling of a "large" trader.

1. INTRODUCTION

In a seminal paper [2], Aumann observed that the concept of perfect competition can be made precise by considering an atomless measure space of agents. He interpreted his model as an "expression of (the) economic idea...(that) the influence of each individual participant is negligible." Since Aumann's paper, extensive work has been done on such markets, and it has been shown that they have interesting properties not shared by economies with a finite number of agents. This work has then inspired investigation into the question as to whether these properties hold in some approximate sense for large but finite economies. As a result of this investigation, valuable insights have been obtained. For example,[1] it has been shown that approximate competitive equilibria exist for large

* This paper is an outgrowth of some tentative results presented in Chapter VII of my Ph.D. dissertation submitted to Yale University in June 1973.

[1] The following propositions serve as other examples: For nonconvex exchange economies it has been shown that (i) Pareto-optimal allocations can be sustained as approximate efficiency equilibria (see [13, 22], among others); (ii) approximate cores are nonempty (see [25, 14, 15, 21], among others). It has also been shown that (ii) core allocations can be "priced out" under varying generality as approximate competitive equilibria (see [12, 26, 10, 17, 16, 1, 3, 8, 9, 18], among others); (iv) core enlarges "only slightly" if restrictions are placed on the size or characteristics of the blocking coalitions (see [20]).

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but finite exchange economies when the preferences of the agents are not necessarily convex (see [28, 11, 1, 4, 15, 19], among others).[2]

Since competition is indeed far from perfect in many real markets, Aumann also stressed the importance of studying economies with both negligible and nonnegligible agents. This suggestion has given rise to work on economies with a non-atomless measure space of agents with the atoms being interpreted as "large" traders or as oligopolists. For such economies, Shitovitz [27] has recently pursued the following inquiry: "To what extent can the large trader exploit the small ones, and how much can the small ones gain from possible competition between the large traders?" The question arises as to whether Shitovitz' results carry over, albeit in an approximate sense, to a large but finite economy. In the words of Hildenbrand, what is "the relevance of the ideal to the finite case?" This paper establishes this relevance and shows that Shitovitz' results hold in an approximate sense for large but finite economies in which each trader, except those belonging to a fixed finite set, is "negligible." We believe that these results represent a first attempt to provide asymptotic counterparts to results originally proved for an economy with a non-atomless measure space of agents.[3] However, we do not engage in this exercise only as an end in itself but rather in the added hope that some light will be shed not only on Shitovitz' work but also on the interpretation and modeling of a large trader.

Readers primarily interested in large but finite economies need read only Sections 2 and 3. We have tried to make these self-contained. Section 2 is a rather stark presentation of the principal results of the paper and Section 3 attempts to discuss them in the context of previous work on this topic.

Those readers who do not want to work through the proofs but are nevertheless interested in the general idea underlying them need read in addition Sections 4 and 5. Section 4 illustrates in general terms how a result pertaining to a suitably formulated idealized limit economy can be used to derive results for a large but finite economy. We have tried to keep this section as free of technical details as possible. Section 5 presents the results for the limit economy, the nonstandard exchange economy. There are no proofs in this section and it is hoped that especially readers familiar with a measure space of agents will find this section easy to read. They will come across familiar concepts and results, only that the analysis of the

[2] This is of course not to say that all these papers took continuous markets, or indeed any other limit economy, as their starting point. We bunch them together because ex post facto all these investigations can be viewed as providing approximations to results valid for ideal limit economies. This observation is also valid for Footnote 1.

[3] Such a measure space is of course assumed to have an atomless part.

paper requires that they be presented in a different language; one using infinitely large, and infinitesimally small numbers.

Sections 6 and 7 are the mathematical core of the paper. Readers interested only in the nonstandard limit economy and in nonstandard techniques may confine themselves primarily to these sections. Together with Section 5, these are mathematically self-contained.

2. The Model and Results

Let R be the set of real numbers, We shall be working in the space R^n, the n-fold Cartesian product of R, where n denotes a finite number of commodities. Let Ω^n denote the nonnegative orthant of R^n and N the set of positive integers. Before presenting the model, it is well to lay out some preliminary notation. For any x, y in R^n (i) $x \geqslant y$ means $x_i \geqslant y_i$ for all coordinates $i = 1, 2,..., n$; (ii) $x > y$ means $x \geqslant y$ and $x \neq y$; (iii) $x \gg y$ means $x_i > y_i$ for all i. e denotes an element of R^n all of whose coordinates are unity. For any x, y in R^n, $x \ominus y = (x_1 \ominus y_1, ..., x_n \ominus y_n)$ where $x_i \ominus y_i$ denotes Max $(0, x_i - y_i)$. For any set S, $|S|$ denotes the number of elements in S and for any two numbers u, v, $|u - v|$ denotes Max $(u - v, v - u)$. Finally, in the sequel we shall assume that ϵ is an arbitrary small positive number.

A finite exchange economy, E^k, consists of a set of traders whose initial endowments and preferences are restricted to the commodity space Ω^n. Each trader has, in addition to his preferences and his endowment, a nonnegative number designating his weight. Thus E^k can be formally written as a finite sequence of triples $(I(t), \succ_t, \lambda(t))_{t=1}^k$ with $I(t) \in \Omega^n$ $\succ_t \subseteq \Omega^n \times \Omega^n$ and $\lambda(t) \in \Omega$ denoting the tth trader's endowment, preference relation and weight, respectively. Let the set of traders T^k be divided into two groups T_1^k and T_0^k where $T_1^k = (1, 2,..., m)$ and $T_0^k = (m + 1,..., k)$. We will assume that m is independent of k, and thus, denote T_1^k by T_1. This set will subsequently be interpreted as the set of "large" traders. We will also refer to traders in T_0^k as "small" traders. We now formalize notions of core and price equilibria for E^k.

D.1. A function X from T^k into Ω^n is an *allocation* if

$$\sum_{t \in T^k} X(t)\,\lambda(t) = \sum_{t \in T^k} I(t)\,\lambda(t).$$

D.2. An allocation Y *ϵ-blocks* an allocation X if there exists a nonempty coalition S such that

(a) $Y(t) \succ_t X(t) \qquad (\forall t \in S)$,

(b) $\sum_{t \in S} Y(t)\,\lambda(t) \leqslant [\sum_{t \in S} I(t)\,\lambda(t)] + \epsilon e$.

Note that the definitions of an allocation and of feasibility as in D.2. (b) are unorthodox in the context of a finite economy. We will have more to say on this in Section 3. The next two definitions are standard.

D. 3. An allocation Y *blocks* an allocation X if it ϵ-blocks it with $\epsilon = 0$.

D. 4. An allocation X is in the ϵ-*core* (*core*) if there exists no allocation which ϵ-blocks (blocks) it.

We now move on to the notions of price equilibria that we shall be using. Let an allocation X and a price system p be a Walrasian or competitive equilibrium for E^k. Then, as conventionally defined, the following statement is true for *every* trader t:

$$|pX(t) - pI(t)| = 0; \qquad y \succ_t X(t) \Rightarrow py > pI(t). \tag{1}$$

For our first price equilibrium concept we modify the above to say that the following statement is true for *almost every* trader t:

$$|pX(t) - pI(t)| < \epsilon; \qquad y \succ_t X(t),\ y \in K_\epsilon \Rightarrow py > pI(t) \ominus \epsilon. \tag{2}$$

The prices are normalized and K_ϵ represents a cube of size $(X(t) + Me)/\epsilon$ where M is an arbitrary large number. Thus, instead of requiring that "all preferred commodity bundless cost more than the exact value of the endowment," we have in part modified (1) to say that "a large set of preferred commodity bundless cost more than the approximate value of the endowment." Under (2), a typical trader will be in a position illustrated by Fig. 1.

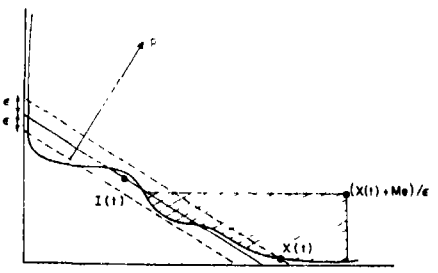


FIGURE 1

The principal weakness of (2) is the restriction to the cube K_ϵ. This restriction could be removed if we make some assumption requiring the "similarity" of the agents' preferences. Indeed, the pricing of core allocations can be articulated in several ways; in the context of our approach, see [18]. In this paper we have tried to make the results of

Shitovitz relevant for finite economies, and as such, have tried to make assumptions parallel to his and pick the simplest formulations. We return to this point in Section 3.

We can now formally define our basic concept.

D. 5. A *weak ϵ-equilibrium*[4] is defined as a pair (p, X) where $p \in \Omega^n$, $p > 0$ and X an allocation such that $\lambda(H_\epsilon^p(X)) < \epsilon$ where

$$H_\epsilon^p(X) = K_\epsilon^p(X) \cup L_\epsilon^p(X),$$

$$K_\epsilon^p(X) = \{t \in T^k \mid (\exists y \in \Omega^n)(py \leqslant pI(t) \ominus \epsilon) \Lambda(y \succ_t X(t)) \Lambda(y \leqslant (X(t) + Me)/\epsilon))\},$$

$$L_\epsilon^p(X) = \{t \in T^k \mid |pX(t) - pI(t)| > \epsilon\}.$$

Note that the idea of a set S comprising "almost all traders" has been formalized as requiring $\lambda(S) \geqslant 1 - \epsilon$.

If we now modify D. 5 to require that a typical small trader have his core bundle approximately within his bedget set, i.e., $pX(t) < pI(t) + \epsilon$, we get our next concept.

D. 6. An *ϵ-exploitative equilibrium*[4] is defined as a pair (p, X) where $p \in \Omega^n$, $p > 0$ and X an allocation such that $\lambda(\hat{H}_\epsilon^p(X)) < \epsilon$ where

$$\hat{H}_\epsilon^p(X) = \hat{K}_\epsilon^p(X) \cup \hat{L}_\epsilon^p(X),$$

$$\hat{K}_\epsilon^p(X) = \{t \in T^k \mid (\exists y \in \Omega^n)(py \leqslant pX(t) \ominus \epsilon) \Lambda(y \succ_t X(t)) \Lambda(y \leqslant (X(t) + Me)/\epsilon))\},$$

$$\hat{L}_\epsilon^p(X) = \{t \in T_0^k \mid pX(t) \geqslant pI(t) + \epsilon\}.$$

Figure 2 illustrates a possible equilibrium position of a typical trader in T_0^k under D. 6.

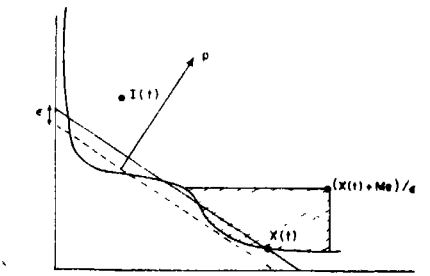


FIGURE 2

[4] Strictly speaking, all these equilibrium notions should be prefixed, in addition to ϵ, by M. We disregard this precision in the interest of becoming more readable. In any case, it should be noted that M is redundant if $X(t) \gg 0$.

For our final price equilibrium notion we shall need some additional concepts. Two traders are said to be of the same *type* if they have the same preferences and the same endowment. They are said to be of the same *kind* if they also have the same weight. Note that a "large" trader can be of the same type as a trader in T_0^k but cannot be of the same kind[5]. Finally, let $A_i (i = 1, 2, \ldots, d)$ be subsets of T_1 such that each trader in A_i is of the same kind.

We now require that almost all traders in T_0^k, i.e., the set of small traders, be in an equilibrium position as in D. 5, i.e., as illustrated by Fig. 1. Further, traders in T_1, i.e., the set of large traders, are required to be in a position illustrated in Fig. 2 with $I(t)$ deleted. In addition, traders belonging to A_i are required to have approximately equal values of their core bundles. More formally:

D. 7. An *ϵ-equitable equilibrium*[4] is defined as a pair (p, X) where $p \in \Omega^n, p > 0$ and X an allocation such that $\lambda(\tilde{H}_\epsilon^p(X)) < \epsilon$ and $R_\epsilon^p(X) = \phi$ where

$$\tilde{H}_\epsilon^p(X) = H_\epsilon^p(X) \cap T_0^k,$$
$$R_\epsilon^p(X) = \tilde{K}_\epsilon^p(X) \cup Q_\epsilon^p(X),$$
$$\tilde{K}_\epsilon^p(X) = K_\epsilon^p(X) \cap T_1,$$
$$Q_\epsilon^p(X) = \bigcup_i \{s, t \in A_i \mid |pX(t) - pX(s)| > \epsilon\}.$$

Let $\mathcal{G} = \{E^k\}_{k \in N - T_1}$ be an unbounded sequence of such economies. It is unbounded in the sense that for any natural number, however large, there exists an economy with the number of traders in it equal to that number. As we shall see below, this sequence has the further characteristic that for any number, however small, there exists an economy with all traders in it, except for a fixed finite set, having a weight less than that number. We will assume that $\mathcal{G}$ satisfies the following conditions.

(1) There exists a positive number $\bar{r}$ such that for all economies E^k in $\mathcal{G}$,

$$\sum_{t \in T^k} I(t)\lambda(t) \leqslant \bar{r}e.$$

(2) There exists a positive number $\underline{r}$ such that for all traders t in E^k and for all economies E^k in $\mathcal{G}$, $I(t) > \underline{r}e$.

(3) For each trader t in E^k and for all economies E^k in $\mathcal{G}$, $\succ_t$ is[6] (a) irreflexive, (b) transitive, (c) monotonic, and (d) continuous. It t belongs to T_1, $\succ_t$ is, in addition, (e) convex, and (f) complete.

[5] These concepts are straightforward analogs of those used by Shitovitz [27].

[6] All the terms used below are by now well known. For definitions, see, for example, Debreu's "Theory of Value." The particular version of monotonicity that we use is $x \geqslant y$ implies $x \succ_t y$ for all x, y in Ω^n.

(4) Given any positive number δ, there exists a number v such that for all traders t in T_0^k and for all economies E^k in $\mathcal{G}$ such that $|T_0^k| > v$, $\lambda(t) \leqslant \delta$. Further, there exists a positive number z such that for all t in $T_1(T_1^k)$ and for all economies E^k in $\mathcal{G}$, $\lambda(t) \geqslant z$. Finally, for all economies E^k in $\mathcal{G}$, $\sum_{t \in T^k} \lambda(t) = 1$.

It is this assumption that formalizes the idea of all traders, except those belonging to a finite set, getting negligible. As an illustrative example let

$$\mathcal{G} = \{E^{k+1}\}_{k \geqslant 1}$$

where $T_1 = \{1\}$, $\lambda(1) = \frac{1}{3}$, and $\lambda(t) = 4/5k$ for all t in T_0^{k+1}, for all E^{k+1} in $\mathcal{G}$.

For the first theorem, we can weaken (2) to:

(2)′ There exists a positive number $\underline{r}$ such that for all economies E^k in $\mathcal{G}$, $\sum_{t \in T^k} I(t)\, \lambda(t) \geqslant \underline{r}e$.

We can now state:

THEOREM 1. *If $\mathcal{G}$ satisfies* (1), (2)′, (3), *and* (4), *then for all $\epsilon > 0$ there exists $k_0 \in N$ such that corresponding to an allocation X in the core of $E^i (E^i \in \mathcal{G}, |E^i| > k_0)$, there exists p such that (p, X) is an ϵ-exploitive equilibrium*[7].

Now consider a sequence $\{X^k\}_{k \in N - T_1}$, where for each k, X^k is an allocation of an economy E^k. We shall say that such a sequence is *uniformly bounded from above*, if there exists a positive number L such that for all traders t in E^k and for all economies E^k in $\mathcal{G}$, $X^k(t) \leqslant Le$. Denote by B^k the set of allocations of E^k which also belong to such a sequence.

For the next theorem we need to assume:

(1)′ The sequence of initial endowments is unformly bounded from above.

(5)′ There are at least two traders in T_1, all of whom are of the same type.

We now state:

THEOREM 2. *If $\mathcal{G}$ satisfies* (1)′, (5)′, (2)–(4), *then for all $\epsilon > 0$, there exists $k_0 \in N$ such that corresponding to any allocation X chosen from B^i and which is in the ϵ-core of $E^i (E^i \in \mathcal{G}, |E^i| > k_0)$, there exists p such that (p, X) is a weak ϵ-equilibrium.*

[7] Let an allocation Y $\mathcal{F}_1$-*block* an allocation X if it blocks it via the coalition S where $T_1 \subseteq S$. Based on this notion of blocking we can define the $\mathcal{F}_1$-*core*. Note that core $\subseteq$ $\mathcal{F}_1$-*core*. Theorem 1 generalizes to allocations in the $\mathcal{F}_1$-*core*.

(5) Let $T_1 = \bigcup_{i=1}^{d} A_i$ such that each A_i contains $m \geqslant 2$ traders of the same kind.

THEOREM 3. *If $\mathcal{G}$ satisfies* (1)–(5), *then for all $\epsilon > 0$ there exists $k_0 \in N$ such that corresponding to any allocation X in the ϵ-core of $E^t(E^t \in \mathcal{G}, |E^t| > k_0)$, there exists p such that (p, X) is an ϵ-equitable equilibrium.*

Now let T_1 satisfy condition (5) and let W_i^j be the jth trader in A_i for all $E^k \in \mathcal{G}$. Then the set of *split traders* will be denoted by $\{W_i^j(t)\}_{t=1,\ldots,k}$ where $\sum_{t \in W_i^j} \lambda(t) = \lambda(W_i^j)$. Each of these split traders will be of the same type as W_i^j. Let $\hat{\mathcal{G}}$ be the sequence obtained from $\mathcal{G}$ by splitting some of of the "large" traders in all but one of the A_i. More formally, let $\hat{T}_0^k$ and $\hat{T}_1^k$ be the traders in $\hat{E}^k$, $\hat{E}^k \in \hat{\mathcal{G}}$. Then $\hat{T}_0^k = \{T_0^k\} \cup (\bigcup_{i=1}^{d-1} \bigcup_j W_i^j)$.

THEOREM 4. *If $\mathcal{G}$ and $\hat{\mathcal{G}}$ satisfy* (1)′, (2)–(5), *then for all $\epsilon > 0$ there exists $k_0 \in N$ such that corresponding to any allocation X chosen from B^t and which is in the ϵ-core of $E^t(E^t \in \mathcal{G}, |E^t| > k_0)$, there exists p such that (p, X) is a weak ϵ-equilibrium.*

3. Economic Interpretations and Relationship to Other Work

For the economic interpretations of the results, it will be useful to keep the following theorem as a benchmark.

THEOREM 0. *If $T_1 = \varnothing$ and $\mathcal{G}$ satisfies* (1)′, (2), (3a), (3c), *and* (4), *then for all $\epsilon > 0$ there exists $k_0 \in N$ such that corresponding to any allocation X in the core of $E^t(E^t \in \mathcal{G}, |E^t| > k_0)$, there exists p such that (p, X) is a weak ϵ-equilibrium.*

This theorem[8] is a straightforward generalization of [18, Corollary to Theorem 1]. Theorems 1–4 can be seen as generalizations of Theorem 0 to situations in which there are large traders. Note that conceptually this is analogous to Shitovitz' work which attempted to see how much could be salvaged of Aumann's equivalence theorem in an economy with atoms. We are trying to see how much of Theorem 0 can be salvaged in a finite economy with large traders. However, the pricing-out of core allocations of large but finite economies without large traders has been articulated in various ways. Thus, we have a choice of benchmarks. As mentioned in Section 2 in the context of the cube K_ϵ, in terms of our own work we can

[8] It can be easily checked that [18, Proof of Theorem 1] applies with slight reinterpretations to Theorem 0.

use instead of Theorem 0 a result phrased in terms of strong ϵ-equilibria or ϵ-ball equilibria (see [18]). Alternatively, one can presumably shift in principle to an entirely different approach and use Bewley's results [3], or those of Arrow and Hahn [1], or of Kannai and Hildenbrand [17, 16]. In terms of the motivation behind this exercise, this would lead to nothing new and the same kinds of results as our Theorems 1–4 obtained. Only the assumptions would be different since we would be using stronger equilibrium concepts and different mathematical techniques.

We now turn to the economic interpretation. We begin with Theorem 2. Consider outcomes achieved through bargaining between a group of traders, a finite number of whom have a disproportionately greater "political or sociological" leverage than the others. The question arises as to whether each of these outcomes could be interpreted as one achieved through the maximization of individual preferences at some given set of prices. Put differently, does there exist a system of prices at which individual traders would not want to trade in their core bundles for some others or alternatively demand back their initial endowments and use these for further trading? If there are two or more large traders of the same type, Theorem 2 answers this question by a qualified no; but only for those outcomes X for which[9] $\| X(t) - I(t) \|$ is not arbitrarily large for any trader t. Thus we have to restrict ourselves to allocations in the ϵ-core for which no trader has a "corner" on the market for any commodity. The qualifications are three: (a) There needs to be imposed a lump-sum tax on transactions; (b) traders are restricted in their choice of preferred bundles; (c) there is a set of deviants for which the answer may be yes inspite of (a) and (b). Consider a trader t not belonging to this set of deviants. His typical equilibrium position is as in Fig. 1. If he has to pay a tax of 2ϵ and is confined to the shaded area in his choice of preferred bundles, he is not interested in exchanging the ϵ-core bundle $X(t)$. Note that ϵ is parametrizing all the factors (a), (b), and (c)[10]. Thus for any ϵ however small, there exists a number k_0 such that all economies with the number of traders greater than k_0 can have a subset of their ϵ-cores conditionally priced out, with ϵ denoting the magnitude of each condition. As a final point note that since ϵ can be made as small as possible, for a large enough finite economy, none of the large traders can belong to the set of deviants.

Theorem 4 extends Theorem 2 to situations with more than one type of large traders. It considers a situation when the large traders can be decomposed into a finite collection of groups A_i such that each group has at least two traders of the same kind. Then Thoerem 2 carries over to the

[9] For any x in R^n, $\| x \|$ denotes $\mathrm{Max}_i\, x_i$.

[10] This is of course only on the pricing side; ϵ also parametrizes blocking by virtue of the ϵ-core.

ϵ-core of such an economy when some of the traders in all but one of the groups have been "split." Shitovitz has interpreted the idealized version of this theorem as applicable to "trust-busting." The message is that not all the large traders need be "busted" for the situation to become competitive. The only additional insight which our Theorem 4 offers is that this "splitting-up" of the large traders is dependent on the size of the ϵ. The larger the ϵ the less finely split the large trader has to be.

What about situations with no restrictions on the types of large traders? This is studied in Theorem 1. The interpretation of Theorem 1 is quite analogous to that of Theorem 2. It should be noted however, that unlike Theorem 2, we can talk of all outcomes achieved through bargaining and are not confined to subsets of ϵ-core. Further, in the pricing-out of these outcomes, we allow the traders only to trade in their core bundles and not demand back their initial endowments. In this case, for all small traders except for a set of deviants, $pX(t) < pI(t) + \epsilon$, keeping open the possibility that all these traders may have $pX(t) < pI(t)$. Thus, in D. 6 of an ϵ-exploitive equilibrium, $\hat{L}_\epsilon{}^p(X)$ represents the set of small traders who are "monetarily contented" with their core bundles. Such a set can be made arbitrarily small. It is in this sense that we can make precise the idea of large traders monetarily exploiting the small ones in a large but finite economy. Figure 2 illustrates the position of a typical small trader.

Theorem 3 sharpens Theorem 1 to a situation with finitely many kinds of large traders but with at least two traders of each kind. We have again to fall back on the ϵ-core but do not have to be restricted to a subset of it. For allocations in the ϵ-core, there exist prices at which all the small traders are in a position as in Theorem 2. All the large traders are in a position illustrated in Fig. 2 with $I(t)$ deleted. We can say nothing about the value of the endowments of one large trader relative to the value of his core bundle. All we can say is that the values of the core bundles of large traders of the same kind are approximately equal. Note that this theorem depicts a situation in which the small traders are not monetarily exploited by the large traders but we can say nothing between the relative positions of the large traders vis-à-vis themselves.

We now make some observations which have more to do with modeling than with interpretation. To begin with, note that for Theorems 2–4 we have been able to discuss only ϵ-cores. An ϵ-core is a set of unblocked outcomes when blocking is subsidized at a rate of ϵe per coalition. As such, the ϵ-core is contained in the core and tends to it when $\epsilon \to 0$. The fact that such cores are, in general, empty without convex preferences is rather awkward.

A further point that we have to go into is the inclusion of the weights in the definitions of both allocation and blocking. Whereas these definitions

are analogous to those used for a continuous economy, they are certainly unorthodox from the viewpoint of finite economies. A way out of these difficulties is to use the same interpretation as in the idealized case. To quote Shitovitz [27], "If X is an allocation, then $X(t)$ is *not* the bundle that is assigned to t but the "bundle density". The actual bundle that is assigned to t is, in the case of an oligopolist, $x(t)\,\mu(t)$". Note that whereas this begs the question as to the actual bundle assigned to t when t is not an oligopolist, since in this case $x(t)\,\mu(t) = 0$, we have no such difficulties in the finite case. However, as in the ideal case we retain the problem of reinterpretation of our equilibrium concepts since the preferences are defined on bundle densities. To avoid going into these new waters, we keep the unorthodox feasibility restraints and avoid talking of densities in the sequel.

Note that the above problem does not arise in the asymptotic interpretation of models without atoms, say Aumann's work. The reason for this is that the crucial element of such work is that each agent is getting negligible—the actual magnitude of an agent's weight is unimportant. Thus, we can regard the weight of the tth trader as the reciprocal of the size of the economy, i.e., $\lambda(t) = (1/|\,T^k\,|)$ for all t in T^k. This allows us to get rid of the weight at the end of the analysis. In the asymptotic interpretation of Shitovitz' work this cannot be done without further assumptions since the relative magnitudes of the weights are of primary significance.

A way out of these difficulties would be to get rid of the weights altogether and distinguish between large and small traders by the magnitude of their endowments. From the economic point of view this would be more satisfactory since it formulates nonnegligibility of a trader in terms of a factor that is most relevant, his endowment. On the other hand it allows us to get rid of a concept that has no economic underpinning. However, this opens an entirely different line of inquiry, one for which results pertaining to non-atomless economies are not necessarily relevant since these results too are crucially dependent on a concept without an economic underpinning, namely, the measure of an atom.

The point that emerges so far is that, in general terms, our Theorems 1–4 are analogous to those of Shitovitz' Theorems A to D. However, it is precisely in instances when Shitovitz can show that the core of a non-atomless economy is a competitive equilibrium, his Theorems B and D, that the difference between the finite and ideal cases is most marked. In our Theorems 2 and 4 we are reaching an analogous conclusion for a large, finite economy but have to assume that both the initial endowments and the core allocations are uniformly bounded from above. This implies, as mentioned earlier, that for no trader is $\|\,X(t) - I(t)\|$ unbounded. Shitovitz

needs to assume only that X and I are integrable. However, in this context it is instructive to quote some of the uneasiness Shitovitz feels as regards his Theorem B.

> For definiteness think of a town with a single large hotel. We have seen... that the monopolist can exploit the small traders.... The result is intuitively reasonably,... Suppose now a small entrepreneur enters the scene with a small Hotel or Pension. If the Pension is very small, one feels that the situation should not change very much. But this is false; no matter how small the additional oligoplist is, the large hotel has suddenly lost all its power and must take competitive prices as if the town was full of all kinds of hotels. One feels that intuitively the original monopolist has not, in fact, lost his power. The large hotel can, and probably will simply ignore the new entrepreneur.

It is our contention that requiring $\| X(t) - I(t)\|$ to be bounded for each trader rules out precisely the situation described above by Shitovitz, and as such we dispel some of the awkwardness of the ideal case. Thus, the asymptotic versions of Theorems B and D bring us back to our earlier point as regards the formulation of a large trader in terms of his endowment. It would be interesting to see what Shitovitz' Theorems B and D in particular look like in such an approach. This must remain an open problem for the time being.

4. Methodology of Proofs

To prove Theorems 1–4, we shall be using a branch of model theory known as nonstandard analysis. This is not to say that these theorems, or indeed their generalizations, might not at some future date be reproved using other simpler methods. However, what does need to be emphasized is that Theorems 1–4 have no necessary connection with nonstandard analysis either in their statement or in their logic.

The proofs of Theorems 1–4 can be decomposed into two parts, one simple and the other more difficult. The first part of the proof is as follows. Suppose Theorem 1 (say) is false. Then there exists a positive number ϵ and a large enough finite economy in $\mathscr{G}$ such that the core of such an economy cannot be priced out as an ϵ-exploitive equilibrium. At this point we bring in nonstandard analysis which allows us to say that any statement which is true in a standard world is also true under appropriate reinterpretations in the nonstandard world. We can thus say that there exists a positive standard number ϵ and a large enough star-finite economy in $^*\mathscr{G}$ such that the *core of such an economy cannot be priced out as an

ϵ^*-exploitative equilibrium. The starred entities are the reinterpreted nonstandard extensions of the corresponding unstarred ones. If we now have a result for the limit economy in a nonstandard universe which contradicts this, we are home and the proof of Theorem 1 is furnished. The hard part of the proof is to obtain such a result for the nonstandard limit economy.

The methodology is worth contrasting with that of Hildenbrand and other authors in the measure theory set up. Note that the simple part of our proof corresponds in their context to showing the convergence of a sequence of measures. Since each measure corresponds to a finite economy, this convergence implies that a property which holds in the limit will also hold for a large enough finite economy. In our case, once a particular property is shown to hold for a limit economy, its asymptotic version can be almost automatically derived as above.

We now attempt to obtain results for our limit economy. Using them we can then prove Theorems 1–4 as stated above, though we shall leave the reader to supply the statements himself. For an original illustration, see [9].

5. The Limit Economy

We formulate the limit economy as a nonstandard exchange economy. This formulation is originally due to Brown and Robinson [7]. Before a formal presentation about the concepts involved, it is well to make some introductory remarks about the nonstandard universe.

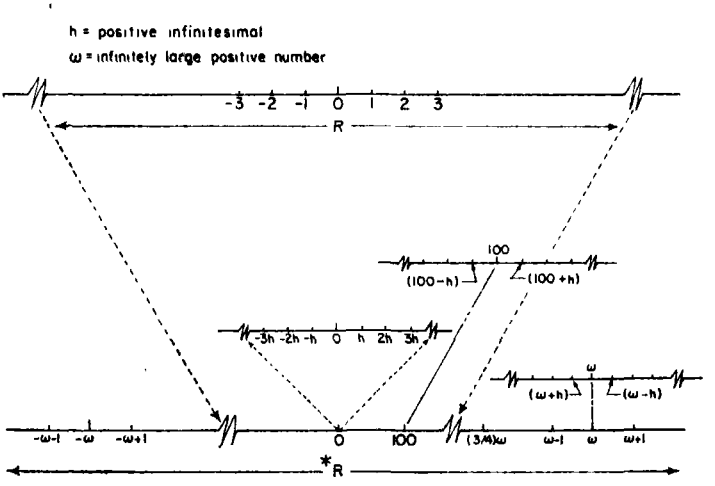


FIGURE 3

Consider the set of real numbers R. It is depicted in the top half of Fig. 3. We indicate the fact that R is unbounded from above and below by the jagged edges. R contains N, the set of positive integers, which is unbounded from above. Nonstandard analysis is concerned with the extension of R to another set $^*R$ which has the property that it contains, in addition to R, infinitely large numbers and infinitesimally small numbers. These infinitely large numbers are greater than any number in R, and are illustrated in Fig. 3 as ω, $\omega + 1$, and so on. All elements of $^*R$ obey the same operations as elements of R. The reader can appreciate the advantage of working in $^*R$. We can formulate economies with an infinitely large number of agents such that all except a finite set have a positive infinitesimal weight.

$^*R^n$ is the nonstandard extension[11] of R^n and $^*N$ the extension of N. Note that $^*R^n$ is a space of n-tuples some of whose elements can also be infinitesimal. Thus for x and y in $^*R^n$, we need in addition to $\geqslant$, $>$, $\gtrsim$, further notation. We shall take $x \simeq y$ to mean x differs by an infinitesimal from y in all coordinates: $x \gtrapprox y$ to mean x is noninfinitesimally greater than y in at least one coordinate and not less in any other; $x \gg y$ to mean x is noninfinitesimally greater than y in all coordinates; $x \gtrsim y$ to mean $x \geqslant y$ or $x \simeq y$.

Let $T \subseteq {}^*N$ be a set $(1, 2, \ldots, \omega)$ where $\omega \in {}^*N - N$. Let $T_1 \subseteq T$ be the set $(1, 2, \ldots, m)$ where m is a standard natural number. Let $T_0 = T - T_1$. T is to be interpreted as the set of traders, a finite subset T_1 of whom are large in the sense to be specified below. We can now define:

D. 1. *A nonstandard exchange economy*[12], $\mathscr{E}$, consists of a triple of functions I, P and λ where $I : T \to {}^*\Omega^n$, $P : T \to \mathscr{P}({}^*\Omega^n \times {}^*\Omega^n)$ and $\lambda : T \to {}^*\Omega$.

We shall denote these functions respectively as $I(t), \succ_t$ and $\lambda(t)$ where for all t in T, $I(t)$ is the tth traders endowment, $\succ_t$ his preference relation, and $\lambda(t)$ his weight in the economy. We shall have nothing to say about how this weight is determined. We will assume that $\mathscr{E}$ satisfies the following properties:

A.1. The function I indexing the initial endowment is internal.

A.2. $\sum_{t \in T} I(t)\,\lambda(t) < \bar{r}$ where $\bar{r}$ is a standard vector.

A.3. $I(t) \gtrapprox 0$ for all t in T.

A.3′. $\sum_{t \in T} I(t)\,\lambda(t) \gtrapprox 0$.

[11] For an informal introduction to nonstandard analysis, the reader is referred to [7–9]. For a full treatment, see [24].

[12] Nonstandard exchange economies with a nonuniform weights but without large traders were first defined by Brown [6]. $\mathscr{P}(A)$ denotes power set of A.

A.4. The relation Q, where $Q = \{(t, \succ_t) | \ t \in T, \succ_t \subseteq {}^*\Omega^n \times {}^*\Omega^n\}$, is internal for all t in T. In addition,[13]

(α) $\succ_t$ is irreflxive, transitive, monotonic, and continuous for all t in T.

(β) $\succ_t$ is convex and complete for all t in T_1.

A.5. The function λ indexing the weights is internal and $\sum_{t \in T} \lambda(t) = 1$.

A.6. $\lambda(t)$ is a positive infinitesimal for all t in T_0 and noninfinitesimally positive for all t in T_1.

It is in the sense of A.6. that T_1 is the set of large traders. We shall need the following concepts for our limit economy.

D.2. An *assignment* is an internal function from T into ${}^*\Omega^n$.

D.3. An *allocation* is an assignment Y such that

$$\sum_{t \in T} Y(t)\,\lambda(t) = \sum_{t \in T} I(t)\,\lambda(t).$$

D.4. *A coalition* S is an internal subset of T. S is said to be *nonnegligible* if $\lambda(S) = \sum_{t \in S} \lambda(t) \gtrsim 0$.

It is important to note this definition of negligibility. A coalition S may be "numerically negligible" (i.e. $|S|/|T| \simeq 0$)[14], or "economically negligible" (i.e., $\sum_{t \in S} I(t)\,\lambda(t) \simeq 0$ if $I(t) \simeq 0$ for all t in S), without being negligible in the sense of D.4. Note that this situation is parallel for a continuous economy with atoms. If λ now denotes the measure and τ is an atom, then $\int_\tau I(t)\,d\lambda(t) = 0$ if $I(\tau) = 0$, and τ is numerically negligible since it is one of uncountable many traders.

D.5. An allocation Y *blocks* an allocation X if there exists a non-negligible coalition S such that

(a) $Y(t) \succ_t X(t) (\forall t \in S)$,

(b) $\sum_{t \in S} Y(t)\,\lambda(t) \lesssim \sum_{t \in S} I(t)\,\lambda(t)$.

D.6. An allocation X is in the *core* if there exists no allocation which blocks it.

D.7. A *price system* p, is a standard vector in ${}^*R^n$ such that $p > 0$.

D.8. The tth trader's *budget set*, $B^p(t)$, is $\{x \in {}^*\Omega^n \mid px \lesssim pI(t)\}$.

[13] All the terms given below are defined in exactly the same way as for the standard case (cf. Footnote 6).

[14] Recall that $|S|$ denotes the number of elements in the set S.

D.9. $X(t)$ is *maximal* in $B^p(t)$ if $X(t) \in B^p(t)$ and $y \succ_t X(t)$ implies $py \gtrsim pI(t)$ where $(y - X(t))/\lambda(t)$ is infinitesimal.

D.10. A *competitive equilibrium* is defined as a pair (p, X) where p *is* a price system and X an allocation such that $X(t)$ is maximal for almost all t in T.

Note that the set K is almost equal to the set T if $T - K$ is a negligible set.

We are now ready to state our theorems.

THEOREM L1[15]. *If* $\mathscr{E}$ *satisfies* A.1.–A.6. *with* A.3′ *substituted for* A.3, *then corresponding to any allocation* X *in the core, there exists a price system* p *such that*

(a) $y \succ_t X(t), (y - X(t))\lambda(t)$ *infinitesimal* $\Rightarrow py \gtrsim pX(t)$ *for almost all* t *in* T,

(b) $pX(t) \lesssim pI(t)$ *for almost all* t *in* T_0.

As in Section 2, two traders are said to be of the same *type* if they have the same preferences and the same endowment. They are said to be of the same *kind* if they also have the same weight. We will say that an allocation X is *standardly bounded* if there exists a standard number L such that for all t in T, $X(t) \leqslant Le$.

THEOREM L2. *Let there be at least two traders in* T_1 *all of the same type and let* I *be standardly bounded. Then, under the assumptions of Theorem* L1, *corresponding to any standardly bounded allocation* X *in the core, there exists a price system* p *such that* (p, X) *is a competitive equilibrium.*

Remark. Note that unlike Shitovitz [27] we have not been able to prove this theorem for all allocations in the core.

THEOREM L3. *Let* $T_1 = \bigcup_{i=1}^{d} A_i$ *such that each* A_i *contains* $m \geqslant 2$ *atoms of the same kind. Then, under the assumptions of Theorem* 1, *corresponding to any allocation in the core, there exists a price system such that*

(a) $y \succ_t X(t)$, $(y - X(t))\lambda(t)$ *is infinitesimal* $\Rightarrow py \gtrsim pX(t)$ *for almost all* t *in* T,

(b) $pX(t) \simeq pI(t)$ *for almost all* t *in* T_0,

(c) $pX(t) \simeq pX(s)$ *for all* $s, t \in A_i$, *for all* i.

[15] One can define the $\mathscr{T}_1$-*core* of $\mathscr{E}$ in a manner analogous to the $\mathscr{T}_1$-core of E^k (cf. Footnote 7). Then Theorem L1 goes through for allocations in the $\mathscr{T}_1$-core.

Let w be any large trader or atom in $\mathscr{E}$. The *split atom*, W, is a star-finite set such that all t in W are of the same type as w and

$$\lambda(w) \simeq \sum_{t \in W} \lambda(t).$$

The economy obtained after splitting some of the large traders in all but one of the A_i will be denoted by $\mathscr{E}^*$.

THEOREM L4. *Let $\mathscr{E}$ and $\mathscr{E}^*$ satisfy the assumptions of Theorem* L3. *In addition, let I be standardly bounded. Then, corresponding to any standardly bounded allocation X in the core of $\mathscr{E}^*$, there exists a price system p such that (p, x) is a competitive equilibrium.*

We end this section with some concluding remarks relating the above results to the work of Brown and Robinson [7], and to the sections dealing with nonstandard economies in [18]. In all of this work, the nonstandard limit economy is defined so that each trader has the same infinitesimal weight $1/\omega$. The nonstandard economy of this paper is a generalization in that it not only admits large traders but also allows traders to have non-uniform weights. Ignoring this aspect, note that the analogy of the relationship of Shitovitz' work to that of Aumann [2] does not carry through completely. If there are no atoms, Shitovitz' Theorem A implies the Aumann theorem on the coincidence of the core and the set of competitive equilibria. On the other hand, under the condition that $T_1 = \varnothing$, a correspondingly modified version of our Theorem L1 does not imply the Brown–Robinson theorem [18, Theorem 9]. The difficulty has to do with the fact that in our approach the integral over a negligible set may be a real number. More specifically, $pX(t) \lesssim pI(t)$ for almost all t in T does not imply $pX(t) \simeq pI(t)$ for almost all t in T, since for some $\tau \in T$, there is a possibility that $pX(\tau)\,\lambda(\tau)$ may be much greater than $pI(\tau)\,\lambda(\tau)$. In other words, the commodity bundle that τ gets in the core may be so large that it swamps the effect of the infinitesimal weight. In the continuum approach, $pX(t) \leqslant pI(t)$ for almost all t in T implies $pX(t) = pI(t)$ for almost all t in T. It is this point that is the source of the differences between our results and those of Shitovitz.

6. Proofs of Results for the Limit Economy

We can now furnish the proofs of Theorems L1–L4. The arguments follow those used by Shitovitz [27]. We begin with the following lemma.

LEMMA 1. *Let X be an allocation. Then under* A.2. *and* A.6. *the set* $\{t \in T_0 \mid X(t)\,\lambda(t) \not\simeq 0$ *or* $I(t)\,\lambda(t) \not\simeq 0\}$ *is contained in an internal set V such that* $\lambda(V) \simeq 0$.

Proof. The proof follows that of Lemma 1 in [18, Section V].

Proof of Theorem L1. Let $T' = T_0 - V$ where V is as in Lemma 1. Then T' is an internal set such that $\lambda(T') \simeq \lambda(T_0)$. Let X be an allocation in the core. Define the following sets:

$$F(t) = \{x(t) \in {}^*\Omega^n \mid x(t) \succ_t X(t);\ (x(t) - X(t))\,\lambda(t) \text{ infinitesimal}\},$$

$$G(t) = \{F(t) - X(t)\} \cup \{I(t) - X(t)\} \qquad (\forall t \in T'),$$

$$= \{F(t) - X(t)\} \qquad (\forall t \in T_1),$$

$$= \{I(t)\} \qquad (\forall t \in V).$$

By Brown's theorem (see Appendix), $\sum_{t\in T} G(t)\,\lambda(t)$ is S-convex[16]. By A.4., $\sum_{t\in T_1} G(t)\,\lambda(t)$ is also S-convex. Thus, $\mathscr{L} = \sum_{t\in T} G(t)\,\lambda(t)$ is S-convex. It is also nonempty and near standard.

We can also show that $0 \notin S\text{-Int}\,(\mathscr{L})$. Suppose not, i.e., there exists an internal function h such that for all t in T, $h(t) \in G(t)$ and $\sum_{t\in T} h(t)\,\lambda(t) = (-\alpha e)$ for some $\alpha \gtrsim 0$. Let $S = \{t \in T \mid h(t) = X(t) - I(t)\}$. S and therefore $T' - S$ are internal. Define

$$Y(t) = h(t) + X(t) \qquad (\forall t \in T_1 \cup (T' - S)),$$

$$= I(t) + (1/\lambda(S))\,\alpha e \qquad (\forall t \in S \cup V).$$

Let $B = T_1 \cup (T' - S)$. Then $\lambda(B) \gtrsim 0$ and

$$\sum_{t\in B} (Y(t) - I(t))\,\lambda(t) = \sum_{t\in T} X(t)\,\lambda(t) + (-\alpha e) - \sum_{t\in S} (I(t) - X(t))\,\lambda(t)$$

$$- \sum_{t\in B} I(t)\,\lambda(t) - \sum_{t\in S} X(t)\,\lambda(t)$$

$$= (-\alpha e) + \sum_{t\in T} (X(t) - I(t))\,\lambda(t)$$

Thus Y is an allocation that blocks X, a contradiction. If $V, S = \varnothing$, then substitute any negligible set W for V. The above argument then goes through.

Thus by the S-separation lemma (see Appendix), there exists a standard vector $p \neq 0$ such that $px \gtrsim 0$ for all x in $\mathscr{L}$. Now $px(t) \gtrsim 0$ for all $x(t)$

[16] A set B in ${}^*R^n$ is said to be *S-convex* if for all $x, y, \in B$ and all $\mu \in {}^*(0, 1)$, there exists $z \in B$ such that $z \simeq \mu x + (1 - \mu)y$. Note that by replacing $\simeq$ by $=$ we get the definition of a *convex* set.

in $G(t)$ for almost all t in T. Suppose not, i.e., there exist $z(t) \in G(t)$ for all t in S where $\lambda(S) \gtrsim 0$ such that $pz(t) \lesssim 0$. Let

$$\begin{aligned} Y_n(t) &= X(t) + (1/n)\, e \qquad (\forall t \in T - S), \\ &= X(t) + z(t) \qquad (\forall t \in S). \end{aligned}$$

Certainly $\sum_{t \in T} (Y_n(t) - X(t))\, \lambda(t) = m_n \in \mathscr{L}$. However,

$$pm_n = p \sum_{t \in S} z(t)\, \lambda(t) + (1/n).$$

For small enough $n \in N$, we have $pm_n \lesssim 0$, a contradiction.

It can be easily seen that $p > 0$. Suppose not, i.e., $p_1 \lesssim 0$ for, say, commodity 1. (Since p is a standard vector, none of its coordinates can be infinitesimals.) By monotonicity of preferences

$$X(t) + (M, \epsilon, \epsilon, \ldots, \epsilon) \succ_t X(t),$$

where $M \gtrsim 0$, and ϵ is a positive infinitesimal. Thus

$$pX(t) + p_1 M + \epsilon \sum_{i=2}^{n} p_i \gtrsim pX(t)$$

from the above paragraph. By making M large enough we can get a contradiction.

Finally, $(I(t) - X(t)) \in G(t)$ implies that $pX(t) \lesssim pI(t)$ for almost all t in T_0. Q.E.D.

Let T_1 be partitioned into d sets $A_1, A_2, \ldots, A_d$ such that all the traders in A_i are of the same type and there exists α with $0 < \alpha < 1$ such that for each i, there is a coalition $B_i \subseteq A_i$ with $\lambda(B_i) = \alpha\lambda(A_i)$. Let $B = \bigcup_i B_i$ and X be an allocation in the core such that $X(t) = X_i (\forall t \in A_i)$ for all i. Define the following correspondence.

$$\begin{aligned} H(t) &= \{F(t) - X(t)\} \cup \{I(t) - X(t)\} \cup \{X(t) - I(t)\} \qquad (\forall t \in T'), \\ &= \{F(t) - X(t)\} \qquad (\forall t \in T_1), \\ &= ((1 - \alpha)/\gamma)(I(t) - X(t)) \qquad (\forall t \in V). \end{aligned}$$

Let $\mathscr{L}_H$ denote the integral of this correspondence.

LEMMA 2. $0 \notin S\text{-Int}\,(\mathscr{L}_H)$.

Proof. Let $\gamma = \text{Min}(\alpha, 1 - \alpha)$. We shall show that $0 \notin S\text{-Int}(\gamma \mathscr{L}_H)$. It can be easily seen that this implies the lemma.

Note that

$$\sum_{t\in T_1} H(t)\,\lambda(t) = \sum_i \sum_{t\in A_i} H(t)\,\lambda(t) = \sum_i H_i\lambda(A_i)$$
$$= \sum_i H_i(\lambda(B_i)/\alpha) = (1/\alpha)\sum_{t\in B} H(t)\,\lambda(t),$$

and that $(\gamma/\alpha)\sum_{t\in B} H(t)\,\lambda(t) \subseteq \sum_{t\in B} H(t)\,\lambda(t)$. The last assertion follows from convexity of $\succ_t$, since $g \in H(t)$ implies $g + X(t) \succ_t X(t)$, which by convexity implies $\beta g + X(t) \succ_t X(t)$ for some $0 \leqslant \beta \leqslant 1$. Thus $\beta g \in H(t)$. Choose $\beta = \gamma/\alpha$.

Suppose $0 \in S\text{-Int}\,(\gamma\mathscr{L}_H)$. Then there exists an internal function h such that $h(t) \in H(t)$ for all t in T and

$$\sum_{t\in B} h(t)\,\lambda(t) + \gamma\sum_{t\in T'} h(t)\,\lambda(t) + (1-\alpha)\sum_{t\in V}(I(t) - X(t)) = (-\delta e) \quad \delta \gtrless 0.$$

Let $D = \{t \in T' \mid h(t) = X(t) - I(t)\}$ and $C = T' - D$. Form

$$\mu(L) = \left\langle \sum_{t\in L} I(t)\,\lambda(t), \sum_{t\in L} X(t)\,\lambda(t), \sum_{t\in L} h(t)\,\lambda(t)\right\rangle.$$

The hypotheses of Loeb's theorem are fulfilled for all $L \subseteq T'$. Thus, there exist internal sets C_1, C_2, D_1, D_2 such that $\mu(C_1) \simeq \gamma\mu(C)$; $\mu(C_2) \simeq (1-\alpha)\mu(C)$; $\mu(D_1) \simeq (\gamma/1-\alpha)\mu(D)$ and $\mu(D_2) \simeq (1-\alpha)\mu(D - D_1)$. Let

$$a_1 = \sum_{t\in C_2}(X(t) - I(t))\,\lambda(t) \simeq (1-\alpha)\sum_{t\in C}(X(t) - I(t))\,\lambda(t),$$

$$a_2 = (1-\alpha)\sum_{t\in D_1}(X(t) - I(t))\,\lambda(t),$$

$$a_3 = \sum_{t\in D_2}(X(t) - I(t))\,\lambda(t) \simeq (1-\alpha)\sum_{t\in D-D_1}(X(t) - I(t))\,\lambda(t).$$

Thus,

$$a_1 + a_2 + a_3 \simeq (1-\alpha)\sum_{t\in D\cup C}(X(t) - I(t))\,\lambda(t)$$
$$= (1-\alpha)\sum_{t\in T_1\cup V}(I(t) - X(t))\,\lambda(t)$$
$$\simeq (1-\alpha)\sum_{t\in V}(I(t) - X(t))\,\lambda(t) + \sum_{t\in B'}(I(t) - X(t))\,\lambda(t)$$

where $B' = T_1 - B$. Now define $S = C_1 \cup C_2 \cup D_2 \cup T_1$, and let

$$Z(t) = X(t) + h(t) \qquad (\forall t \in B \cup C_1),$$
$$= I(t) \qquad (\forall t \in B' \cup C_2 \cup D_2),$$
$$= X(t) \qquad (\forall t \in (T - S)).$$

Then

$$\sum_{t\in T}(Z(t)-X(t))\,\lambda(t) \simeq a_1+a_2+a_3-a_1-a_3+\sum_{t\in B\cup C_1} h(t)\,\lambda(t) - (1-\alpha)\sum_{t\in V}(I(t)-X(t))\,\lambda(t)$$

$$\simeq \sum_{t\in B} h(t)\,\lambda(t)+\gamma\sum_{t\in T} h(t)\,\lambda(t) = (-\delta e).$$

Let $\psi = \sum_{t\in T}(Z(t)-I(t))\,\lambda(t)$. ψ is a vector which is negative in all its components. Form

$$Z'(t) = Z(t) - (\psi/\lambda(T-S)) \qquad (\forall t\in T-S),$$

$$= Z(t) \qquad \text{all other } t \text{ in } T.$$

Then Z' is an allocation which blocks X via the coalition $B\cup C_1\cup(T-S)$, a contradiction. Q.E.D.

Let X be an allocation in the core. For each $i(1\leqslant i\leqslant d)$, define

$$Z_i = (1/\lambda(A_i))\sum_{t\in A_i} X(t)\,\lambda(t).$$

LEMMA[17] 3. *$X(t)\gtrsim_t Z_i$ for all t in A_i, for all i.*

Proof. We shall first prove this for $d=1$. Suppose the lemma is false, i.e., $\lambda(W)\gtrsim 0$ where $W=\{t\in T_1 \mid Z_1\succ_t X(t)\}$. Let $\alpha=\lambda(W)/\lambda(T_1)$. Either $\alpha=1$ or $\alpha\lesssim 1$. If $\alpha=1$, form

$$Y(t) = Z(t)-\delta e \qquad (\forall t\in T_1),$$

$$= X(t)+(\delta e)(\lambda(T_1)/\lambda(T_0)) \qquad (\forall t\in T_0)$$

where δ is such that $S(Z(t),\delta)\succ_t X(t)$ for all t in T_1. By continuity of preferences, we know that it exists. Monotonicity of preferences insures that Y blocks X. Note that Y is an allocation.

Thus suppose that $\alpha\lesssim 1$. Let S be an internal set such that $\sum_{t\in S}(X(t)-I(t))\,\lambda(t)\simeq\alpha\sum_{t\in T'}(X(t)-I(t))\,\lambda(t)$. By Loeb's theorem, we know that it exists. Now define

$$Y(t) = X(t)+he \qquad (\forall t\in S)$$

$$= Z_1 \qquad (\forall t\in W)$$

$$= I(t)-\psi \qquad (\forall t\in T-(S\cup W))$$

[17] Since $\succ_t$ is complete for all t in T_1, we can define $x\sim_t y$ as $x\not\succ_t y$ and $y\not\succ_t x$. Then $x\gtrsim_t y$ means $x\succ_t y$ or $x\sim_t y$.

where h is a positive infinitesimal and

$$\psi = \sum_{t \in S \cup W} [(Y(t) - I(t))\, \lambda(t)]/\lambda(S \cup W).$$

It can easily be seen that $\psi \simeq 0$. Thus Y is an allocation that blocks X, a contradiction.

Now let there be d *kinds* of large traders such that there are m traders of each kind. Suppose the lemma is false; then for at least one i it is true that $\lambda(C_i) \gtrapprox 0$ where $C_i = \{t \in A_i \mid Z_i \succ_i X(t)\}$. For each such i pick an atom, W_i, and for all other i let W_i be an atom such that $Z_i \gtrsim_i X(W_i)$. Such an atom will exist by convexity and irreflexity of $\succ_i$. Define $W = \bigcup_i W_i$ and let $\alpha = \lambda(W_i)/\lambda(A_i)$. Since we are picking only one trader from each of the A_i, $\alpha = \lambda(W)/\lambda(T_1) = 1/m$. Let S and ψ be defined as above and let

$$\begin{aligned} Y(t) &= X(t) + he && (\forall t \in S), \\ &= Z_i && (t = W_i)(W_i \in C_i), \\ &= Z_i + he && \text{all other } W_i \text{ in } W, \\ &= I(t) - \psi && (\forall t \in T - (S \cup W)). \end{aligned}$$

As above, Y is an allocation that blocks X, a contradiction. Note that for Y to be well defined we need the assumption that $I(t) \gtrapprox 0$ for all t in T. Q.E.D.

LEMMA 4. *$X(t) \sim_i Z_i$ for all t in A_i, for all i.*

Proof. Let $D_i = \{t \in A_i \mid X(t) \succ_i Z_i\}$ and $B_i = A_i - D_i$. Suppose, contrary to the lemma, that $\lambda(D_i) \gtrapprox 0$ for some i. Let $\alpha = \lambda(D_i)/\lambda(A_i)$ and define $H = \{y \mid y \succ_i Z_i\}$ and $E = \{y \mid y \gtrsim_i Z_i\}$. By convexity of $\succ_i$, $a = (1/\lambda(D_i)) \sum_{t \in D_i} X(t)\, \lambda(t) \in H$ and $b = (1/\lambda(B_i)) \sum_{t \in B_i} X(t)\, \lambda(t) \in E$. Therefore, $Z_i = \alpha a + (1 - \alpha)\, b \succ_i Z_i$, a contradiction. Q.E.D.

We first prove Theorem L3.

Proof of Theorem L3. Let X be an allocation in the core. Form

$$\begin{aligned} Z(t) &= X(t) && (\forall t \in T_0), \\ &= Z_i && (\forall t \in A_i) \text{ for all } i. \end{aligned}$$

Certainly Z is an allocation in the core. If not, we can get a contradiction to the fact that X is in the core by virtue of Lemma 4. Define $H(t)$ in terms of Z and let $\mathscr{L}_H$ be as before. Then $\mathscr{L}_H$ is nonempty, S-convex, and finite. Since Z satisfies the condition of Lemma 2, $0 \notin S\text{-Int}(\mathscr{L}_H)$. Thus, by the S-separation lemma, there exists a standard vector $p \neq 0$ such that $px \gtrsim 0$ for all x in $\mathscr{L}_H$. Just as in the proof of Theorem 1, we can show

that this implies $px(t) \gtrsim 0$ for all $x(t) \in H(t)$, for almost all t in T. It can also be shown that $p > 0$. Finally $(I(t) - Z(t)) \cup (Z(t) - I(t)) \in H(t)$ implies that $pZ(t) \simeq pI(t)$ for almost all t in T_0.

We can now show that $pZ(t) \simeq pX(t)$ for all t in T. Note that it is trivially so for t in T_0. For any other t, $X(t) + he \succ_t X(t) \sim_t Z_t$ for some infinitesimal h. Thus, $pX(t) \lesssim pZ_t$. Since $\sum_{t \in A_i} pX(t)\lambda(t) = \sum_{t \in A_i} pZ_t$, and there are a finite number of large traders in each A_i, the assertion follows. This completes the proof. Q.E.D.

Proof of Theorem L2. Let $d = 1$. Note that the above arguments carry through. Since $pX(t) \simeq pX(s)$ for all $t, s \in T_1$ and since

$$\sum_{t \in T_0} (pX(t) - pI(t))\lambda(t) \simeq \sum_{t \in T'} (pX(t) - pI(t))\lambda(t)$$

under the condition of the theorem, $pX(t) \simeq pI(t) = pI_1$ for all t in T_1. Q.E.D.

Proof of Theorem L4. Without loss of generality assume that that atoms in A_d are not split. Let W_i^j be the set of split atoms in A_i where $j = 1, 2, \ldots, f_i$ and $i = 1, 2, \ldots, d-1$. Let X be an allocation in the core of $\mathscr{E}$. Then, by Theorem 3, there exists a price system p such that

(a) $y \succ_t X(t) \Rightarrow py \gtrsim pX(t)$ for almost all t in T,

(b) $pX(t) \simeq pI(t)$ for almost all t in T_0 and all t in W_i^j, all j, all $i = d$,

(c) $pX(t) \simeq pX(s)(\forall s,\ t \in (A_i - \bigcup_j W_i^j))(i = 1, \ldots, k-1)(\forall s,\ t \in A_d)$.

Form the mapping $Z : T \to {}^*\Omega^n$ where

$$Z(t) = Z_{W_i^j} \qquad (\forall t \in W_i^j)\,(j = 1, 2, \ldots, f_i)\,(i = 1, \ldots, d-1),$$

$$= X(t) \qquad \text{all other } t.$$

Note that we have compressed the mapping X to a smaller domain, and that $Z_{W_i^j} = (1/\lambda(W_i^j)) \sum_{t \in W_i^j} X(t)\lambda(t)$. Then (b) implies that $pZ_{W_i^j} \simeq pI_i$ for all j, all $i = d$, where I_i is the endowment of the traders in A_i.

We can now assert that $\{t \in W_i^j \mid X(t) \not\sim_t Z_{W_i^j}\}$ is empty for all j and for all $i = d$. The argument is exactly as in Lemmas 3 and 4. Thus the allocation Z is in the core of $\mathscr{E}$. Given this, (a) implies that

$$y \succ_t Z_{W_i^j} \Rightarrow y \succ_t X(t) \Rightarrow py \gtrsim pX(t) \simeq pI_i \simeq pZ_{W_i^j}.$$

Since Z is in the core of $\mathscr{E}$, Lemma 4 gives us the fact that $Z(t) \sim_i Z(s)$ for all $s, t \in A_i$, for all i. Thus $pZ(t) \simeq pZ(s)$ for all $s, t \in A_i$, for all i. Thus, except for the traders in A_d, for all t in A_i, $pX(t) \simeq pZ_{W_i^j} \simeq pI_i$. Since X

and I are standardly bounded, we also have $pX(t) \simeq pI_d$ for all t in A_d. Hence, $pX(t) \simeq pI(t)$ for almost all t in T, and (p, X) is a competitive equilibrium. Q.E.D.

7. Appendix

THEOREM (LOEB). *Let T be a start-finite set and for each $t \in T$. Let $v(t)$ be a vector in n-space $^*R^n$ with $nv(t) \simeq 0$. Then the following is true: Given internal sets $B \subseteq T$ and $C \subseteq T$ and given $\mu \in {}^*R$ with $0 < \mu < 1$, there is an internal set $D \subseteq T$ with $S(D) \simeq \mu S(B) + (1 - \mu) S(C)$ where for any internal set B, $S(B) = \sum_{t \in B} v(t)$.*

Proof. See [23].

THEOREM (ROBINSON). *Let (s_n) be an internal sequence such that s_n is infinitesimal for all finite n. Then there exists an infinite natural number ν such that s_n is infinitesimal for all $n < \nu$.*

Proof. (See [24, p. 65].)

Let $F : T \to \mathscr{P}(^*R^n)$ be a correspondence whose integral is defined as $\sum_{t \in T} f(t)\, \lambda(t)$ where $f \in \mathscr{F}_F$, the set of all internal selections from $\prod_{t \in T} F(t)$.

THEOREM (BROWN). *If for all t in T, $F(t)\, \lambda(t) \simeq 0$ then $\mathscr{L}$ is S-convex.*

Proof. The proof is a trivial modification of in [5, Proof of Theorem 2].

S-SEPARATION LEMMA. *If A is a near standard, S-convex set in $^*R^n$ and $0 \notin S\text{-Int}(A)$, then there exists a standard vector $p \neq 0$ such that for all $y \in A$, $py \gtrsim 0$.*

Proof. See [18] which shows how the proof of a similar lemma in [7] can be modified to cover this case.

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Optimal Planning under Transaction Costs: The Demand for Money and Other Assets*

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A dynamic, continuous-time, inventory optimization model of household behavior is developed and analyzed. In this model, the household is allowed to hold stocks of money and of a single consumption good. A flow of consumption is achieved by depleting the inventory of the consumption good. The household maximizes the present value of the flow of utility, a function only of the rate of consumption. It is found that if the initial stock of the good is positive, there will be a positive demand for both the good and cash balances at each point in time. Further, once purchasing of the good has begun, it occurs continuously thereafter, and the rate of purchase increases with time. The consumption good inventory and the rate of consumption decrease monotonically. Comparative dynamics results are described for a certain class of transaction cost and utility functions.

1. INTRODUCTION

This paper deals with a partial equilibrium model of household behavior under transaction costs with a view to describing the household's demand for inventories of money and other assets in an inflationary environment under perfect-foresight expectation.

Analyses of the demand for money by households have taken two basic approaches. In the simplest approach, real balances are assumed to be an argument in the utility function, and the analysis proceeds along the usual theory of value line (see, for example, [14, 12, 10, 1]).

The second basic approach is one of inventory optimization under transaction costs. These models assume that all transactions involve

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instantaneous stock transfers. These are accompanied by transaction costs that may depend on the amounts transferred. Consumers are generally assumed to minimize the cost (transaction costs plus foregone interest) of financing a given flow of consumption. The basic theorem is that if transaction costs contain a fixed component, transactions will occur only at discrete instants of time. This result follows from the assumption that stocks can be transferred instantaneously, so that transactions require no time to complete. Since consumption is assumed to be a continuous flow, if stocks of the consumption good cannot be held, a demand for cash balances is guaranteed (see [4, 16]). If a stock of the good can be held, a demand for this stock is assured (see [6]).

The model of this paper is an extension, in several directions, of the inventory optimization approach. First, stocks are assumed to adjust over time through a continuous flow of transactions. Transaction costs accrue at a continuous rate whenever transactions occur at a nonzero rate. Second, the consumer's objective is to maximize the present value of a flow of utility derived from an endogenous flow of consumption. Third, the dynamics of the inventory optimization problem are taken explicitly into account. The analysis yields a characterization of the optimal paths of both flow variables (the rate of consumption, the rate of spending) and stock variables (cash balances, consumption good inventories).

Some of the main results are:

(i) Demand for cash balances is positive even though both consumption and transactions are assumed to occur as continuous flows. Desired cash balances first increase with the age of the consumer, then decline.

(ii) The desired stock of consumption goods declines monotonically with age.

(iii) Once a consumer begins to purchase goods, he thereafter continues to do so. The rate of purchase increases with age, while the rate of consumption falls. Thus, near the beginning of the planning period, consumption occurs mainly by depleting one's inventory of goods. Near the end, consumption is financed mainly by purchases out of accumulated cash balances.

In addition, several results of a qualitative "comparative dynamics" nature are derived. Some of these differ from those obtained in previous models.

The next section is devoted to formally defining the model. In Section 3, the main results are discussed, while the formal statements and proofs are deferred to Section 4. Section 5 contains concluding remarks.

2. The Model of the Household

The household is assumed to maximize a sum over the flow of future utility. This flow of utility is derived from a flow of consumption over a finite lifetime. The flow of consumption occurs by depleting a stock of a single consumption good. The good has storage costs (including exogeneous depreciation) that are proportional to the stock and known with certainty.

There is only one asset other than the consumption good which may be held in this model. This asset, which I shall call money, can be exchanged at any rate for the consumption good. Similarly, the good can be exchanged for money. The price at which money and the good may be exchanged is assumed to rise over time at a constant percentage rate. This behavior of the price of the consumption good in terms of money is fully anticipated with certainty, and taken to be exogenous by the household. No interest is paid on the stock of money (or the stock of the consumption good). Income is paid as a continuous flow in the form of money; its path is anticipated with certainty by the household.

Transactions are defined to be occuring whenever money and the good are being exchanged. Income received is assumed to be net of any costs involved with its collection. While a transaction is occurring, transaction costs accrue at some positive rate that depends on the rate at which goods and money are being exhanged and may have a fixed component independent of this rate. These costs deplete the household's stock of money, and the real rate at which these costs accrue (in units of consumption good per unit of time) is assumed to depend only on the rate at which the stock of goods is being augmented, or reduced by selling money (this rate is also expressed in units of the good per unit of time). Transaction costs are symmetric with respect to buying or selling the good. The transaction cost function is assumed to be stationary over time and to be known by the household.

Finally, the instantaneous flow of utility is assumed to be a stationary function of the rate of consumption of the good only (money is not consumed). This function is known by the household.[1]

[1] With all the knowledge and certainty which has been assumed, one might ask (see [7]) why the household does not arrange for the entire path of goods purchases (or sales) at the outset, if economies of scale in transactions are assumed. This has been ruled out by the assumption that the transaction must occur while the good is being bought (or sold). The omission of the possibility of accumulating a stock of "transaction capital" which may be used to effect transactions without further costs (above the accumulation costs), means that forward contracts are not allowed. This assumption may be viewed as assuming that storage costs on transaction capital are prohibitive. Since these costs are a function of the degree of uncertainty of the future, there is an

The following notation will be used throughout this study. Let

T be the planning horizon, i.e., the household is only concerned with the time interval $[0, T]$;

I, $[0, T]$;

$c(t)$, the instantaneous rate of consumption of the single good at time $t \in I$;

$u(c(t))$, the instantaneous rate of accumulation of utility at time t if goods are consumed at rate $c(t)$ at t;

$K(t)$, *stock* of the single good at time t;

$m(t)$, *real stock* of money (in goods units) at time t;

$\rho(t)$, instantaneous rate of purchase ($\rho > 0$) or sale ($\rho < 0$) of goods (in goods units per unit time) at time t;

δ, exogenous storage costs per unit of the good (units of one per unit time);

π, exogenous rate of inflation in the price of the good in terms of money;

$f(\rho)$, real rate of transactions costs if goods are being purchased at rate ρ (or sold at rate $-\rho$);

$R(t)$, rate at which real income is received at time t.

The transaction cost function f is assumed to have the following properties.

$$f(0) = 0, \tag{2.1}$$

$$f(\rho) = f(-\rho) \qquad \text{for all } \rho \ (f \text{ is symmetric with respect to } \rho = 0), \tag{2.2}$$

$$\bar{f}(\rho) \text{ has a unique global minimum at } \hat{\rho} > 0 \tag{2.3}$$

where $\bar{f}(\rho) = |f(\rho)/\rho|$. I also assume that f, f', and f'' are continuous except possibly at $\rho = 0$. Note from (2.2) and (2.3)

$$f'(\rho) \gtrless 0 \qquad \text{as } \rho \gtrless 0, \tag{2.4}$$

and there exists $0 \leqslant \tilde{\rho} \leqslant \hat{\rho}$ such that $f''(\tilde{\rho}) = 0$, and if $\tilde{\rho} > 0$

$$\begin{aligned} f''(\rho) &> 0 \qquad \text{for} \quad \rho \in (-\tilde{\rho}, 0) \cup (\tilde{\rho}, \infty), \\ f''(\rho) &< 0 \qquad \text{for} \quad \rho \in (-\infty, -\tilde{\rho}) \cup (0, \tilde{\rho}). \end{aligned} \tag{2.5}$$

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implicit uncertainty here. Let me add, however, that the existence of an increasing component of transaction costs would limit the accumulation of such capital even if its storage costs were not prohibitive.

With respect to the utility function u, I assume that u, u', u'' are continuous for every $c \geqslant 0$ and $u'(c) > 0$, $u''(c) < 0$ for every $c \geqslant 0$. Also assume $u'(0) = +\infty$.

The state variables for this problem are K and m. Their equations of motion are

$$\dot{K}(t) = \rho(t) - c(t) - \delta K(t), \tag{2.6}$$

and $d(p(t)m(t))/dt = p(t)\,R(t) - p(t)\,[\rho(t) + f(\rho(t))]$ where $p(t)$ is the price of goods in terms of money at time t, or

$$\dot{p}m + \dot{m}p = pR - p(\rho + f(\rho)),$$

or, dividing by p and using $\dot{p}/p = \pi$, we have

$$\dot{m}(t) = R(t) - [\rho(t) + f(\rho(t))] - \pi m(t). \tag{2.7}$$

Equation (2.7) reflects the assumption that income is paid as money, transaction costs are paid for out of real balances, and the price of the good in terms of money rises at the constant exponential rate π. Equation (2.6) is self-explanatory.

In addition, the control (c, ρ) must lie, for every $t \in I$, in $R_+ \times R$, and the state (K, m) must lie, for every $t \in I$, in $R_+ \times R_+$ (borrowing is forbidden). Also (c, ρ) must be piecewise continuous on I.

Let $\mathscr{R}$ be the set of arcs $(K(t), m(t)), (c(t), \rho(t))$ which satisfy the above conditions [including Eqs. (2.6) and (2.7)] and such that

$$K(0) = K_0\,,\ m(0) = m_0$$

where $K_0 \geqslant 0$ and $m_0 \geqslant 0$ are given numbers. Then the household's problem is to

$$\text{maximize } \Omega = \int_0^T u(c(t))\,dt$$

over the set of arcs $\mathscr{R}$.

3. The Optimal Paths: Description and Discussion

In this section I discuss, without proof, the major implications of the model. The proofs are set forth in the next section.

The first result, which is not completely straightforward, is that terminal stocks of both money and goods are zero. That the terminal stock of goods is zero is not surprising since any goods remaining near the terminal time may be consumed at no cost, thus increasing utility. The reason that the

terminal stock of money is zero, in spite of transaction costs of converting it into the consumption good, is that if one has accumulated any money, even if transactions costs are extremely high, one can always spend at least for a short period of time and get some consumption from his real balance.

In the remaining results, I assume that real income is constant throughout the life-cycle and that the depreciation rate of goods exceeds the rate of inflation. The latter assumption seems quite reasonable for most types of consumption goods. It is then shown that if the consumer starts with a positive inventory of goods, he will maintain a positive inventory which declines over the life-cycle. It follows that one will always maintain a positive cash balance except at the terminal age and possibly at the initial age. It is also shown that once one begins to purchase goods, he does so continuously for the remainder of the period, i.e., in this model, one does not purchase goods for a time, then retire from the market for a time, then return to purchasing, etc. The reason for this is as follows. The difference in storage costs for real balances and the good grows with the storage time. Therefore if it "pays" to store wealth as money until, say, time t' (rather than as the good), it will be even more "profitable" to store money instead of the good until $t > t'$, because while the savings in storage costs increase with time, the transaction costs of converting the money into the good do not. Thus, if one wishes to store wealth as money until t', then convert it to the good, he will be even more willing (i.e., store and convert more money) to do this until $t > t'$. It also follows that the rate of purchase will increase with age while the rate of consumption will decline.

The general flavor of these results can be motivated intuitively as follows. If the initial stock of the good is positive, a declining rate of consumption can be achieved by consuming in excess of income in the beginning, then, as inventories are depleted, lowering the rate of consumption. This could be done without accumulating cash balances by purchasing goods at a rate which will just exhaust income. A higher consumption path can be achieved, however, by purchasing below income in the early stages and accumulating money, then spending it off later if (as I assume) money is cheaper to store than the good. In this case, one can consume at a higher rate in the beginning even though he is spending less than income by depleting his goods inventories more rapidly. Later, he can also consume at a higher rate than in the case where all income is spent since he has, essentially, replaced his more rapidly depleted stock of the good with a stock of money, a more "durable" asset. Loosely speaking, then, the problem is to plan the path of cash balances so that, at the margin, the savings in storage costs are just offset by the transaction costs of converting the cash back into consumption goods.

When the initial stock of the consumption good is zero, somewhat different results obtain. First it is shown that a stock of goods will not be accumulated. In this case, since there are economies of scale in transacting, if one consumed slowly at first while building up a cash balance, then ran this balance down by consuming more rapidly, he could get more total consumption than if he simply consumed his income (or if $m_0 > 0$, consumed at a decreasing rate, more rapidly than receipt of income at least initially). Although accumulating then decumulating cash allows higher total consumption, with concave utility, neutral time preference, and a negative interest rate ($-\pi$), this would be a less preferred time pattern for consumption as compared with a constant or decreasing rate of consumption. It is shown that the second effect dominates, i.e., if real balances are also initially zero, none will be accumulated. If real balances are initially positive, the desired stock will decline monotonically over the life-cycle.

In addition to the above general results, it is possible to derive several interesting comparative dynamics results for a specific class of transaction cost functions and utility functions.[2]

First, as expected, an increase in real income causes an increase in the rate of spending at each age. The marginal propensity to spend is, however, less than unity at the beginning of the period. Further, an increase in the initial stock of real balances results in an increased rate of spending at each age. An increase in the storage cost of goods results in an increase in spending in the early part of the life-cycle, with a subsequent decline in spending. An increase in the rate of inflation also causes the rate of spending to increase initially.

Desired cash balances increase at each age with an increase in their initial level or with an increase in real income. An increase in the cost of storing goods or a fall in the anticipated rate of inflation also results in increased desired real balances at each age.

Desired real balances fall at each age when transactions become more expensive (either the fixed or the variable component of transaction costs rises). This result is contrary to the results of some previous studies (see, e.g., [2]). Intuitively, the increase in transaction costs has two effects on desired cash balances: a negative real wealth effect and a positive effect through a reduction in the rate of spending. The former effect dominates.

[2] Derivations of these results are available from the author on request. The transaction cost function is assumed to be of the form

$$f(\rho) = 0, \quad \rho = 0$$
$$= \phi + a\rho^{\beta}, \quad \rho \neq 0, \quad \text{some } \phi > 0,\ a > 0,$$

and utility is of the form $u(c) = c^{\lambda}$ where $0 < \lambda < 1$.

The rate of consumption falls at each age (and, therefore, total welfare falls) with an increase in the anticipated rate of inflation and with a decrease in real income. Total welfare also declines as a result of increased storage costs of goods. An increase in the cost of transacting also results in a decline in consumption at each age.

Since, in the present model, transactions occur over time, it is perfectly natural to identify the fixed component of transaction costs with the real wage, i.e., the opportunity cost of time. If this is done, it follows that an increase in the real wage, holding real income constant, reduces desired real balances at each age. Again, this result is the opposite of that obtained in [2].

4. The Optimal Paths: Formal Derivations

In this section, I state formally the necessary conditions for an optimum for the problem stated in Section 2 and derive the results described in Section 3. The necessary conditions are derived from those set forth in [13, pp. 257–316; 8, pp. 352–374] for the general optimal control problem with bounded state variables.[3]

4.1 *Conditions When* $\bar{m} > 0, \bar{K} > 0, \bar{\rho} \neq 0, \bar{c} > 0$

Let $y(t) = (y_1(t), y_2(t))$ be the costate variable for $(K(t), m(t))$. If $(\bar{K}(t), \bar{m}(t)), (\bar{c}(t), \bar{\rho}(t))$ solves our problem, then for every t such that $\bar{K}(t) > 0$, $\bar{m}(t) > 0$, $\bar{\rho}(t) \neq 0$, and $\bar{c}(t) > 0$, we have $y_1 \geqslant 0$, $y_2 \geqslant 0$,

$$u'(c) = y_1, \tag{4.1.1}$$

$$y_1 = y_2(1 + f'(\bar{\rho})) \quad \text{or} \quad f'(\bar{\rho}) = (y_1 - y_2)/y_2 = (y_1/y_2) - 1, \tag{4.1.2}$$

$$\dot{y}_1 = \delta y_1, \tag{4.1.3}$$

$$\dot{y}_2 = \pi y_2. \tag{4.1.4}$$

These conditions imply

$$\dot{\bar{c}} = \delta u'(\bar{c})/u''(\bar{c}) < 0, \tag{4.1.5}$$

$$\dot{\bar{\rho}} = (\delta - \pi)[1 + f'(\bar{\rho})]/f''(\bar{\rho}). \tag{4.1.6}$$

Since from (4.1.2), $f'(\bar{\rho}) \geqslant -1$, if $\delta > \pi$, $\dot{\bar{\rho}} > 0$ whenever $\bar{\rho} \neq 0$, $\bar{K} > 0$, $\bar{m} > 0$, and $\bar{c} > 0$.

[3] In general, the differential equations for the costate variables hold only on intervals of continuity of the controls. In this case, however, the controls do not appear in these equations, and they hold at all points in I.

4.2. *Conditions When* $\bar{m} > 0, \bar{K} > 0, \bar{\rho} = 0, \bar{c} > 0$

If $t \in I$ is such that $\bar{m}(t) > 0, \bar{K}(t) > 0$, and $\bar{c}(t) > 0$, then $\bar{\rho}(t) = 0$ if and only if either

$$\begin{aligned} &\text{(a)} \quad y_1 = y_2 \text{ ; or} \\ &\text{(b)} \quad y_1 > y_2 \text{ and } \bar{f}(\hat{\rho}) > (y_1 - y_2)/y_2 \text{ ; or} \\ &\text{(c)} \quad y_1 < y_2 \text{ and } -\bar{f}(\hat{\rho}) < (y_1 - y_2)/y_2 . \end{aligned} \tag{4.2.1}$$

It is also easily shown that if $\bar{\rho} \neq 0$, then either $\bar{\rho} > \tilde{\rho}$ or $\bar{\rho} < -\tilde{\rho}$. This follows from the second-order condition for a maximum of the Hamiltonian associated with the problem defined in Section 2. Thus if $\bar{\rho} > 0$, we have $f'(\bar{\rho}) = (y_1 - y_2)/y_2 \geqslant \bar{f}(\hat{\rho})$. But $\hat{\rho}$ is such that $f'(\hat{\rho}) = \bar{f}(\hat{\rho})$ (see definition of $\hat{\rho}$ in Section 2). Thus, since $\bar{\rho} > \tilde{\rho}$, by (2.5), $\bar{\rho} \geqslant \hat{\rho}$. Similarly, if $\bar{\rho} < 0$, we have $f'(\bar{\rho}) = (y_1 - y_2)/y_2 \leqslant -f(\hat{\rho})$, and the same argument yields $\bar{\rho} \leqslant -\hat{\rho}$. This proves the following lemma.

LEMMA 4.2.2. *If* $\bar{\rho} > 0$, *then* $\bar{\rho} \geqslant \hat{\rho}$, *and if* $\bar{\rho} < 0$, *then* $\bar{\rho} \leqslant -\hat{\rho}$.

4.3 *Conditions When* $\bar{m} = 0$ *and/or* $\bar{K} = 0, \bar{c} > 0$

Let

$$Z_k = \{t \in [0, T] : \bar{K}(t) = 0\},$$
$$Z_m = \{t \in [0, T] : \bar{m}(t) = 0\}.$$

Since $\bar{m}$ and $\bar{K}$ are continuous, Z_k and Z_m will each be a union of closed intervals (some of which may be a single point). Thus, we may write $Z_k = \bigcup_{i=1}^{\infty} [t_{k_i}^-, t_{k_i}^+]$ and $Z_m = \bigcup_{j=1}^{\infty} [t_{m_j}^-, t_{m_j}^+]$ where

$$0 \leqslant t_{k_1}^- \leqslant t_{k_1}^+ < t_{k_2}^- \leqslant t_{k_2}^+ < \cdots < t_{k_r}^- \leqslant t_{k_r}^+ < \cdots,$$

$$0 \leqslant t_{m_1}^- \leqslant t_{m_1}^+ < t_{m_2}^- \leqslant t_{m_2}^+ < \cdots < t_{m_s}^- \leqslant t_{m_s}^+ < \cdots.$$

Now let $\bar{Z}_k = Z_k - Z_m$ and $\bar{Z}_m = Z_m - Z_k$. Then there exist functions $\lambda_1 : Z_k \to R$ and $\lambda_2 : Z_m \to R$ such that for every $t \in \bar{Z}_k$,

$$\lambda_1(t) \geqslant 0, \text{ nonincreasing, and continuous on } [t_{k_i}^-, t_{k_i}^+] \text{ for every } i. \tag{4.3.1}$$

$$u'(\bar{c}) = y_1 + \lambda_1 . \tag{4.3.2}$$

If

$$\bar{\rho} \neq 0, \qquad f'(\bar{\rho}) = (y_1 + \lambda_1 - y_2)/y_2 = ((y_1 + \lambda_1)/y_2) - 1. \tag{4.3.3}$$

If $\bar{\rho} = 0$, then either

$$\begin{aligned} &\text{(a)} \quad y_1 + \lambda_1 = y_2 \text{ ; or} \\ &\text{(b)} \quad y_1 + \lambda_1 > y_2 \text{ and } \bar{f}(\bar{\rho}) > (y_1 + \lambda_1 - y_2)/y_2 \text{ ; or} \\ &\text{(c)} \quad y_1 + \lambda_1 < y_2 \text{ and } -\bar{f}(\bar{\rho}) < (y_1 + \lambda_1 - y_2)/y_2 . \end{aligned} \tag{4.3.4}$$

$$\bar{\rho} = \bar{c} \text{ on Interior } \bar{Z}_k . \tag{4.3.5}$$

Note that since $\bar{\rho}$ is piecewise continuous, from (4.3.5), $\bar{\rho} > 0$ whenever $\bar{c} > 0$ except possibly at 0 or T. Since we will show that $\bar{c} > 0$ on I (see Lemma 4.3.20), conditions (4.3.4) will hold at most on $\bar{Z}_k \cap \{0, T\}$.

$$\dot{y}_1 = \delta(y_1 + \lambda_1). \tag{4.3.6}$$

$$\dot{y}_2 = \pi y_2 \qquad \text{[same as (4.1.4)]}. \tag{4.3.7}$$

$$y_1(t_{k_i}^- - 0) = y_1(t_{k_i}^- + 0) \qquad \text{unless} \quad t_{k_i}^- = 0 \text{ or } T, \tag{4.3.8}$$

and for every $t \in \bar{Z}_m$,

$$\lambda_2(t) \geqslant 0, \text{ nonincreasing, and continuous on each } [t_{m_j}^-, t_{m_j}^+]. \tag{4.3.9}$$

$$u'(\bar{c}) = y_1 . \tag{4.3.10}$$

If

$$\bar{\rho} \neq 0, \qquad f'(\bar{\rho}) = y_1/(y_2 + \lambda_2) - 1. \tag{4.3.11}$$

If $\bar{\rho} = 0$, then either

$$\begin{aligned} &\text{(a)} \quad y_1 = y_2 + \lambda_2 \text{ ; or} \\ &\text{(b)} \quad y_1 > y_2 + \lambda_2 \text{ and } \bar{f}(\bar{\rho}) > (y_1 - y_2 - \lambda_2)/(y_2 + \lambda_2); \\ &\text{(c)} \quad y_1 < y_2 + \lambda_2 \text{ and } -\bar{f}(\bar{\rho}) < (y_1 - y_2 - \lambda_2)/(y_2 + \lambda_2). \end{aligned} \tag{4.3.12}$$

$$R = \rho + f(\rho) \text{ on Int } \bar{Z}_m \qquad (\text{or } \bar{\rho} = \rho_R \text{ where } \rho_R > 0 \text{ is such that } \rho_R + f(\rho_R) = R). \tag{4.3.13}$$

$$\dot{y}_1 = \delta y_1 . \tag{4.3.14}$$

$$\dot{y}_2 = \pi(y_2 + \lambda_2). \tag{4.3.15}$$

$$y_2(t_{m_j}^- - 0) = y_2(t_{m_j}^- + 0) \qquad \text{unless} \quad t_{m_j}^- = 0 \text{ or } T, \tag{4.3.16}$$

and for every $t \in Z_k \cap Z_m$,

$$\text{if } \bar{\rho} \neq 0, \qquad f'(\bar{\rho}) = ((y_1 + \lambda_1)/(y_2 + \lambda_2)) - 1; \tag{4.3.17}$$

if $\bar{\rho} = 0$, *then either*

$$\begin{aligned}&\text{(a)}\quad y_1 + \lambda_1 = y_2 + \lambda_2\,;\ \text{or}\\ &\text{(b)}\quad y_1 + \lambda_1 > y_2 + \lambda_2 \text{ and } \hat{f}(\hat{\rho}) > ((y_1 + \lambda_1)/(y_2 + \lambda_2)) - 1;\ \text{or}\\ &\text{(c)}\quad y_1 + \lambda_1 < y_2 + \lambda_2 \text{ and } -\hat{f}(\hat{\rho}) < ((y_1 + \lambda_1)/(y_2 + \lambda_2)) - 1,\end{aligned} \tag{4.3.18}$$

as well as (4.3.1), (4.3.2), (4.3.5), (4.3.6), (4.3.8), (4.3.9), (4.3.13), (4.3.15), and (4.3.16).

There is one fairly general remark which is in order here. Suppose $\bar{\rho} < 0$. Then from (4.3.10) $\bar{\rho} \leqslant -\hat{\rho}$. Now I claim that $\rho f'(\rho)/f(\rho)$ is decreasing at $-\hat{\rho}$ since using the fact that $\hat{\rho}f'(\hat{\rho})/f(\hat{\rho}) = 1$, we have

$$\frac{d}{d\rho}\left[\frac{\rho f'(\rho)}{f(\rho)}\right]_{\rho=-\hat{\rho}} = \frac{-\hat{\rho}f''(-\hat{\rho})}{f(-\hat{\rho})} < 0.$$

Thus, for $\rho \leqslant -\hat{\rho}$, we have $f'(\rho) \leqslant f(\rho)/\rho$ and, in particular, $f'(\bar{\rho}) \leqslant f(\bar{\rho})/\bar{\rho}$ for $\bar{\rho} < 0$. But it is clear from (4.1.2), (4.3.3), and (4.3.11) that $f'(\bar{\rho}) > -1$. Hence, for $\bar{\rho} < 0$, $f(\bar{\rho})/\bar{\rho} \geqslant -1$ or $f(\bar{\rho}) + \bar{\rho} \leqslant 0$. This proves

LEMMA 4.3.19. *If* $\bar{\rho} < 0$, *then* $\bar{\rho} + f(\bar{\rho}) \leqslant 0$.

The interpretation of this lemma is simply that if one is selling goods for money, one will not sell so rapidly that the transaction costs more than use up the money obtained.

LEMMA 4.3.20. *If* $(\bar{c}, \bar{\rho})$, $(\bar{K}, \bar{m})$ *solves the problem of Section* 2, *then* $\bar{c}(t) > 0$ *for every* $t \in I$.

Proof. From the maximal principle, we have that for every $t \in I$, $\bar{c}(t)$ maximizes

$$\begin{aligned}H(c, \bar{\rho}(t), \bar{K}(t), \bar{m}(t), y_1(t), y_2(t), \lambda_1(t), \lambda_2(t)) &= u(c)\\ &+ (y_1(t) + \lambda_1(t))(\bar{\rho}(t) - c - \delta\bar{K}(t))\\ &+ (y_2(t) + \lambda_2(t))(R - \bar{\rho}(t) - f(\bar{\rho}(t)) - \pi\bar{m}(t))\end{aligned}$$

where λ_1 and/or λ_2 may be zero. Therefore, if for some $t \in I$, $c(t) = 0$, we must have $\partial H/\partial c\,|_{c=0} \leqslant 0$. But

$$\partial H/\partial c\,|_{c=0} = u'(0) - (y_1(t) + \lambda_1(t)) > 0$$

since we assume $u'(0) = +\infty$. Q.E.D.

4.4. *Boundary Conditions*

Since $K(T)$ and $m(T)$ are free, the transversality condition is

$$y_1(T) = y_2(T) = 0.$$

Since $y_1(t) > 0$ for every t such that $\bar{K}(t) > 0$, we must have $\bar{K}(T) = 0$. This is expected since any goods left at $t = T - 0$ could be consumed without incurring transaction costs, and no utility is derived from consumption occurring after $t = T$.

It is also true that $\bar{m}(T) = 0$; however, a more subtle argument is required. Suppose that $\bar{m}(T) > 0$ and $\tilde{\rho}(T) \neq 0$. Then

$$y_2(T) = \lambda_1(T)/(1 + f'(\tilde{\rho}(T)) \qquad \text{since } y_1(T) = \bar{K}(T) = 0.$$

Thus $\lambda_1(T) = 0$ and by (4.3.2), $u'(\tilde{c}(T)) = 0$, contrary to the assumption that $u' > 0$ for every $c \geqslant 0$. This implies that if $\bar{m}(T) > 0$, $\tilde{\rho}(T) = 0$. But if $\bar{m}(T) > 0$, since $\lambda_1(T) > 0$ and $y_2(T) = 0$, it is clear from (4.3.4b) that $\tilde{\rho}(T) \neq 0$. This proves

LEMMA 4.4.1. $\bar{K}(T) = \bar{m}(T) = 0$.

4.5 *General Results*

LEMMA 4.5.1. $Z_m{}^0 \subset Z_k$, *where* $Z_m{}^0 = Z_m - \{0\}$, *i.e.*, $\bar{m}(t) = 0$ *implies* $\bar{K}(t) = 0$ *for every* $t > 0$.

Proof. Suppose $Z_m{}^0 - Z_k \neq \Phi$. Then from Eqs. (4.3.13)–(4.3.15) for every $t \in \bar{Z}_m$, $\dot{y}_1 = \delta y_1$, $\dot{y}_2 = \pi(y_2 + y_2)$, and for every $t \in \operatorname{Int} \bar{Z}_m$, $\tilde{\rho} = \rho_R$ where ρ_R is such that $\rho_R + f(\rho\tau) = R$ (note that $\rho_R > 0$ by Lemma 4.3.19). Now for every $t \in \bar{Z}_m$ from (4.3.11), $(y_2 + \lambda_2)(1 + f'(\tilde{\rho})) = y_1$. Claim that $y_2 + \lambda_2 > 0$ since we know $y_2 + \lambda_2 \geqslant 0$, and if $y_2 + \lambda_2 = 0$, then $y_1 = 0$. But for $t \in \bar{Z}_m$, $u'(c) = y_1 \neq 0$. Therefore, for every $t \in \bar{Z}_m$, $f'(\rho) = y_1/(y_2 + \lambda_2) - 1$. Now suppose that $\operatorname{Int} \bar{Z}_m \neq \Phi$. Then for every $t \in \operatorname{Int} \bar{Z}_m$, $\tilde{\rho}$ is constant so that $y_1/(y_2 + \lambda_2) - 1$ is constant, or

$$\dot{y}_1/y_1 = (\dot{y}_2/(y_2 + \lambda_2)) + (\dot{\lambda}_2/(y_2 + \lambda_2)),$$

or

$$\delta = \pi + (\dot{\lambda}_2/(y_2 + \lambda_2)).$$

But $\dot{\lambda}_2 \leqslant 0$ and $y_2 + \lambda_2 > 0$ (see above), so we have $\delta \leqslant \pi$, which is a contradiction.

Finally if $\operatorname{Int} \bar{Z}_m = \Phi$, and $Z_m{}^0 - Z_k \neq \Phi$, then

$$Z_m{}^0 - Z_k = \{t_1, t_2, \ldots\},$$

and at $t_i \in Z_m{}^0 - Z_k$, we have $\bar{\rho}(t_i - 0) \geqslant \rho_R$ and $\bar{\rho}(t_i + 0) \leqslant \rho_R$. If equality holds in both cases, we can apply the above argument, so assume that $\bar{\rho}$ is discontinuous at t_i. But for $t \neq t_i$ in a neighborhood of t_i, $f'(\bar{\rho}) = y_1/y_2 - 1$, while at $t_i, f'(\bar{\rho}) = y_1/(y_2 + \lambda_2) - 1$. Now y_2 is continuous at t_i [see (4.3.16)] and so $\bar{\rho}$ cannot be piecewise continuous in a neighborhood of t_i. This contradiction implies that $Z_m{}^0 - Z_k = \Phi$, i.e., $Z_m{}^0 \subset Z_k$. Q.E.D.

THEOREM 4.5.2. *$\bar{K}$ is nonincreasing on I.*

Proof. Suppose there exists $t' \in I$ such that $\bar{K}(t') > 0$. Then there is a neighborhood of t', $N_{t'}$, such that $t \in N_{t'}$ and $t > t'$ implies that $\bar{K}(t) > 0$. Let $N_{t'}^c = I - N_{t'}$ and suppose

$$A = Z_k \cap N_{t'}^c \cap (t', T) \neq \Phi.$$

First suppose that Z_k is closed. Then since t' is in $N_{t'}$, A is closed. Therefore, let $t'' = \min A$. Then $t'' > t', \bar{K}(t'') = 0$, and $\bar{K} > 0$ on (t', t''). From the last fact and Lemma 4.5.1, we have $\bar{m} > 0$ on (t', t''). But since

$$\dot{\bar{K}}(t') > 0,$$

$$\bar{\rho}(t') > \bar{c}(t') + \delta \bar{K}(t') > 0.$$

Hence (4.1.5) and (4.1.6) hold on (t', t'') and so with $\pi < \delta$ and $\delta > 0$, we have that

$$\dot{\bar{c}} < 0 < \dot{\bar{\rho}}$$

on (t', t''). Therefore $\bar{\rho} > \bar{c}$ on (t', t''). But we can solve (2.6) to get

$$\bar{K}(t'') = \bar{K}(t')\, e^{-\delta(t''-t')} + e^{-\delta t''} \int_{t'}^{t''} [\bar{\rho}(s) - \bar{c}(s)]\, e^{\delta s}\, ds.$$

So we see that with $\bar{\rho} > \bar{c}$ on $(t', t''), \bar{K}(t'') > 0$. This contradicts the definition of t''. We conclude that $A = \Phi$. Now if Z_k is not closed, t'' may not be a member of A. In this case, however, t'' is a cluster point of A, so we may choose t''' as close as we like to t'' and such that t''' is in A. The proof that A is empty then goes through with t''' replacing t'' since the interval (t'', t''') can be sufficiently small that $\bar{K}(t''') > 0$ even though $\bar{\rho}$ may be less than $\bar{c}$ on (t'', t'''). But we cannot have $A = \Phi$, since from Lemma 4.4.1, T is in A. Therefore, t' cannot exist, i.e., $\dot{\bar{K}} \leqslant 0$ on I. Q.E.D.

In the following theorem, I show that if the initial stock of the good is positive, a positive inventory will be maintained until the end of the period.

THEOREM 4.5.3. *If $K_0 > 0$, then $\bar{K} > 0$ on $[0, T)$, i.e., $Z_k = \{T\}$.*

Proof. Suppose $t_{k_1}^- < T$. Then $t_{k_1}^+ = T$, and so, y_1 is continuous on I since we know that it is continuous on $[0, t_{k_1}^-)$, on $(t_{k_1}^-, t_{k_1}^+]$, and at $t_{k_1}^-$. But $y_1(t_{k_1}^-) > 0$ and $y_1(T) = 0$, so $\dot{y}_1$ must be negative over some subinterval of $[t_{k_1}^-, T]$. Now on $[t_{k_1}^-, T]$, $\dot{y}_1 = \delta(y_1 + \lambda_1)$ and $y_1 + \lambda_1 = u'(\bar{c}) > 0$. Thus, $\dot{y}_1 > 0$ on $[t_{k_1}^-, T]$. The contradiction implies $t_{k_1}^- = T$. Q.E.D.

COROLLARY 4.5.4. *If $K_0 > 0$, then $\bar{m} > 0$ on $(0, T)$, i.e., $Z_m \subset \{0, T\}$.*

Proof. Follows immediately from Lemma 4.5.1 and Theorem 4.5.3.

THEOREM 4.5.5. *Define $\rho^+ = \{t \in I: \bar{\rho}(t) > 0\}$. If $K_0 > 0$ and $\rho^+ \neq \Phi$ and $\hat{t} = \inf \rho^+$, then $\bar{\rho}(t) > 0$ for every $t \in (\hat{t}, T)$.*

Proof. Since $K_0 > 0$, $\bar{m} > 0$, and $\bar{K} > 0$ on $(0, T)$. Then for every $t \in (\hat{t}, T)$, Eq. (4.1.6) holds, so $\dot{\bar{\rho}} > 0$ on $(\hat{t}, T)$. Since $\rho^+ \neq \Phi$, by the definition of $\hat{t}$, $\bar{\rho}(t) > 0$ for every $t > \hat{t}$ in some neighborhood of $\hat{t}$. Therefore, since $\dot{\bar{\rho}} \geqslant 0$ on $(\hat{t}, T)$, $\bar{\rho} > 0$ on $(\hat{t}, T)$. Q.E.D.

LEMMA 4.5.6. *If $K_0 = 0$, then $\dot{\bar{\rho}} < 0$ whenever $\bar{m} > 0$, i.e., on $Z_k = I - Z_m$.*

Proof. Since $\bar{\rho} \equiv \bar{c}$ on Z_k, $\dot{\bar{\rho}} \equiv \dot{\bar{c}}$. From Eq. (4.3.2),

$$u''(\bar{c})\dot{\bar{c}} = \delta u'(\bar{c}) + \lambda_1,$$

and from (4.3.3),

$$f''(\bar{\rho})\dot{\bar{\rho}} = ((1 + f'(\bar{\rho}))/u'(\bar{c}))[u''(\bar{c})\dot{\bar{c}} - \pi u'(\bar{c})].$$

Combining these two equations and using $\bar{\rho} \equiv \bar{c}$ and $\dot{\bar{\rho}} \equiv \dot{\bar{c}}$ gives

$$\dot{\bar{\rho}} = -\pi(1 + f'(\bar{\rho}))/[f''(\bar{\rho}) - (u''(\bar{\rho})(1 + f'(\bar{\rho}))/u'(\bar{\rho}))]$$

Since $f' > 0$, $f'' > 0$, $u' > 0$, and $u'' < 0$, we see that $\dot{\bar{\rho}} < 0$. Q.E.D.

THEOREM 4.5.7. *If $K_0 = 0$, $\dot{\bar{m}} \leqslant 0$ on I.*

Proof. The proof is essentially identical to that of Theorem 4.5.2. Suppose $t' \in I$ is such that $\dot{\bar{m}}(t') > 0$. Then there is a neighborhood, $N_{t'}$, such that $t \in N_{t'} \cap (t', T)$ implies that $\bar{m}(t) > 0$. As before, if $A = Z_m \cap N_{t'}^c \cap (t', T)$ and Z_m is closed, then A is closed. Suppose $A \neq \Phi$ and let $t'' = \min A$. Then $t'' > t'$, $\bar{m}(t'') = 0$ and $\bar{m} > 0$ on (t', t''). Now since $\dot{\bar{m}}(t') > 0$, we have

$$R > \bar{\rho}(t) + f(\bar{\rho}(t)),$$

all $t \in (t', t'')$. But solving Eq. (2.6) gives

$$\bar{m}(t'') = \bar{m}(t')\, e^{-\pi(t''-t')} + e^{-\pi t''} \int_{t'}^{t''} [R - \tilde{\rho}(s) - f(\tilde{\rho}(s))]\, e^{\pi s}\, ds,$$

so that $\bar{m}(t'') > 0$. This contradicts the definition of t''; therefore we conclude that $A = \Phi$. As before, if Z_m is not closed and t'' is not in A, then t''' is a cluster point of A and t''' can be chosen in a manner analogous to the proof of Theorem 4.5.2. But from lemma 4.4.1, we have $T \in A$. Thus t' cannot exist. Q.E.D.

The following corollary is immediate:

COROLLARY 4.5.8. *If $K_0 = m_0 = 0$, then $\bar{K} \equiv \bar{m} \equiv 0$, and $\tilde{\rho} \equiv \tilde{c} \equiv \rho_R$ on I.*

THEOREM 4.5.9. *If $K_0 = 0$, $m_0 > 0$, then $\bar{m} > 0$ on $[0, T)$, i.e., $Z_m = \{T\}$.*

Proof. The proof is identical to that of Theorem 4.5.3.

5. CONCLUSIONS

Perhaps the most striking result of the household model is contained in Theorem 4.5.5, namely, that once purchasing of the good has begun, it continues until the end of the period. This is in sharp contrast to the results of previous models in which one is alternately "in and out of the market." Related to this difference is the "nonsawtooth" nature of the paths of the stock of real balances and of the good. In the present model, these paths are smooth and will generally be unimodal and decreasing, respectively. This result is, I believe, due to the assumption that consumption and transactions can occur simultaneously in the flow model. It is my conjecture that if we introduce, in addition to the consumption of goods, another utility-generating activity, the consumption of leisure, and assume that the cost of transacting is the loss of consumption of leisure, then the rate of purchase will exhibit this "on-again–off-again" behavior.

It may also be argued that transaction costs have both stock and flow aspects. That is, some transactions occur more or less instantaneously, and the levels of the stocks involved change instantaneously. Other transactions can best be characterized as occurring over time so that the levels of the stocks involved are changed continuously over time. While previous inventory optimization models have analyzed only the

stock aspects, the present model analyzes only the flow aspects. It may be of interest, therefore, to embed this model in one that comprehends both the stock and flow natures of transaction costs. Here, one would envision the household as paying, in addition to the transaction costs of the present model, a fixed cost each time the rate of purchase went from zero to some nonzero value. As I have shown, for the model analyzed here, if the initial stock of the good is positive, then this cost would be paid at most once at the beginning of the planning period. Thus, if it is paid at all, i.e., if purchases are not identically zero, it would have no real effect on the analysis. If, as has been conjectured above, in a model with an explicit leisure–transaction tradeoff, the rate of purchase is alternately zero and nonzero, then an analysis of this model could be applied to the periods when the rate of purchase is nonzero, and a stock transaction cost analysis could be used to determine the length and spacing of the periods when the rate of purchase is zero. A dynamic analysis would be necessary to determine the optimal values of the stocks at the end of each period of nonzero purchasing.

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A Model of Optimal Depletion of Renewable Resources

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1. INTRODUCTION

The basic issue in resource economics is the question of how far renewable and nondepletable resources can be substituted for depletable, nonrenewable resources. Projections like those of Meadows [6] have been criticized by Nordhaus [7] and Beckerman [1] for ignoring the effects of price changes on depletion rates. Daly [3] suggests that recycling of depletable resources, and increasing use of nondepletable energy sources, may permit higher levels of welfare than are enjoyed at present. In the model presented below, the state of technique in obtaining energy from resources, and the stock of energy yielding resources, are controlled over time by a level of energy consumption and a measure of search for higher thermal efficiency. The criterion is a function of consumption over a planning horizon, plus the present value of steady state consumption possibilities at a terminal time. A complete solution of the model proved impossible, but we could generate a realistic example of the model, for which a complete solution was possible. The model suggests that along an optimal path, the level of search should fall over time, and that the optimal rate of energy consumption should rise and then fall, if the initial state of technique in converting resources into energy is primitive in relation to some optimal steady state level of technique. This optimistic model is akin to earlier golden age models of economic growth.

2. THE MODEL

A realistic model of optimal resource depletion and inventive activity would involve physical capital, pollution, resources, knowledge and population as state variables. The study of optimal trajectories with n state variables, by the methods of modern control theory, involves $2n$ differential equations, with mixed initial and terminal boundary conditions.

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The main analytical device is the phase diagram, which is only applicable to simplified models (see [5] for examples of this problem). Even in the simplified model presented below, a complete solution is possible only for the special case, discussed in Section 4.

We define the following variables, at time t:

$Q(t)$, rate of homogeneous output;
$E(t)$, energy consumption;
$X(t)$, level of search activity;
$R(t)$, stock of natural resources;
$M(t)$, state of technique;
$Y(t)$, public and private consumption of output;

and the following functions:

production function, $Q = f(E)$;
thermal efficiency function, $g(M)$;
cost of search function, $C(X)$.

$R(t)$ is a stock of energy yielding resources, assumed homogeneous and measured in "barrels." In that improvements in technique alter the energy available from different resources by differing ratios, this involves an index number problem, which we ignore. Search for higher thermal efficiency $X(t)$ is measured in man-hours per year, and comprises exploration for new energy sources and investment (physical and research) in new techniques which use less energy per unit of output, as well as the search for higher thermal efficiency. A reduction in the energy content of output is assumed to be equivalent to using fewer resources per unit of energy, at a constant energy content per unit of output.

The state of technique $M(t)$, measured in man-hours, is how much search has occurred in the past. In that new workers must be trained in best practice techniques, and energy conversion devices maintained, we assume that a state of technique M requires search at a level δM to maintain that state. Hence,

$$\dot{M} = X - \delta M \tag{1}$$

The state of technique M determines, via the thermal efficiency function $g(M)$, how many units of energy, measured in therms, can be obtained per barrel. We assume that

$$g(M) > 0 \text{ for } M > m^0,\ g'(M) > 0,\ g''(M) < 0,\ g'(m^0) = M^1,\ g'(\infty) = 0 \tag{2}$$

where m^0 is "very small" and M^1 is "very large." If E therms per year are generated, then $E/g(M)$ units of resource are consumed.

Energy consumption $E(t)$ is the only argument in the production function $Q = f(E)$. A constant, fully employed labor force is implied, technical progress is captured by $g(M)$, and the absence of capital stock in $E(t)$ implies a one-commodity economy, with capital consumption permitted, and with a continuously optimal capital stock determined by the myopic rule. We assume that

$$f(E) > 0 \text{ for } E > e^0, f'(E) > 0, f''(E) < 0, f'(e^0) = E^0, f'(\infty) = 0 \qquad (3)$$

where again e^0 is "very small" and E^0 is "very large."

Resource consumption $E/g(M)$, or more precisely the services of a stock of resources, differs from resource depletion, in that some resources may be renewable and/or nondepletable. Fossil fuels are depletable, and on any subgeological time-scale, nonrenewable. Depletable sources such as wood or sewage are renewable, and given some high price for oil, gas and coal, trees would be planted for burning. Tides, winds, hydroelectric, geothermal, thermonuclear, and solar sources of power are nondepletable in that an installation can be operated indefinitely without extraction of any substance from the earth's crust. Depletion rates of various types of resources would be

$\dot{R} = -E/g(M)$	for a depletable, non-renewable resource;
$\dot{R} = 0$	for a nondepletable resource;
$\dot{R} = h(n) - E/g(m)$	for a depletable, renewable resource, where n is some measure of renewal activity.

The search for higher thermal efficiency includes attempts to develop nondepletable and renewable sources of energy, which would allow indefinite consumption of energy and output. To capture features common to all three types of resource discussed above, we adopt as our resource depletion function a simple amalgam of the three processes

$$\dot{R} = \beta - E/g(m). \qquad (4)$$

Hence, β is a measure of the (natural or man-made) renewal powers of the stock of energy-yielding resources. If search X takes, in part, the form of increasing β, we assume that its effect is via $g(M)$. The model presented below relies crucially on the assumption that $\beta > 0$.

The level of search $X(t)$ has a cost, in terms of immediate consumption foregone, of $C(X)$, and consumption of net output Q is therefore $f(E) - C(X)$. We assume that

$$C(0) = 0, C(X) > 0 \text{ for } X > 0, C'(X) > 0, C''(X) > 0. \qquad (5)$$

An increasing marginal cost of search, in terms of consumption forgone, implies an interior optimum with search and consumption positive almost everywhere along an optimal path. A constant marginal cost of search would imply a bang-bang solution for search and consumption.

3. Optimal Paths

Given a state of technique M, the function

$$\phi(M) = f\{\beta g(M)\} \tag{6}$$

is, in view of (4), that level of output sustainable indefinitely, without further resource depletion. Properties (2) and (3) imply

$$\phi(M) > 0 \text{ for } M > m',\ \phi'(M) > 0,\ \phi''(M) < 0,\ \phi'(m') = \phi^0,\ \phi'(\infty) = 0 \tag{7}$$

where m' is "very small" and ϕ^0 is "very large." Since a state of technique M can only be maintained by a level of search of δM, at a cost in consumption of $C(\delta M)$, we define

$$S(M) = \phi(M) - C(\delta M) \tag{8}$$

as the steady-state consumption possibility associated with M. Properties (5) and (7) imply that $S(M)$ reaches a maximum at a unique steady-state $\hat{M}$, given by $\phi'(\hat{M}) = \delta C'(\delta \hat{M})$. Such a state is analogous to the golden age (see [8], where the marginal product of capital equals the sum of the rates of depreciation and population growth. (This anology is not accidental. I am indebted to a referee for pointing out that the model can be interpreted in the standard production function sense, with M a stock of capital, X investment, $C(X)$ the cost of converting output to capital, and $Q = f\{g(m)(\beta - \dot{R})\}$ the production function.)

Steady-state consumption $S(M)$ does not depend on the stock of resources R, and for a while consumption might exceed the golden age level. Optimal resource depletion in our model implies exhausting the stock of depletable nonrenewable resources to finance consumption and research into better methods of using renewable and nondepletable sources of energy.

The objective is to select time-paths for consumption $E(t)$ and search $X(t)$, so as to maximize

$$J = \int_0^T e^{-rt}\{f(E) - C(X)\}\,dt + (e^{-rT}/r)\{\phi(M) - C(\delta M)\}_{t=T} \tag{9}$$

where the rates of change of the state variables $M(t)$ and $R(t)$ are given by (1) and (4), and subject to nonnegativity requirements and initial conditions

$$X \geqslant 0,\ E \geqslant 0,\ R \geqslant 0,\ M \geqslant 0,\ R(0) = R^0,\ M(0) = M^0 \qquad (10)$$

(We omit the time-index where to do so would not lead to ambiguity.) The criterion J is the present value of consumption over the interval $(0, T)$, with T unspecifield, plus the present value of the steady-state consumption possibilities associated with the terminal state of technique $M(T)$, where r is the social rate of time preference. In what follows, we can ignore the nonnegativity constraints on X, E, and M. Let μ and $e^{-rt}\lambda(t)$ be Lagrange multipliers associated with the constraints $R(T) \geqslant 0$, $R(t) \geqslant 0$, and let $e^{-rt}w(t)$ and $e^{-rt}y(t)$ be auxiliary variables (shadow prices) associated with $M(t)$ and $R(t)$.

Form the augmented Hamiltonian (see [4]

$$H = e^{-rt}\{f(E) - C(X) + w(X - \delta M) + y(\beta - (E/g(M))) + \lambda R\}. \qquad (11)$$

The auxiliary variables must obey the following differential equations:

$$\dot{w} = (r + \delta)\, w - (\, yEg'(M)/g(M)^2), \qquad (12)$$

$$\dot{y} = ry - \lambda, \qquad (13)$$

and the boundary conditions

$$w(T) = (1/r)\{\phi'(M) - \delta C'(\delta M)\}_{t=T}, \qquad (14)$$

$$y(T) = e^{rT}\mu. \qquad (15)$$

Maximizing H by choice of controls X and E implies that

$$f'(E^*) = (\, y/g(M)), \qquad (16)$$

$$C'(X^*) = w \qquad (17)$$

where E^* and X^* denote values along optimal time-paths.

The nonnegativity constraints on $R(t)$ imply that

$$R(t)\,\lambda(t) = 0,\ R(T)\,\mu = 0. \qquad (18)$$

The terminal date T is not specified, and a value T that maximizes J is given by the following transversality condition (see [22]):

$$f(E^*) - C(X^*) + w(X^* - \delta M) + y(\beta - (E^*/g(M))) + \lambda R - \phi(M) + C(\delta M) = 0\ _{|t+T}. \qquad (19)$$

A complete solution of the above model, using phase diagrams, eluded the author, since four differential equations are involved. We discuss a partial solution in the rest of this section, and give an example, which can be solved completely, in Section 4. Since $R(T)$ does not appear as an argument in J, an optimal path would imply depletion of resources at time T. ($R(T) > 0$ would imply $\mu = 0$, $R(t) > 0$ for $t < T$ would imply $\lambda(t) = 0$, which from (13) and (15) would imply $y(t) = 0$; hence, from (16) $f'(E^*) = 0$.) Hence

$$R(t) > 0 \text{ for } t < T, R(T) = 0;$$

hence, from (15) and (16)

$$\dot{y} = ry, y(t) = e^{rt}\mu. \tag{20}$$

That the shadow price of a scarce resource should rise exponentially over time is well-known (see [9]).

Since $R(T) = 0$, $E^* \leqslant \beta g(M)$ must hold at time T, and the absence of $R(T)$ from the expression for J implies

$$E^* = \beta g(M)_{t=T}. \tag{21}$$

From (6), (12), (16), and (21), we have

$$\dot{w} = (r + \delta)\, w - \phi'(M)|_{t=T}, \tag{22}$$

and the transversality condition (19) simplifies, in view of (6), (16) and (18) to

$$w(X^* - \delta M) - C(X^*) + C(\delta M) = 0. \tag{23}$$

We can prove that the terminal conditions (14) and (23) have a unique solution at the stationary equilibrium $\dot{w}(T) = \dot{M}(T) = 0$. From (14)

$$dw/dM = (1/r)\{\phi''(M) - \delta^2 C''(\delta M)\}_{t=T} < 0$$

in view of (5) and (7). Substituting $C'(X^*) = w$ in (23) and differentiating with respect to X^*,

$$dM/dX^* = C''(X^*)(X^* - \delta M)/\delta\{C'(X^*) - C'(\delta M)\} > 0$$

in view of (5), and since $dw/dX^* > 0$, $dw/dM > 0$ along the locus of (23). Hence, the loci of (14) and (23) can have only one point of intersection.

If $\dot{M} = 0$ then $X^* = \delta M$ from (1), hence (23) is satisfied. If $\dot{M} = \dot{w} = 0$ then $w = \phi'(M)/(r + \delta) = C'(\delta M)$, and hence (14) is satisfied.

So at time T, the state of technique M attains a modified golden age equilibrium M^*, given by $\phi'(M^*) = (r + \delta)\, C'(\delta M^*)$. If $r > 0$ then

$M^* < \hat{M}$. The approach path to this equilibrium, given by (12), is difficult to analyze, since the $\dot{w}$ locus in $w - M$ space depends on the behaviors of E^* and y. Further analysis would require phase diagrams of more than two dimensions. It is, fortunately, easy to construct a realistic example which avoids this problem, and this is done in the next section.

4. EXAMPLE

We choose the following functional forms for $g(M)$, $f(E)$ and $C(X)$:

$$g(M) = a \log_e M, \tag{24}$$

$$f(E) = b \log_e E, \tag{25}$$

$$C(X) = cX^2. \tag{26}$$

Apart from having marginal products that diminish more rapidly than for most production functions (i.e., $Ef'(E)$ is constant), these forms appear intuitively reasonable, and satisfy conditions (2), (3), and (5). We assume that a and b are small enough to ensure that $M > 1$, $E > 1$ everywhere.

From (16), (24), and (25), we have

$$E^* = ab \log M/y; \tag{27}$$

hence, from (12)

$$\dot{w} = (r + \delta)\, w - (b/M \log M). \tag{28}$$

From (1), (17), and (26) we have

$$\dot{M} = (w/2c) - \delta M. \tag{29}$$

From (4), (20), (24), and (27) we have

$$\dot{R} = \beta - (be^{-rt}/\mu);$$

hence,

$$R(t) = R^0 - (b/r\mu) + \beta t + (be^{-rt}/r\mu), \tag{30}$$

and from (21) we have

$$\mu = e^{-rT} b/\beta. \tag{31}$$

The peculiar forms adopted for $g(M)$ and $f(E)$ allow us to solve two separate sets of differential equations independently. Equations (28) and (29) determine w and M, and (30) and (31) allow us to solve for μ and T. We know that $\dot{w}(T) = \dot{M}(T) = 0$; hence, the solution for w and M is given in the phase diagram of Fig. 1. (The monotonic form of the $\dot{w} = 0$ locus depends on the functional forms adopted for $g(M)$ and $f(E)$.

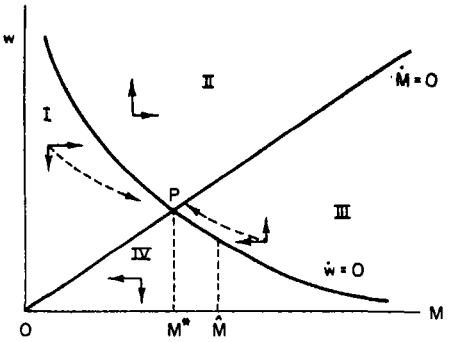


FIGURE 1

The only possible approach paths to the modified golden age equilibrium M^* (point P) are the dotted arrowed curves in sectors I and III. We shall only discuss the case where $M(0) = M^0 < M^*$. Here $w(t)$ falls over time; hence, $X^*(t)$ falls. The final value of M^* is, from (28) and (29), given by

$$M^{*2} \log M^* = b/2c\delta(r + \delta). \tag{32}$$

From (20) and (27) we have

$$dE^*/dt = (ab/y)\{X^*/M) - \delta - r \log M\}, d_2E^*/dt^2 < 0.$$

Since $\dot{M} > 0$, there exists a critical value of $M(0)$ at which $dE^*/dt > 0$. Hence if the initial state of technique is very low, an optimal path for energy consumption will rise, and then fall (since $X^*(T) = \delta M(T)$) as T is approached. If E^* does rise at first, consumption $Y = f(E) - C(X)$ will also rise, since $dX^*/dt < 0$. Consumption will, however, fall as T is approached. To see this

$$\frac{dY}{dt} = \frac{b}{\log M}\left\{\frac{X}{M} - \delta - r \log M\right\} - 2cX\left\{(r + \delta)X - \frac{b}{2cM \log M}\right\},$$

and since $X^*(T) = \delta M^*$, and using (32),

$$dY/dt\,|_{t=T} = -rb < 0.$$

We now determine T and μ. Since $R(T) = 0$, from (30) and (31), the optimal terminal date T is the solution to

$$R^0 = (\beta/r)\{e^{rT} - 1 - rT\} = \beta\{(rT^2/2!) + (r^2T^3/3!) + \cdots\}, \tag{33}$$

from which

$$\partial T/\partial r\,|_{R^0 \text{ const}} < 0, \qquad \partial T/\partial R^0\,|_{r \text{ const}} > 0.$$

Hence, the lower the social rate of time preference, and the larger the initial stock of resources, the longer the resources should last. Since $e^{rT} > (=) 1 + rT$ for $rT > (=) 0$, so long as $R^0 > 0$, for any $T > 0$ (however large), there exists an $r > 0$ such that (33) holds. Also for any $r > 0$ (however small), there exists a $T > 0$ such that (33) holds. Hence, as r gets arbitrarily small, T gets arbitrarily large.

5. Conclusions

The assumption that energy-yielding resources are renewable, or that a large stock of nondepletable resources are available, is crucial to the model. It allows us to resurrect, in a different form, the earlier golden age models. The lower the social rate of time preference, the higher the terminal state of technique, and the longer the period over which resources are depleted. The counter-intuitive statement that energy use and consumption may rise at first applies only once the economy has moved onto the optimal path. These levels of consumption may be lower than the levels prevailing before the economy moved onto its optimal trajectory. The empirical question is whether renewable resources (backstop technology) can ever be as cheap (in terms of human labor per therm) to use as the depletable resources relied on today. Nothing has been said in the paper about population, although the assumption that there is a maximum sustainable level of total output implies that in the long run consumption per head must fall unless a stationary level of population is reached.

Acknowledgment

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A Complete Characterization of Efficiency for a General Capital Accumulation Model

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1. INTRODUCTION

The problem of characterizing efficient programs was first introduced in the now famous paper of Malinvaud [5]. As a partial characterization it was shown that, for a competitive program, if the limiting *value of capital* $p_t x_t$ vanishes,

$$\lim_{t\to\infty} p_t x_t = 0,$$

the program is efficient.

We will say that a sequence of real numbers (α_t) *grows too fast* if

$$\sum_{t=0}^{\infty} 1/\alpha_t < \infty.$$

The first complete characterization for a class of programs was achieved by Cass [3]; for the neoclassical one-good models in which the production functions satisfy certain restrictions on the first and second derivatives, a program is inefficient if and only if the prices p_t grow too fast. Cass [4] has extended the result to certain production models with several capital goods and a single consumption good. Mitra [6] further extended the result to a set of production models with several consumption goods. In the models used by Mitra, production is again described by a production function and the results depend crucially upon first- and second-order conditions on the function.

All of the aforementioned characterizations restrict attention to programs in which the inputs are bounded above and below away from zero. This assumption is very restrictive in practice, for, as we may observe in our own economy, capital stocks can grow very large, and it is exactly this state of capital overaccumulation which we are trying to characterize. It might be thought that these assumptions can be circumvented by simply

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"normalizing" the input size. But, as was pointed out, the results depend crucially on first- and second-order conditions on the production functions and such normalizations would, in most cases, cause these conditions to be violated. In the recent paper by Benveniste and Gale [1], it was observed that for the one-sector model, if restrictions are imposed on the *first and second elasticities*, such normalizations will not affect these conditions. Under these circumstances the gauge which measures the efficiency of a program is the rate of growth of the value of capital; a program is inefficient if and only if the value of capital $p_t x_t$ grows too fast.

One might observe that when the size of the capital is bounded, the above condition is equivalent to the condition of Cass. However, when the size of capital is unrestricted, the x_t in the denominator is a contributing factor.

The present paper presents a complete characterization of inefficient programs for a very general capital accumulation model which subsumes those of [1, 3, 5, 6]. It can be viewed as a generalization of the above result to several capital goods although the model is broader in the following sense: It may be interpreted to include consumers as well as producers and distribution as well as production. For such models an efficient program is usually called *Pareto-optimal*, meaning that no other program, beginning with the same initial stocks, provides more satisfaction to every agent in every period.

Lest I be accused of claiming too much, it should be pointed out that the present exposition does not handle the question of characterizing programs which maximize consumers' lifetime satisfaction. Some recent works of Bose [2] suggests that theorems of the Cass' type may also extend to this situation.

2. The Model

Definition 1. An *action* σ consists of a pair of *inputs* $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^{2n}_+$, together with a set $P(\bar{x}_1, \bar{x}_2)$ in $\mathbb{R}^{2n}_+$ which consists of pairs (x_1, x_2) called *inputs preferred to* $(\bar{x}_1, \bar{x}_2)$.

Assumptions.

P_1. $P(\bar{x}_1, \bar{x}_2)$ is convex.

P_2. Monotonicity: If either $x_1 \geqslant \bar{x}_1$[1] and $x_2 \leqq \bar{x}_2$, or $x_1 \geqq \bar{x}_1$ and $x_2 \leqslant \bar{x}_2$, then $(x_1, x_2) \in P(\bar{x}_1, \bar{x}_2)$.

P_3. $(\bar{x}_1, \bar{x}_2) \notin P(\bar{x}_1, \bar{x}_2)$.[2]

[1] The vector notation will be $x \geqslant y$ if $x_i \geqslant y_i$ for all i and $x \neq y$, $x \geqq y$ if $x_i \geqslant y_i$ for all i and $x \gg y$ if $x_i > y_i$ for all i.

[2] There is no assumption made about a transitive preference relation.

LEMMA 1. $(\bar{x}_1, \bar{x}_2) \in \overline{P(\bar{x}_1, \bar{x}_2)}$ [3] $- P(\bar{x}_1, \bar{x}_2)$.

Proof. Consider any vector $\epsilon \geqslant 0$. It follows from P_2 that for any $\delta > 0$, $(\bar{x}_1 + \delta\epsilon, \bar{x}_2) \in P(\bar{x}_1, \bar{x}_2)$. Therefore $(\bar{x}_1, \bar{x}_2) \in \overline{P(\bar{x}_1, \bar{x}_2)}$. Because of P_3, $(\bar{x}_1, \bar{x}_2) \notin P(\bar{x}_1, \bar{x}_2)$. ∎

Remark. The above lemma shows that $(\bar{x}_1, \bar{x}_2)$ lies on the boundary of its preferred set. For economic interpretation, one can think of the closure of $P(\bar{x}_1, \bar{x}_2)$ as those pairs of inputs (x_1, x_2) *preferred or indifferent* to the given action.

3. EXAMPLES

The following examples will illustrate the full generality of this model.

(1) *The Malinvaud Model*

Let $(\bar{x}_1, \bar{y}_2)$ be the input output pair in period 1. We assume that this pair is *efficient* meaning that there is no pair $(\bar{x}_1, y_2)$ in the technology τ with $y_2 \geqslant \bar{y}_2$. Let $\bar{x}_2$ be the inputs of period 2, then the activity described above is given by the pair $(\bar{x}_1, \bar{x}_2)$ and

$$P(\bar{x}_1, \bar{x}_2) = \{(x_1, x_2) | \exists (x_1, y_2) \in \tau \text{ with } y_2 - x_2 \geqslant \bar{y}_2 - \bar{x}_2\}.$$

Condition P_1 is implied by the convexity of τ, P_2 by free disposal, and P_3 by the assumption that $(\bar{x}_1, \bar{y}_2)$ is efficient.

(2) *A Model with Consumers*

The model consists of m consumers, each endowed with an *opportunity set* $C_i \subseteq R^{n_i}$ which contains descriptions of all possible ways the ith consumer can spend his time. For example, a typical element c_i of C_i might consist of 6 hours working, 8 hours sleeping, 2 hours watching television, etc. Associated to each $c_i \in C_i$ there is a set $d(c_i) \subseteq C_i$, of *more desirable opportunities* in the opportunity set. The set $\overline{d(C_i)}$ consists of the opportunities which are *just as good* as c_i. We assume that $d(c_i)$ is convex for all i.

An *activity of the consumers* in the first period is made of an m vector of opportunities $\xi = (c_1, \ldots, c_m)$ $c_i \in C_i$, which describes the way all consumers spend their time. To each activity ξ there arises a natural set of *better activities.*

$$D(\xi) = \{\xi' = (c_1', \ldots, c_m') : c_i' \in \overline{d(c_i)} \ \forall i \text{ and } c_j' \in d(c_j) \text{ for some } j\}.$$

[3] This is the symbol for the closure of $P(\bar{x}_1, \bar{x}_2)$.

Hence, an activity ξ' in $D(\xi)$ is one which makes all consumers at least as well off and improves the position of at least one.

The technology τ consists of a set of triples (x_1, x_2, ξ) where x_1, x_2 are nonnegative n-vectors of *initial* and *terminal capital stocks* and ξ is an activity of the consumers. Thus, the triples are to the thought of as consisting of those activities which are possible starting from initial stocks x_1 and ending with terminal stocks x_2. As in the Malinvaud model, a triple $(\bar{x}_1, \bar{x}_2, \bar{\xi})$ will be called *efficient* if there is no triple $(\bar{x}_1, \bar{x}_2, \xi)$ with $\xi \in D(\bar{\xi})$.

We are now in a position to describe the set of actions. To each efficient $(\bar{x}_1, \bar{x}_2, \bar{\xi}) \in \tau$, there corresponds the action made up of the pair $(\bar{x}_1, \bar{x}_2)$ and the preferred set

$$P(\bar{x}_1, \bar{x}_2) = \{(x_1, x_2): \text{there is some } \xi \in D(\bar{\xi}) \text{ such that } (x_1, x_2, \xi) \in \tau\}.$$

By assuming the proper convexity and monotonicity assumptions on the sets $D(\xi)$ and τ, conditions P_1 and P_2 are satisfied. P_3 is satisfied because we are only considering efficient activities.

4. Competitivity of an Action

Definition 2. An action σ with inputs $(\bar{x}_1, \bar{x}_2)$ and preferred set $P(\bar{x}_1, \bar{x}_2)$ is *competitive* if there exist *prices* $(\bar{p}_1, \bar{p}_2)$ such that

$$\bar{p}_2\bar{x}_2 - \bar{p}_1\bar{x}_1 > \bar{p}_2x_2 - \bar{p}_1x_1$$

for all (x_1, x_2) in $P(\bar{x}_1, \bar{x}_2)$.

The requirement of strict inequality might appear strange when compared with the usual condition of profit maximization. The condition will become more natural when competitivity is illustrated for the examples of the previous section. Competitivity will be shown for the model with consumers with the Malinvaud model being done as a special case.

The Model With Consumers

Recall that the technology in this model consists of triples (x_1, x_2, ξ) with $x_1, x_2 \in \mathbb{R}_+^n$ and $\xi \in \mathbb{R}^{\sum_{i=1}^{m} n_i}$. A vector of prices $(\bar{p}_1, \bar{p}_2, \bar{q})$ is a *competitive equilibrium* for $(\bar{x}_1, \bar{x}_2, \bar{\xi})$ if

(1) $\bar{p}_2\bar{x}_2 - \bar{p}_1\bar{x}_1 + \bar{q}\bar{\xi} \geqslant \bar{p}_2x_2 - \bar{p}_1x_1 + \bar{q}\xi$ for all $(x_1, x_2, \xi) \in \tau$,

and

(2) $\bar{q}\xi > \bar{q}\bar{\xi}$ for all ξ in $D(\bar{\xi})$.

Condition (1) is the usual profit maximization, while condition (2) represents optimization subject to a budget constraint, i.e., any better activity must cost the consumer more.

When we consider this activity in the framework of the general model, the prices $(\bar{p}_1, \bar{p}_2)$ are competitive prices. To show this we must show that they satisfy Definition 2, i.e.,

$$\bar{p}_2\bar{x}_2 - \bar{p}_1\bar{x}_1 > \bar{p}_2x_2 - \bar{p}_1x_1 \tag{4.1}$$

for all $(x_1, x_2, \xi) \in \tau$ with $\xi \in D(\bar{\xi})$. This follows immediately from the above conditions. From condition (2),

$$\bar{q}\xi > \bar{q}\bar{\xi}, \tag{4.2}$$

and by condition (1),

$$\bar{p}_2\bar{x}_2 - \bar{p}_1\bar{x}_1 + \bar{p}\bar{\xi} \geqslant \bar{p}_2x_2 - \bar{p}_1x_1 + \bar{q}\xi. \tag{4.3}$$

Subtracting (4.2) from (4.3) yields (4.1).

For the Malinvaud model, an action consists of some $(\bar{x}_1, \bar{y}_2) \in \tau$ and the inputs in the next period $\bar{x}_2$, from which we determine a consumption vector $\bar{c} = \bar{y}_2 - \bar{x}_2$. The complete description of the two-period technology consists of triples $(\bar{x}_1, \bar{x}_2, \bar{c})$. If we replace $\bar{\xi}$ above by $\bar{c}$ and let $D(\bar{c}) = \{c : c \geqslant \bar{c}\}$, competitive equilibrium is exactly as above and again the prices $(\bar{p}_1, \bar{p}_2)$ are the competive prices for the action.

If (p_1, p_2) are the usual profit maximizing prices for the production activity $(p_2\bar{y}_2 - p_1\bar{x}_1 \geqslant p_2y_2 - p_1x_1\ (x_1, y_2) \in \tau)$ and $p_2 \gg 0$, then (p_1, p_2) are competitive prices for the action given by the triple $(\bar{x}_1, \bar{x}_2, \bar{c})$ with $\bar{c} + \bar{x}_2 = \bar{y}_2$.

To show this, it is sufficient to show that (p_1, p_2, p_2) are competitive equilibrium prices. Hence, we must show that

$$p_2\bar{x}_2 - p_1\bar{x}_1 + p_2\bar{c} \geqslant p_2x_2 - p_1x_1 + p_2c \tag{4.4}$$

for all $(x_1, x_2 + c) \in \tau$. And

$$p_2c > p_2\bar{c} \qquad c \geqslant \bar{c}. \tag{4.5}$$

Using $c = y_2 - x_2$, (4.4) reduces to the usual profit-maximizing condition, and hence, is satisfied. Equations (4.5) are immediate since $p_2 \gg 0$.

5. The Geometric Concepts of Strictness and Smoothness

The essential conditions for the results of this paper involve (as in previous characterizations) second-order or quadratic conditions on the preferred sets. For models where production is described by a function, as

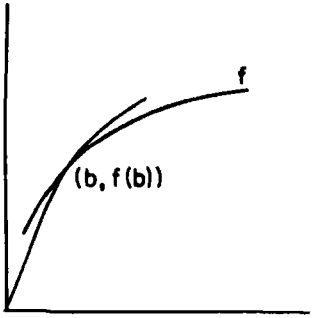
in the neoclassical model, the curvature conditions take the form of negative bounds on the second derivative of the production function f. For the neoclassical model, this implies (as shown by Cass):

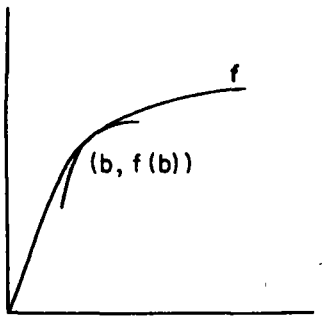
If $f \in c^2$ *on* $\mathbb{R}_+$ *satisfies* $-M \leqslant f''(x) \leqslant m < 0$ *for all* $x \in (a, b) \subseteq (0, \infty)$ *then*

$$(m/2)(b - x)^2 \leqslant f(b) - f(x) - f'(b)(b - x) \leqslant (M/2)(b - x)^2 \quad (5.1)$$

for all $x \in (a, b)$.

The inequalities in (5.1) can be interpreted geometrically. The left side represents a parabola through the point $(b, f(b))$ and lying completely above the graph of f in $\mathbb{R}^2$. The right side is a parabola tangent to the graph of f at $(b, f(b))$ and lying entirely below it.





For higher dimensions it is more convenient to consider convex sets rather than concave functions. A convex set $X \subseteq \mathbb{R}^2$ is *q-strict* at x on the boundary of X if there is a tangent parabola at the point x which contains the set X. X is *q-smooth* at x if there is a tangent parabola at x, contained in X. For example, the set

$$X = \{(x, y): -1 \leqslant x \leqslant 1, y \geqslant x^{4/3}\}$$

is strict but *not* q-strict, and the set

$$X = \{(x, y): -1 \leqslant x \leqslant 1, y \geqslant x^4\}$$

is smooth but *not* q-smooth.

For sets in higher dimension recall that a $2n$-vector P is a support at the point $\bar{y}$ of the set $X \subseteq \mathbb{R}^{2n}$ if $P(\bar{y} - y) \geqslant 0$ for all y in X. The higher dimensional analog to parabolas are "paraboloids." Consider a set $X \subseteq \mathbb{R}^{2n}$ and a point $y = (\bar{x}_1, \bar{x}_2)$ in X with support $P = (p, q) \in \mathbb{R}^{2n}$. A

paraboloid through the point $(\bar{x}_1, \bar{x}_2)$ is given by the locus of points $\{(x_1, x_2)\}$ satisfying

$$p(\bar{x}_1 - x_1) + q(\bar{x}_2 - x_2) = M[p(\bar{x}_1 - x_1)]^2. \tag{5.2}$$

Notice that the tangent plane is the locus of points such that the left side of (5.2) is zero. The second-order term on the right forces the locus of points to move away from this plane in a quadratic manner.

A convex set $X \subseteq \mathbb{R}^{2n}$ is *q-strict* or *q-smooth* at a boundary point if one is able to fit a tangent paraboloid above or inside the set X.

For an action $(\bar{x}_1, \bar{x}_2)$, $P(\bar{x}_1, \bar{x}_2)$ the $2n$ vector of prices $(-\bar{p}_1, \bar{p}_2)$ is a support of $P(\bar{x}_1, \bar{x}_2)$ at $(\bar{x}_1, \bar{x}_2)$ if

$$\bar{p}_2(\bar{x}_2 - x_2) - \bar{p}_1(\bar{x}_1 - x_1) \geqslant 0$$

for all $(x_1, x_2) \in P(\bar{x}_1, \bar{x}_2)$.

DEFINITION 3. The set $P(\bar{x}_1, \bar{x}_2)$ is *q-strict* at $(\bar{x}_1, \bar{x}_2)$ with respect to the support vector $(\bar{p}_1, \bar{p}_2)$ if there is some $m > 0$ such that

$$\bar{p}_2(\bar{x}_2 - x_2) \geqslant \bar{p}_1(\bar{x}_1 - x_1) + m[\bar{p}_1(\bar{x}_1 - x_1)]^2 \tag{5.3}$$

for all (x_1, x_2) in $P(\bar{x}_1, \bar{x}_2)$ with $x_1 \not\geqslant \bar{x}_1$.[4] The set $P(\bar{x}_1, \bar{x}_2)$ is *q-smooth* at $(\bar{x}_1, \bar{x}_2)$ if there is some $M > 0$ such that if the nonnegative $2n$ vector (x_1, x_2) satisfies

$$\bar{p}_2(\bar{x}_2 - x_2) \geqslant \bar{p}_1(\bar{x}_1 - x_2) + M[\bar{p}_1(\bar{x}_1 - x_1)]^2, \tag{5.4}$$

then (x_1, x_2) is in $P(\bar{x}_1, \bar{x}_2)$.

The measures of curvature in the above are the coefficients m and M. Analogous to the observation in the paper by Benveniste and Gale [1] that the measures of curvature should not be "unit sensitive," the coefficients should not be "scale sensitive." Notice that if all prices or all goods vectors are multiplied by some constant it will change the values of m, M, since the last term in (5.3) and (5.4) is quadratic rather than linear in p and x. To get an accurate measure of maximum or minimum curvature we can easily get around this by normalizing. Namely, define $\overline{m}$ as the supremum of all m such that

$$\bar{p}_2(\bar{x}_2 - x_2) \geqslant \bar{p}_1(\bar{x}_1 - x_1) + (m/\bar{p}_1\bar{x}_1)[\bar{p}_1(\bar{x}_1 - x_1)]^2 \tag{5.5}$$

can replace (5.3) in Definition 3. Likewise $\overline{M}$ is the inf of all M such that

$$\bar{p}_2(\bar{x}_2 - x_2) \geqslant \bar{p}_1(\bar{x}_1 - x_1) + (M/\bar{p}_1\bar{x}_1)[\bar{p}_1(\bar{x}_1 - x_1)]^2 \tag{5.6}$$

can replace (5.4) in Definition 3.

[4] This is necessary since without it the monotonicity assumptions would allow for no strictness.

It is now clear that multiplying all prices or quantities by a constant will not change the value of $\bar{m}$ or $\bar{M}$ so that these are appropriate measures of curvature for the n-vector case. We will refer to these quantities $\bar{m}$ and $\bar{M}$ as the *coefficients of strictness and smoothness*, respectively.

6. Programs and the Concept of Efficiency

Thus far we have described the actions during one period. We now extend this.

During any period t there are a set of actions A_t made up of pairs of inputs (x_t, x_{t+1}) with preferred sets $P(x_t, x_{t+1})$.

Definition 4. A program $\bar{\pi}$ is a sequence of inputs $(\bar{x}_t)$ and preferred sets $(P(\bar{x}_t, \bar{x}_{t+1}))$, $t = 0, 1, \ldots$, where in each period t the pair $(\bar{x}_t, \bar{x}_{t+1})$ and set $P(\bar{x}_t, \bar{x}_{t+1})$ is some $\sigma_t \in A_t$.

Programs are made up of a sequence of actions satisfying the essential economic condition that inputs required to sustain the $(t+1)$st action are available at the end of the tth period.

Definition 5. A program $\bar{\pi}$ with actions (σ_t) made up of inputs $(\bar{x}_t)$ and preferred sets $P(\bar{x}_t, \bar{x}_{t+1})$ is *inefficient* if there is a sequence $(x_t) \geqslant 0$ such that $x_0 = \bar{x}_0$ and

(1) $(x_t, x_{t+1}) \in \overline{P(\bar{x}_t, \bar{x}_{t+1})}$ for all t,

(2) $(x_{t_1}, x_{t_1+1}) \in P(\bar{x}_{t_1}, \bar{x}_{t_1+1})$ for at least one t_1.

Otherwise the program is called *efficient*. An inefficient program can be described as a given plan for an economy to which there is an alternative plan available (since initial inputs are identical) which is at least as good in each period and better in at least one period.

Associated to each program $\bar{\pi}$ is a sequence of coefficients of strictness $(\bar{m}_t)$ and coefficients of smoothness $(\bar{M}_t)$.

7. Competitivity

In this section the concept of short-run efficiency is defined and its relation to the existence of the appropriate prices is given.

Definition 6. A program $\bar{\pi}$ with inputs $(\bar{x}_t)$ is *T-period efficient* if there is no set of vectors $(x_t)_{t=0}^{T}$ which satisfy $x_0 = \bar{x}_0$, $x_T = \bar{x}_T$ and

(1) $(x_t, x_{t+1}) \in \overline{P(\bar{x}_t, \bar{x}_{t+1})}$ for $0 \leqslant t < T$,

(2) $(x_{t_1}, x_{t_1+1}) \in P(\bar{x}_{t_1}, \bar{x}_{t_1+1})$ for some $0 \leqslant t_1 < T$.

$\bar{\pi}$ is *short-run efficient* if it is T-period efficient for all T.

DEFINITION 7. A program $\bar{\pi}$ with inputs $(\bar{x}_t)$ is *T-period weakly competitive* if there are n-vectors of prices $(\bar{p}_t)_{t=0}^{T}$, not identically zero, such that for $0 \leqslant t < T$,

$$\bar{p}_{t+1}\bar{x}_{t+1} - \bar{p}_t\bar{x}_t \geqslant \bar{p}_{t+1}x_{t+1} - \bar{p}_t x_t$$

for all $(x_t, x_{t+1}) \in P(\bar{x}_t, \bar{x}_{t+1})$. $\bar{\pi}$ is *T-period competitive* if there are prices $(\bar{p}_t)_{t=0}^{T}$ such that for $0 \leqslant t < T$,

$$\bar{p}_{t+1}\bar{x}_{t+1} - \bar{p}_t\bar{x}_t > \bar{p}_{t+1}x_{t+1} - \bar{p}_t x_t$$

for all $(x_t, x_{t+1}) \in P(\bar{x}_t, \bar{x}_{t+1})$. $\bar{\pi}$ is *weakly competitive* if there is a sequence of prices $(\bar{p}_t)$ $t = 0,1,\ldots$ that are T-period weakly competitive prices for all T, and *competitive* if such a sequence exists that is T-period competitive for all T.

PROPOSITION 1. *Let $\bar{\pi}$ be a program with inputs $(\bar{x}_t)$. If $\bar{\pi}$ is weakly competitive with prices $(\bar{p}_t)$, then $\bar{p}_t \geqslant 0$ for all t. Likewise, if π is competitive, then $\bar{p}_t \gg 0$ for all t.*

Proof. Let $(\bar{p}_t)$ be weakly competitive prices for $\bar{\pi}$. By the monotonicity assumption P_2, if ϵ_i is the ith unit vector then

$$(\bar{x}_t + \epsilon_i, \bar{x}_{t+1}) \in P(\bar{x}_t, \bar{x}_{t+1})$$

for any t and $1 \leqslant i \leqslant n$. Hence,

$$\bar{p}_{t+1}\bar{x}_{t+1} - \bar{p}_t\bar{x}_t \geqslant \bar{p}_{t+1}\bar{x}_{t+1} - \bar{p}_t(\bar{x}_t + \epsilon_i),$$

and $\bar{p}_t\epsilon_i = \bar{p}_{t_i} \geqslant 0$.

The proof of the second statement follows exactly as above with inequality replaced by strict inequality. ∎

The usual relations between competitivity and short run efficiency are now exhibited.

PROPOSITION 2. *Let $\bar{\pi}$ be a program with inputs $(\bar{x}_t)$. If $\bar{\pi}$ is T-period efficient then $\bar{\pi}$ is T-period weakly competitive.*

Proof. Let B_T be the set of all $(T+1)$ n-tuples

$$(-x_0, \tilde{x}_1 - x_1, \ldots, \tilde{x}_T)$$

where $(x_t, \tilde{x}_{t+1}) \in P(\bar{x}_t, \bar{x}_{t+1})$ for $0 \leqslant t < T$. It follows from the convexity of $P(\bar{x}_t, \bar{x}_{t+1})$ that B_T is convex.

The proof will be broken into steps:

(i) The vector $(-\bar{x}_0, 0, \ldots, 0, \bar{x}_T)$ is in the closure of B_T but not in B_T. From Lemma 1 $(\bar{x}_t, \bar{x}_{t+1}) \in \overline{P(\bar{x}_t, \bar{x}_{t+1})}$ for all t. Therefore,

$$(\bar{x}_0, 0, \ldots, 0, \bar{x}_T) = (-\bar{x}_0, \bar{x}_1 - \bar{x}_1, \ldots, \bar{x}_{T-1} - \bar{x}_{T-1}, \bar{x}_T)$$

is in the closure of B_T. Furthermore, because $\bar{\pi}$ is T-Period efficient, this vector cannot be in B_T.

(ii) By the "fundamental theorem on convexity" there exists a $(T+1)n$ vector $(\bar{p}_0, \ldots, \bar{p}_T) \neq 0$ which "separates" $(-\bar{x}_0, 0, \ldots, 0, \bar{x}_T)$ from $\bar{B}_T$. This yields the relation,

$$-\bar{p}_0 x_0 + \sum_{t=1}^{T-1} \bar{p}_t(\tilde{x}_t - x_t) + \bar{p}_T \tilde{x}_T \leqslant -\bar{p}_0 \bar{x}_0 + \bar{p}_T \bar{x}_T \tag{7.1}$$

where $(x_t, \tilde{x}_t) \in \overline{P(x_t, x_{t+1})}$, $0 \leqslant t < T$.

(iii) The prices $(\bar{p}_t)_{t=0}^T$ are T-period weakly competitive prices for $\bar{\pi}$. Consider any $(x_t, x_{t+1}) \in P(\bar{x}_t, \bar{x}_{t+1})$ with $0 \leqslant t < T$. It follows from $(\bar{x}_t, \bar{x}_{t+1}) \in \overline{P(\bar{x}_t, \bar{x}_{t+1})}$ and (7.1) that

$$-\bar{p}_0 \bar{x}_0 + \bar{p}_t(\bar{x}_t - x_t) + \bar{p}_{t+1}(x_{t+1} - \bar{x}_{t+1}) + \bar{p}_T \bar{x}_T \leqslant -\bar{p}_0 \bar{x}_0 + \bar{p}_T \bar{x}_T,$$

from which we get

$$\bar{p}_{t+1} \bar{x}_{t+1} - \bar{p}_t \bar{x}_t \geqslant \bar{p}_{t+1} x_{t+1} - \bar{p}_t x_t. \quad \blacksquare$$

PROPOSITION 3. *A program $\bar{\pi}$ with inputs $(\bar{x}_t)$ which has competitive prices $(\bar{p}_t)$ is short-run efficient.*

Proof. It must be shown that $\bar{\pi}$ is T-period efficient for all T. If we assume that this is not true, then there is some set of vectors $(x_t)_{t=0}^T$ satisfying $x_0 = \bar{x}_0$, $x_T = \bar{x}_T$, and

$$(x_t, x_{t+1}) \in \overline{P(x_t, x_{t+1})} \qquad \text{for } 0 \leqslant t < T,$$

$$(x_{t_1}, x_{t_1+1}) \in P(\bar{x}_{t_1}, \bar{x}_{t_1+1}) \qquad \text{for some } 0 \leqslant t_1 < T.$$

Since the prices $(\bar{p}_t)$ are competitive,

$$\bar{p}_{t+1} \bar{x}_{t+1} - \bar{p}_t \bar{x}_t \geqslant \bar{p}_{t+1} x_{t+1} - \bar{p}_t x_t \qquad \text{for } 0 \leqslant t < T, \tag{7.2}$$

$$\bar{p}_{t_1+1} \bar{x}_{t_1+1} - \bar{p}_{t_1} \bar{x}_{t_1} > \bar{p}_{t_1+1} x_{t_1+1} - \bar{p}_{t_1} x_{t_1}. \tag{7.3}$$

Summing the inequalities in (7.2) and using the strict inequality of (7.3), the telescoping effect yields

$$\bar{p}_T\bar{x}_T - \bar{p}_0\bar{x}_0 > \bar{p}_T x_T - \bar{x}_0 x_0 ,$$

which is a contradiction since $x_0 = \bar{x}_0$ and $x_T = \bar{x}_T$. ∎

8. The Theorems and Proofs

Let $\bar{\pi}$ be a competitive program made up of activities $(\bar{\sigma}_t)$ with inputs $(\bar{x}_t)$ and preferred sets $P(\bar{x}_t , \bar{x}_{t+1})$, and having competitive prices $(\bar{p}_t)$. For the tth activity $\bar{\sigma}_t$, let $\overline{m}_t$ be the coefficient of strictness and $\overline{M}_t$ the coefficient of smoothness.

Necessity Theorem. *If $\bar{\pi}$ is inefficient and there exists some constant $\overline{m} > 0$ such that $\overline{m}_t \geqslant \overline{m}$ for all t, then the capital value grows too fast.*

Sufficiency Theorem. *If there is some $\overline{M} < \infty$ such that $\overline{M}_t \leqslant \overline{M}$ for all t and the capital value grows too fast, then $\bar{\pi}$ is inefficient.*

Proof. Necessity. Since $\bar{\pi}$ is inefficient there exists a sequence of inputs (x_t) with $x_0 = \bar{x}_0$ such that

$$(x_t , x_{t+1}) \in \overline{P(\bar{x}_t , \bar{x}_{t+1})} \qquad \text{for all } t,$$
$$(x_{t_1} , x_{t_1+1}) \in P(\bar{x}_{t_1} , x_{t_1+1}) \qquad \text{for some } t_1 \geqslant 0.$$

Define the sequence of vectors $\delta_t = \bar{x}_t - x_t$. Because $(\bar{p}_t)$ are competitive prices,

$$\bar{p}_{t+1}\bar{x}_{t+1} - \bar{p}_t\bar{x}_t \geqslant \bar{p}_{t+1}x_{t+1} - \bar{p}_t x_t \qquad \text{for all } t. \tag{8.1}$$

$$\bar{p}_{t_1+1}\bar{x}_{t_1+1} - \bar{p}_{t_1}\bar{x}_{t_1} > \bar{p}_{t_1+1}x_{t_1+1} - \bar{p}_{t_1}x_{t_1} \qquad \text{for some } t_1 \geqslant 0. \tag{8.2}$$

Since $x_0 = \bar{x}_0$, $\delta_0 = 0$ and $\bar{p}_0\delta_0 = 0$. By rearranging (8.1) we find that

$$\bar{p}_{t+1}\delta_{t+1} = \bar{p}_{t+1}(\bar{x}_{t+1} - x_{t+1}) \geqslant \bar{p}_t(\bar{x}_t - x_t) = \bar{p}_t\delta_t ,$$

and hence, by induction, $\bar{p}_t\delta_t \geqslant 0$ for all t. Also because of the strict inequality in (7.2), for all $t > t_1$, $\bar{p}_t\delta_t > 0$.

From Proposition 1, the competitive prices $\bar{p}_t \gg 0$, and hence, $\delta_t \not\leqslant 0$ or $x_t \not\geqslant \bar{x}_t$.

By the definition of $\overline{m}_t$, for any $\overline{m} \leqslant \overline{m}_t$, if $(x_t , x_{t+1}) \in \overline{P(\bar{x}_t , \bar{x}_{t+1})}$, $x_t \not\geqslant \bar{x}_t$, then

$$\bar{p}_{t+1}(\bar{x}_{t+1} - x_{t+1}) \geqslant \bar{p}_t(\bar{x}_t - x_t) + (\overline{m}/\bar{p}_t\bar{x}_t)(\bar{p}_t(\bar{x}_t - x_t))^2, \tag{8.3}$$

or

$$\bar{p}_{t+1}\delta_{t+1} \geqslant \bar{p}_t\delta_t + (\bar{m}/\bar{p}_t\bar{x}_t)(\bar{p}_t\delta_t)^2. \tag{8.4}$$

Define a sequence of scalars by $v_t = \bar{p}_t\delta_t/\bar{p}_t\bar{x}_t$. Equation (8.4) can now be written as

$$\bar{p}_{t+1}\bar{x}_{t+1}v_{t+1} \geqslant \bar{p}_t\bar{x}_t v_t + \bar{m}\bar{p}_t\bar{x}_t v_t^2. \tag{8.5}$$

For $t > t_1$, $\bar{p}_t\delta_t > 0$, and hence, $v_t > 0$. Also, for all t, $v_t \leqslant 1$ since $\delta_t \leqslant \bar{x}_t$ and $\bar{p}_t \geqslant 0$. Therefore, when $t \geqslant t_1$, we can invert (8.5) and rewrite it as

$$\frac{1}{\bar{p}_{t+1}\bar{x}_{t+1}v_{t+1}} \leqslant \frac{1}{\bar{p}_t\bar{x}_t v_t} - \frac{\bar{m}}{1+\bar{m}v_t}\cdot\frac{1}{\bar{p}_t\bar{x}_t}. \tag{8.6}$$

Summing (8.6) from $t = t_1 + 1$ to $T - 1$ and cross-cancelling,

$$\frac{1}{\bar{p}_T\bar{x}_T v_T} \leqslant \frac{1}{\bar{p}_{t_1+1}\bar{x}_{t_1+1}v_{t_1+1}} - \sum_{t=t_1+1}^{T-1}\left(\frac{\bar{m}}{1+\bar{m}v_t}\right)\frac{1}{\bar{p}_t\bar{x}_t}. \tag{8.7}$$

Since

$$0 \leqslant v_t \leqslant 1, \qquad \frac{\bar{m}}{1+\bar{m}v_t} \geqslant \frac{\bar{m}}{1+\bar{m}} = \gamma > 0.$$

Hence (8.7) yields

$$\frac{1}{\bar{p}_T\bar{x}_T v_T} \leqslant \frac{1}{p_{t_1+1}x_{t_1+1}v_{t_1+1}} - \gamma\sum_{t=t_1+1}^{T-1}\frac{1}{\bar{p}_t\bar{x}_t},$$

from which it follows that

$$\gamma\sum_{t=t_1+1}^{T-1}\frac{1}{\bar{p}_t\bar{x}_t} \leqslant \frac{1}{\bar{p}_{t_1+1}\bar{x}_{t_1+1}v_{t_1+1}} < \infty,$$

and

$$\sum_{t=0}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t} < \infty. \quad \blacksquare$$

Sufficiency. Since

$$\sum_{t=0}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t} < \infty, \qquad \lim_{t\to\infty}\bar{p}_t\bar{x}_t = \infty.$$

Hence, there is some $\epsilon > 0$ such that for every t there is some input $\bar{x}_{t_t}$ with $\bar{p}_{t_t}\bar{x}_{t_t} \geqslant \epsilon$. This yields a sequence of indices (t_t). Define a sequence of scalars (δ_t) by choosing $\delta_1 > 0$ and

$$\bar{p}_{t+1_t}\delta_{t+1} = \bar{p}_{t_t}\delta_t + (\bar{M}/\bar{p}_t\bar{x}_t)(\bar{p}_{t_t}\delta_t)^2. \tag{8.8}$$

For $t \geqslant 1$ we can invert this and rearrange:

$$\frac{1}{\bar{p}_{t+1_t}\delta_{t+1_t}} = \frac{1}{\bar{p}_{t_t}\delta_t} + \frac{\overline{M}}{\bar{p}_t\bar{x}_t + \overline{M}\bar{p}_{t_t}\delta_t} \geqslant \frac{1}{\bar{p}_{t_t}\delta_t} - \frac{\overline{M}}{\bar{p}_t\bar{x}_t}. \tag{8.9}$$

Summing (8.9) from $t = 1$ to $T - 1$ and cross-cancelling,

$$\frac{1}{p_{T_t}\delta_T} \geqslant \frac{1}{\bar{p}_{1_t}\delta_1} - \overline{M}\sum_{t=1}^{T-1}\frac{1}{\bar{p}_t\bar{x}_t}. \tag{8.10}$$

Because

$$\sum_{t=0}^{\infty}(1/\bar{p}_t\bar{x}_t) = B < \infty,$$

we can choose

$$0 < \delta_1 \leqslant (\epsilon/\bar{p}_{1_t}(1 + \epsilon\overline{M}B))$$

so that from (8.10) we would have that $p_{T_t}\delta_T \leqslant \epsilon$ for all T.

Now consider the sequence of n-vectors given by $x_0 = \bar{x}_0$ and $x_t = \bar{x}_t - \delta_t\epsilon_{t_t}$ where ϵ_{t_t} is t_tth unit vector. Because $\bar{p}_{T_t}\delta_T \leqslant \epsilon \leqslant \bar{p}_{T_t}\bar{x}_{T_t}$, $\delta_T \leqslant \bar{x}_{T_t}$ and $x_t \geqslant 0$ for all t.

From the last part of Section 4, for any $\overline{M} \leqslant \overline{M}_t$ if $(x_t, x_{t+1}) \geqslant 0$ satisfy

$$\bar{p}_{t+1}(\bar{x}_{t+1} - x_{t+1}) \geqslant \bar{p}_t(\bar{x}_t - x_t) + (\overline{M}/\bar{p}_t\bar{x}_t)(\bar{p}_t(\bar{x}_t - x_t))^2, \tag{8.11}$$

then $(x_t, x_{t+1}) \in P(\bar{x}_t, \bar{x}_{t+1})$.

Observe that by (8.8) we chose our δ_t's, and hence, our x_t's so that (8.11) would be satisfied and the program $\bar{\pi}$ is inefficient. ∎

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Notes, Comments, and Letters to the Editor

Two Notes on the Malinvaud Condition for Efficiency of Infinite Horizon Programs

INTRODUCTION

The question of what characterizes a long run efficient program was first studied by Malinvaud [4]. In this paper, Malinvaud observed that a program might be competitive (profit maximizing) in every period, and still be long run inefficient. As a partial characterization, he showed that if, in addition, the competitive prices p_t are positive and the limiting value of the inputs ($\lim_{t\to\infty} p_t x_t$) is zero, then the program is efficient. In proving this, he actually showed that a weaker condition works. Namely, if there is a subsequence $p_{t_i} x_{t_i} \to 0$, then the program is efficient. Interpreted as a necessary condition for inefficient programs, this says that for an inefficient program, $\liminf p_t x_t > 0$.

The first observation is that for one class of production models, the Malinvaud condition completely determines efficient programs. These are the one sector models considered by Gale and Sutherland [3] and others, in which the production function is *strongly productive* ($f'(x) \geqslant a > 1$ for all x). It is shown that the limiting value of inputs always exists and is zero precisely when the program is efficient.

The second result again involves a one sector model in which the production function satisfies a condition of smoothness ($0 \geqslant f''(x) \geqslant -M$). For this model it was shown by Cass [2] that if a program has inputs (x_t) satisfying $0 < k \leqslant x_t \leqslant K$ and prices which grow "too fast," meaning $\sum_{t=0}^{\infty} 1/p_t < \infty$, then the program is inefficient. The upper bound on the inputs was shown to be unnecessary by Benveniste and Gale [1]. It turns out that the lower bound is essential, but, as will be shown, it can be replaced by a variation on the Malinvaud condition. To be more precise, a program whose prices grow too fast and whose input value satisfies $\liminf p_t x_t > 0$ is inefficient.

A DESCRIPTION OF THE PROBLEM

The neoclassical one-good model is a pure production model with a technology given by a function defined on the nonnegative reals. The production possibilities consist of inputs x and outputs $f(x)$, $x \geqslant 0$.

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When time is introduced, the technology in each period is given by a production function f_t where the f_t satisfy

(1) $f_t(0) = 0$,

(2) f_t is concave for all t,

(3) f_t is monotone increasing,

(4) f_t is differentiable. (This is primarily for simplicity. Many of the results can be achieved without it.)

DEFINITION 1. A *program* is a sequence of nonnegative inputs (x_t), $t = 0, 1, \ldots$ The associated *consumption* sequence c_t is given by

$$c_t = f_t(x_t) - x_{t+1} \geqslant 0.$$

DEFINITION 2. The *competitive prices* (p_t) associated with any program (x_t) are given by

(1) $p_0 = 1$,

(2) $p_{t+1} = p_t / f_t'(x_t)$.

These are precisely the prices which maximize profits in each period, i.e.,

$$p_{t+1} f_t(x_t) - p_t x_t \geqslant p_{t+1} f_t(x) - p_t x, \; x \geqslant 0.$$

DEFINITION 3. A program $(\bar{x}_t)$ is *efficient* if there is no program (x_t) with $x_0 = \bar{x}_0$ such that

(1) $f_1(x_1) - x_{t+1} \geqslant f_t(\bar{x}_t) - \bar{x}_{t+1}$, for all t,

(2) $f_{t_1}(x_{t_1}) - x_{t_1+1} > f_{t_1}(\bar{x}_{t_1}) - \bar{x}_{t_1+1}$, for some t_1.

An *inefficient* program is one which is not efficient.

1. THE CASE OF STRONGLY PRODUCTIVE FUNCTIONS

When the derivative of the production function f is bounded below by some number greater than 1, the Malinvaud condition (that the limiting value of capital vanishes) completely characterizes efficient programs.

In this section we will only consider programs with nonnegative consumption, i.e., $x_{t+1} \leqslant f_t(x_t)$. In fact, when negative consumption is allowed, the limiting value of capital need not exist. For many (but not all) of the functions of this type, the condition $\inf p_t x_t > 0$ completely characterizes inefficient programs (x_t). Also, we shall only be concerned with programs involving a fixed production function f for all periods t.

DEFINITION 4. A differentiable production function f is called *productive* if $f'(x) > 1$ for all x. Furthermore if $\inf_{x>0} f'(x) = a > 1$, then f is called *strongly productive*.

Henceforth, we will always let $\inf_{x>0} f'(x) = a$. Also, to simplify the notation, we will let $V_t = p_t x_t$ and $A = \sum_{s=0}^{\infty} 1/a^s$.

LEMMA 1. *Let f be a strongly productive production function. If $(\bar{x}_t)$ is any program with nonnegative consumption, then for all $t \geqslant T$*

$$\bar{V}_t/\bar{V}_T \leqslant ((f(\bar{x}_T) - \bar{x}_T)/a\bar{x}_T)\, A. \tag{1.1}$$

Proof. Let (x_t) be the program such that $x_0 = \bar{x}_0$, and $x_{t+1} = f(x_t)$, or $c_t = 0$ for all t. Since by assumption $(\bar{x}_t)$ is a program with nonnegative consumption and f is monotonically increasing, $x_t \geqslant \bar{x}_t$ for all t. By concavity $0 < f'(x_t) \leqslant f'(\bar{x}_t)$. It then follows that for all t,

$$V_t = \left[1\Big/\prod_{s=0}^{t-1} f'(x_s)\right] x_t \geqslant \left[1\Big/\prod_{s=0}^{t-1} f'(\bar{x}_s)\right] \bar{x}_t = \bar{V}_t\,. \tag{1.2}$$

Define a sequence of scalars (α_t) by $\alpha_t = f(x_t) - x_t$. By the choice of x_t, $\alpha_t = f(x_t) - x_t = x_{t+1} - x_t > 0$. Therefore, by concavity,

$$f'(x_{t+1}) \leqslant \frac{\alpha_{t+1}}{\alpha_t} = \frac{f(x_{t+1}) - f(x_t)}{x_{t+1} - x_t} \leqslant f'(x_t).$$

Using this and $x_t = x_0 + \sum_{s=0}^{t-1} \alpha_s$, it follows that

$$V_t = \left[1\Big/\prod_{s=0}^{t-1} f'(x_s)\right] x_t \leqslant \prod_{s=0}^{t-1} \frac{\alpha_s}{\alpha_{s+1}} \left(x_0 + \sum_{s=0}^{t-1} \alpha_s\right)$$

$$\leqslant \left[\prod_{s=0}^{t-2} \frac{\alpha_s}{\alpha_{s+1}}\right] \frac{1}{a} \left(x_0 + \sum_{s=0}^{t-1} \alpha_s\right).$$

Hence,

$$V_t \leqslant \frac{\alpha_0}{a} \left(\frac{x_0}{\alpha_{t-1}} + \sum_{s=0}^{t-1} \frac{\alpha_s}{\alpha_{t-1}}\right).$$

Since $\alpha_t/\alpha_{t+1} \leqslant 1/f'(x_{t+1}) \leqslant 1/a$ and $\alpha_s/\alpha_{t-1} = \prod_{i=s}^{t-2} (\alpha_i/\alpha_{i+1})$, it follows that $\alpha_s/\alpha_{t-1} \leqslant 1/a^{t-1-s}$ for $0 \leqslant s \leqslant t-1$. Hence, for all t,

$$V_t \leqslant \frac{\alpha_0}{a} \left(\frac{\bar{x}_0}{\alpha_{t-1}} + \sum_{s=0}^{t-1} 1/a^s\right). \tag{1.3}$$

The V_t are monotonically increasing since we see from the concavity of f and the fact that $f(0) = 0$ that

$$V_{t+1}/V_t = p_{t+1}x_{t+1}/p_t x_t = f(x_t)/f'(x_t)\, x_t \geqslant 1.$$

By the definition of $\alpha_t = f(x_t) - x_t$, $\lim_{t\to\infty} \alpha_{t-1} = \infty$, and hence,

$$\lim_{t\to\infty} \frac{\alpha_0}{a}\left(\frac{x_0}{\alpha_{t-1}} + \sum_{s=0}^{t-1} 1/a^s\right) = \frac{\alpha_0}{a} A.$$

It then follows from the monotonicity of the V_t and (1.3) that $V_t \leqslant (\alpha_0/a)\, A$ for all t. By (1.2) $\bar{V}_t \leqslant V_t$, and since $x_0 = \bar{x}_0$,

$$\bar{V}_t \leqslant ((f(\bar{x}_0) - \bar{x}_0)/a)\, A. \tag{1.4}$$

For any T we can define a new program (x_t') with $x_0' = \bar{x}_T$ and $x_t' = \bar{x}_{T+t}$. The associated prices are then $p_t' = \bar{p}_{T+t/\bar{p}_t}$. Applying (1.4),

$$V_t' \leqslant \left(\frac{f(x_0') - x_0'}{a}\right) A = \left(\frac{f(\bar{x}_T) - \bar{x}_T}{a}\right) A.$$

Hence $\forall t \geqslant T$,

$$\frac{\bar{V}_t}{\bar{V}_T} = \frac{\bar{p}_{T+(t-T)}\bar{x}_{T+(t-T)}}{\bar{p}_T\bar{x}_T} = \frac{V'_{t-T}}{\bar{x}_T} \leqslant \left(\frac{f(\bar{x}_T) - \bar{x}_T}{a\bar{x}_T}\right) A. \quad \blacksquare$$

The following corollary is immediate from Lemma 1.

COROLLARY 1. *Let f be a strongly productive production function. If $(\bar{x}_t)$ is any program with nonnegative consumption, then the capital value is bounded, i.e., there is some constant B such that for all t, $\bar{V}_t \leqslant B$.*

PROPOSITION 1. *Let f be a strongly productive production function. If $(\bar{x}_t)$ is any program with nonnegative consumption, then $\lim_{t\to\infty} \bar{V}_t$ exists.*

Proof. There are two cases to check: (a) lim inf $\bar{V}_t = 0$; again there are two possibilities in this category.

The first is the existence of a subsequence $(\bar{x}_{t_i})$ which is bounded above. Then there is some constant M such that $0 < (f(\bar{x}_{t_i}) - \bar{x}_{t_i})\, A \leqslant M$. Since

$$\bar{p}_{T_i} = 1\Big/\prod_{s=0}^{T_i-1} f'(\bar{x}_s) \leqslant 1/a^{T_i},$$

(1.1) implies that for $t \geqslant T_i$,

$$\bar{V}_t \leqslant \left(\frac{f(\bar{x}_{T_i}) - \bar{x}_{T_i}}{a}\right) A\bar{p}_{T_i} \leqslant M/a^{T_i+1}.$$

Hence, $\lim_{t\to\infty} \bar{V}_t = 0$.

If $\bar{x}_t \to \infty$ then $\bar{x}_t \geqslant b > 0$ for all t. Because $f(0) = 0$ and f is concave, $f(x)/x$ is monotonically decreasing. Therefore

$$0 < \left(\frac{f(\bar{x}_t) - \bar{x}_t}{a\bar{x}_t}\right) A \leqslant \frac{1}{a}\left(\frac{f(b)}{b}\right) A = \bar{M}. \tag{1.5}$$

By assumption, there is a subsequence $\bar{V}_{t_i} \to 0$. Because of (1.1) and (1.5), for all $T \geqslant T_i$, $\bar{V}_t \leqslant \bar{M}\bar{V}_{T_i}$. Hence, $\lim_{t\to\infty} \bar{V}_t = 0$. ∎

For the next case we need the following.

LEMMA 2. *If f is a strongly productive production function, then*

$$\lim_{x\to\infty}\left(\frac{f(x) - x}{ax}\right) A = 1. \tag{1.6}$$

Proof. As we saw previously, $f(x)/x \geqslant f'(x)$, is monotonically decreasing and bounded below by a. Hence, $\lim_{x\to\infty} f(x)/x$ exists and is greater than or equal to a. If $\lim_{x\to\infty} f(x)/x = b > a$ then $f(x) \geqslant bx$ for all $x > 0$. For any $\bar{x} > 0$, the concavity of f yields some constant c such that $f'(\bar{x})\, x + c \geqslant f(x) \geqslant bx$. From which it follows that $f'(\bar{x}) \geqslant b$. But this contradicts the assumption that $\inf_{x>0} f'(x) = a$, and hence, $\lim_{x\to\infty} f(x)/x = a$. Equation (1.6) then follows from $A = \sum_{x=0}^{\infty} 1/a^x = a/(a-1)$.

(b) $\liminf \bar{V}_t = l > 0$; then $\bar{V}_t = \bar{p}_t\bar{x}_t \geqslant l > 0$, and hence, $\bar{x}_t \geqslant l/\bar{p}_t \geqslant la^t$, and $\bar{x}_t \to \infty$. Choose any $\epsilon > 0$. Because of (1.6), there is some $\hat{t}$ such that for $t \geqslant \hat{t}$,

$$\left(\frac{f(\bar{x}_t) - \bar{x}_t}{a\bar{x}_t}\right) A \leqslant 1 + \frac{\epsilon}{2l}.$$

By the assumption that $\liminf \bar{V}_t = l$, there is some $T \geqslant \hat{t}$ such that $\bar{V}_T \leqslant l + (\epsilon l/(\epsilon + 4l))$. Then by (1.1), for all $t \geqslant T$,

$$\bar{V}_t \leqslant \left(\frac{f(\bar{x}_T) - \bar{x}_T}{a\bar{x}_T}\right) A\bar{V}_T \leqslant \left(1 + \frac{\epsilon}{2l}\right)\bar{V}_T$$

$$\leqslant \left(1 + \frac{\epsilon}{2l}\right)\left(l + \frac{\epsilon l}{\epsilon + 4l}\right) = l + \epsilon.$$

Hence, $\lim_{t\to\infty} \bar{V}_t = l$. ∎

The main theorem for strongly productive production functions is now given.

THEOREM 1. *Let f be a strongly productive production function. If $(\bar{x}_t)$ is a program with nonnegative consumption, then $(\bar{x}_t)$ is inefficient iff* $\lim_{t\to\infty} \bar{V}_t > 0$.

Proof. If $(\bar{x}_t)$ is inefficient then the theorem proved by Malinvaud [4] says that $\lim_{t\to\infty} \bar{V}_t > 0$.

To go in the other direction, by assumption $\lim_{t\to\infty} \bar{V}_t > 0$, and hence, there are constants b, $B > 0$ such that for all t, $0 < b \leqslant \bar{V}_t \leqslant B$. Choose N large enough so that $ba^N - B = c > 0$. By the definition of $\bar{V}_t$ and the choice of N,

$$\bar{x}_{t+N} - \bar{x}_t = \frac{\bar{V}_{t+N}}{\bar{p}_{t+N}} - \frac{\bar{V}_t}{\bar{p}_t} = \frac{1}{\bar{p}_t}\left(\bar{V}_{t+N}\frac{\bar{p}_t}{\bar{p}_{t+N}} - \bar{V}_t\right)$$

$$\geqslant \frac{1}{\bar{p}_t}(\bar{V}_{t+N}a^N - \bar{V}_t) \geqslant c\prod_{s=0}^{t-1} f'(\bar{x}_s) > 0.$$

Consider the scalars (ϵ_t) given by $\epsilon_t = 0$ for $t \leqslant N-1$, $\epsilon_N = c$ and $\epsilon_{t+N} = c\prod_{s=0}^{t-1} f'(\bar{x}_s)$ for all $t \geqslant 1$. As was shown, $\epsilon_{t+N} \leqslant x_{t+N} - \bar{x}_t$ for all $t \geqslant 0$. Define the sequence (x_t) by $x_t = \bar{x}_t$ for $0 \leqslant t \leqslant N-1$ and $x_t = \bar{x}_t - \epsilon_t$ for $N \leqslant t$. Since $\epsilon_{t+N} \leqslant \bar{x}_{t+N} - \bar{x}_t$, $x_t \geqslant 0$ for all t. Hence, (x_t) defines a program. By concavity and the fact that for $t \geqslant N$, $\epsilon_t \leqslant \bar{x}_t - \bar{x}_{t-N}$,

$$f(\bar{x}_t) - f(\bar{x}_t - \epsilon_t) \leqslant f'(\bar{x}_t - \epsilon_t)\,\epsilon_t \leqslant f'(\bar{x}_{t-N})\,\epsilon_t = \epsilon_{t+1}, \qquad \text{for } t \geqslant N.$$

It then follows that for $t \geqslant N$,

$$f(\bar{x}_t - \epsilon_t) \geqslant f(\bar{x}_t) - \epsilon_{t+1}. \tag{1.7}$$

To show that (x_t) represents a better program, it follows from (1.7) that for $t \geqslant N$,

$$f(x_t) - x_{t+1} = f(\bar{x}_t - \epsilon_t) - (\bar{x}_{t+1} - \epsilon_{t+1}) \geqslant f(\bar{x}_t) - \bar{x}_{t+1}.$$

Also

$$f(x_{N-1}) - x_N = f(\bar{x}_{N-1}) - (\bar{x}_N - c) > f(\bar{x}_{N-1}) - \bar{x}_N.$$

For $t < N$, $x_t = \bar{x}_t$, and so $c_t = \bar{c}_t$ for $t < N-1$. ∎

2. Minimum Capital Value and Cass' Theorem

As was pointed out in the introduction, when a condition of minimum capital value is added to the condition of prices growing too fast, the restriction that inputs be bounded below can be removed in the sufficiency part of Cass' theorem.

LEMMA 3. *If f is a c^2 function and there is $M > 0$ such that $f''(x) \geqslant -M$ for $0 < x \leqslant \bar{x}$, then*

$$f(\bar{x}) - f(x) \leqslant f'(\bar{x})(\bar{x} - x) + \frac{M}{2}(\bar{x} - x)^2, \qquad 0 < x \leqslant \bar{x}. \quad (2.1)$$

Proof. It follows from the mean value theorem that for $0 < \xi \leqslant \bar{x}$, $f'(\bar{x}) - f'(\xi) \geqslant -M(\bar{x} - \xi)$. If we integrate this with respect to ξ and let ξ range from x to $\bar{x}$, (2.1) follows. ∎

THEOREM 2. *Let (f_t) be a sequence of c^2 production functions. If $(\bar{x}_t)$ is a program for which there are constants $l > 0$, $m > 0$ such that*

(1) $f_t'(\bar{x}_t) \geqslant l$ *for all* t,

(2) $f_t''(x) \geqslant -M$, $0 < x \leqslant \bar{x}_t$, *for all* t,

(3) $\sum_{t=0}^{\infty} 1/\bar{p}_t < \infty$,

(4) $\liminf V_t > 0$,

then $(\bar{x}_t)$ is an inefficient program.

Proof. Because of (2) it follows from Lemma 3 that

$$f_t(\bar{x}_t) - f_t(x) \leqslant f_t'(\bar{x}_t)(\bar{x}_t - x) + M/2(\bar{x}_t - x)^2, \qquad 0 < x \leqslant \bar{x}_t. \quad (2.2)$$

By the definition of the prices, $\bar{p}_t/\bar{p}_{t+1} = f_t'(\bar{x}_t)$. Hence, if we multiply (2.2) by $\bar{p}_{t+1}$,

$$\bar{p}_{t+1}(f_t(\bar{x}_t) - f_t(x)) \leqslant \bar{p}_t(\bar{x}_t - x) + \frac{M}{2f_t'(\bar{x}_t)}(\bar{x}_t - x)^2 \bar{p}_t, \qquad 0 < x \leqslant \bar{x}_t.$$

It follows from (1) that $M/2f_t'(\bar{x}_t) \leqslant M/2l = \overline{M}$, and hence,

$$\bar{p}_{t+1}(f_t(\bar{x}_t) - f_t(x)) \leqslant \bar{p}_t(\bar{x}_t - x) + \overline{M}(\bar{x}_t - x)^2 \bar{p}_t, \qquad 0 < x \leqslant \bar{x}_t. \quad (2.3)$$

By (4) there is some $t_1 > 0$ such that for $t \geqslant t_1$, $\bar{p}_t\bar{x}_t \geqslant \delta > 0$. Define a sequence of scalars (ϵ_t) by $\epsilon_t = 0$ for $0 \leqslant t < t_1$, choosing $\epsilon_{t_1} > 0$ and $\bar{p}_{t+1}\epsilon_{t+1} = \bar{p}_t\epsilon_t + \overline{M}\epsilon_t^2\bar{p}_t$ for $t \geqslant t_1$. Since $\epsilon_t > 0$ for $t \geqslant t_1$, we can take reciprocals and break this up algebraically into

$$\frac{1}{\bar{p}_{t+1}\epsilon_{t+1}} = \frac{1}{\bar{p}_t\epsilon_t} - \frac{\overline{M}}{1 + \overline{M}\epsilon_t} \cdot \frac{1}{\bar{p}_t} \qquad \text{for} \quad t \geqslant t_1.$$

Summing this from $t = t_1$ to $T - 1$, the telescoping effect yields

$$\frac{1}{\bar{p}_T\epsilon_T} = \frac{1}{\bar{p}_{t_1}\epsilon_{t_1}} - \sum_{t=t_1}^{T-1} \frac{\overline{M}}{1 + \overline{M}\epsilon_t} \cdot \frac{1}{\bar{p}_t}, \qquad T \geqslant t_1. \quad (2.4)$$

Since $\bar{M}/(1 + \bar{M}\epsilon_t) \leqslant \bar{M}$ and $\sum_{t=t_1}^{\infty} 1/\bar{p}_t < \infty$, (2.4) shows that by choosing ϵ_{t_1} small enough, we can force $\bar{p}_t\epsilon_t \leqslant \delta \leqslant \bar{p}_t\bar{x}_t$ for $t \geqslant t_1$, so that $\epsilon_t \leqslant \bar{x}_t$ for $t \geqslant t_1$.

Now consider the sequence (x_t) given by $x_t = \bar{x}_t - \epsilon_t$. By the choice of ϵ_t, $x_0 = \bar{x}_0$ and $x_t \geqslant 0$. Using (2.3) and the choice of the ϵ_t,

$$f_t(\bar{x}_t) - f_t(x_t) \leqslant \bar{x}_{t+1} - x_{t+1}.$$

It then follows that for all t

$$f_t(\bar{x}_t) - \bar{x}_{t+1} \leqslant f_t(x_t) - x_{t+1}.$$

Also

$$f_{t_1-1}(x_{t_1-1}) - x_{t_1} = f_{t_1}(\bar{x}_{t_1-1}) - (\bar{x}_{t_1} - \epsilon_{t_1}).$$

Hence, we have constructed a better program. ∎

The following example will illustrate the need for some sort of condition in place of the lower bound on inputs.

EXAMPLE. Let $f_t(x) = 4x - x^2$ for all t. This function satisfies all the requirements on the first and second derivative for $0 < x < 2$. The first program will be one which satisfies the condition of prices growing too fast, yet the necessary condition for inefficient programs, that lim inf $V_t > 0$, is violated, and hence, the program is efficient. The second program will satisfy both conditions, and hence, is inefficient by Theorem 3.

(1) Consider the program (x_t) given by $x_0 = 15/8$, $x_t = 15/8$ for $t = 3n + 1$ and $3n + 2$ for all nonnegative n. When $t = 3n$, $n > 0$, choose x_t such that $p_t x_t = 1/t$. When $x_t = 15/8$, $f_t'(x_t) = \frac{1}{4}$, otherwise $f_t'(x_t) \leqslant 4$. Because of the choice of (x_t), there is a subsequence $V_{t_1} \to 0$, hence failing to satisfy the necessary condition of an efficient program that *lim inf* $V_t > 0$. *Since* $1/p_t = \prod_{s=0}^{t-1} f'(x_s)$, *we can show by induction that* $1/p_t \leqslant (1/4)^{[t/3]+r}$ where $[t/3]$ is the greatest integer of $t/3$ and r is the remainder. To see this, assume it is true for $t = 3n$. Then since $f'(x_{t+1})$ and $f'(x_{t+2})$ are $\frac{1}{4}$, the above holds. When $t = 3n + 3 = 3(n + 1)$, $f'(x_t) \leqslant 4$, and hence,

$$1/p_t = \prod_{s=0}^{t-1} f'(x_s) \leqslant 4(\tfrac{1}{4})^{[t-1/3]+2} = (\tfrac{1}{4})^{[t-1/3]+1} = (\tfrac{1}{4})^{[t/3]}.$$

Therefore,

$$\sum_{t=0}^{\infty} 1/p_t \leqslant \sum_{s=0}^{\infty} 3/4^s < \infty.$$

(2) Consider the program (x_t) given by $x_0 = 15/8$, $x_t = 15/8$ for $t = 3n + 1$ and $3n + 2$ for all positive n. When $t = 3n$ choose x_t such that $p_t x_t = 1$. Then by an argument similar to the above $\sum_{t=0}^{\infty} 1/p_t < \infty$. Also lim inf $V_t > 0$; hence, (x_t) is inefficient.

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A Note on the Core of an Income Redistribution Game

We consider a redistribution of income by an n-person game in which each member is levied a proportional income tax under some fixed rule. It is shown that the redistribution that will be achieved by a specific negative income tax is in the core of the game, which means that the redistribution can not be upset by any coalition of members acting strategically to evade the full taxation.

We consider a redistribution of income in a community in which each member is levied an income tax according to the rule that if his income exceeds a certain fixed level, he is taxed an amount proportional to the excess, and if otherwise, nothing. Income redistribution in the community is achieved with the tax gains under a specific negative income tax scheme.

The purpose of this note is to show that the redistribution of income achieved by the negative income tax cannot be upset by any coalition of members, if we allow the members to behave strategically by forming a coalition in which income is rearranged among the members in order to evade the full taxation.

Let $N = \{1, 2, \ldots, n\}$ denote the set of all members, and k_i the income of member $i \in N$. Each member i is taxed an amount

$$\begin{array}{ll} r(k_i - t), & \text{if } k_i \geqslant t, \\ 0, & \text{if } k_i < t, \end{array} \tag{1}$$

where r is a tax rate and t is a fixed level of income below which no tax is levied. N is partitioned into two groups, $A = \{i \in N: k_i \geqslant t\}$ and $B = \{i \in N: k_i < t\}$. We assume that $\sum_{i \in N} k_i - nt \geqslant 0$. A specific negative income tax scheme we consider under the tax rule (1) is represented by the distribution $d = (d_1, d_2, \ldots, d_n)$ of income after tax such that

$$d_i = k_i - r(k_i - t) \qquad \text{for all } i \in N. \tag{2}$$

Thus, income transfer from the members in A to the members in B occurs at the same tax rate r if the negative income tax scheme is employed.

Now, suppose that a member $i \in A$ can find a member $j \in B$ as a partner who agrees, with some side payments being paid, to an arrangement to transfer nominally a part of income from i to j. Then, the member i can

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keep the part of his income untaxed. In other words, both members can secure some gains under the tax rule (1) if the coalition $\{ij\}$ is formed. Thus, any coalition S with $S \cap B \neq \theta$ can secure gains if the members of S rearrange the distribution of income in such a way that the sum of the tax levied on each member $i \in S$ is minimized. The minimum sum is given by

$$r \cdot \max\left(0, \sum_{i \in S} k_i - st\right) \qquad \text{for all} \quad S \subset N, \tag{3}$$

where s denotes the number of members in S.

Thus, we have defined an n-person characteristic function game (N, v),

$$v(S) = -r \cdot \max\left(0, \sum_{i \in S} k_i - st\right) \qquad \text{for all} \quad S \subset N. \tag{4}$$

Note that the game (N, v) is essential if $B \neq \theta$. An imputation of the game is a payoff vector $p = (p_1, p_2, \ldots, p_n)$ satisfying $\sum_{i \in N} p_i = v(N)$, and $p_i \geqslant v(\{i\})$ for all $i \in N$, i.e.,

$$\sum_{i \in N} p_i = -r(K - nt), \qquad \text{where} \qquad K = \sum_{i \in N} k_i,$$

$$p_i \geqslant \begin{cases} -r(k_i - t) & \text{if } i \in A, \\ 0 & \text{if } i \in B. \end{cases}$$

The core of the game is the set of imputations such that

$$\sum_{i \in S} p_i \geqslant v(S) \qquad \text{for all} \quad S \subset N.$$

Given an imputation p, the distribution d of income satisfies

$$d_i = k_i + p_i \geqslant \begin{cases} k_i - r(k_i - t) & \text{if } i \in A, \\ k_i & \text{if } i \in B. \end{cases}$$

We have the following proposition.

PROPOSITION. (i) *An imputation* $p^* = (p_1^*, p_2^*, \ldots, p_n^*)$ *such that*

$$p_i^* = -r(k_i - t) \qquad \text{for all } i \in N$$

belongs to the core of the game (N, v).

(ii) *The core consists of the unique imputation* p^* *if*

$$K - nt > k_i - t \qquad \text{for all } i \in N.$$

Proof. (i) For any $S \subset N$, we have

$$\sum_{i \in S} p_i^* - v(S) = -r\left(\sum_{i \in S} k_i - st\right) + r \cdot \max\left(0, \sum_{i \in S} k_i - st\right) \geqslant 0.$$

(ii) Let $p = (p_1, p_2, \ldots, p_n)$ be in the core, and $p \neq p^*$. Then, there is an $i \in N$ such that $p_i > p_i^*$. By assumption, we have

$$v(N - \{i\}) = -r(K - nt) + r(k_i - t)$$

for all $i \in N$. Hence, we have

$$\begin{aligned} \sum_{j \in N - \{i\}} p_j = v(N) - p_i &< v(N) - p_i^* \\ &= -r(K - nt) + r(k_i - t) \\ &= v(N - \{i\}). \end{aligned}$$

This is a contradiction. Q.E.D.

The distribution (2) which is achieved if the negative income tax scheme is employed is an outcome of the game (N, v) which no coalition can upset by the strategic rearrangement of income in the coalition. Moreover, if t is predetermined sufficiently small so that $t < (K - k_i)/(n - 1)$ for all $i \in N$, then only the outcome that yields the distribution (2) belongs to the core of the game. In this case, the redistribution would be achieved automatically through the strategic behavior of the members.

The members of B, the lower income class, are in a stronger position in the game (N, v). Although this is an unrealistic situation, the result is of some interest in that a value judgement could be replaced with an individualistic motivation.

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The Optimal Development of Resource Pools

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INTRODUCTION

At each instant of time a depletable resource can be drawn from any one of a number of pools. The cost of removing an extra unit from a pool depends on how much has already been taken out of it. What policy supplies a fixed flow of the resource at minimum present discounted cost?

It is easy to characterize the optimal rule in a classical environment where every pool has nondecreasing extraction costs. At any time simply draw the required amount from the source with lowest marginal cost. But what happens if, as in the real world, incremental extraction costs decline in some initial range? In the general case, a marginalist policy of exploiting the least cost source would be suboptimal.

The present paper shows that a natural generalization of the marginalist rule is optimal when resource pools have *arbitrary* extraction costs. A key solution concept turns out to be the "equivalent stationary cost" of exploiting the resource. Properties of an optimal policy are derived and interpreted.

THE MODEL

The present paper is primarily an exercise in trying to characterize, at a high level of abstraction, the basic form of an optimal policy for exploiting natural resources under very general cost conditions, including decreasing costs. This goal can be most conveniently served by treating the underlying prototype problem in a discrete formulation. (A continuous version yields completely analogous results, but the mathematical analysis is intrinsically more technical. Actually, the form of an optimal policy in the continuous case is most easily proved by first treating the corresponding discrete or integer mode and then carefully taking the limit as the step size approaches zero.)

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A "generalized resource" is extracted from "generalized resource pools," indexed by i. The system is in state $\{m_i\}$, where each m_i is a non-negative integer, if amount m_i of the resource has already been extracted from pool i. A function

$$F_i(m)$$

gives the cost of extracting one more unit of the resource from pool i when m units have already been taken from that source. If no more than amount m can be gotten from pool i, we symbolically write $F_i(m) = \infty$.

Let v_i be an indicator variable that takes on value unity if pool i is being exploited, and zero, otherwise. A transition can be made from state

$$\{m_i\}$$

to state

$$\{m_i + v_i\}$$

where

$$v_i = 0 \text{ or } 1,$$

$$\sum_i v_i = 1,$$

at a cost of

$$\sum_i v_i F_i(m_i).$$

There is a constant demand for the resource which must be satisfied. Without loss of generality, time is measured so that demand equals one. Let α be the appropriate one-period discount factor. The basic problem is to determine control variables $\{v_i(t)\}$ which will

$$\text{minimize} \sum_{t=0}^{\infty} \alpha^t \sum_i v_i(t) F_i(m_i(t)) \tag{1}$$

subject to

$$m_i(t+1) = m_i(t) + v_i(t), \tag{2}$$

$$\sum_i v_i(t) = 1, \tag{3}$$

$$v_i(t) = 0 \text{ or } 1. \tag{4}$$

(Implicitly we are assuming that (1)–(4) is well formulated. In fact it is not difficult to specify sufficient conditions for guaranteeing the existence of a solution.)

There are many ways in which the actual economics of extraction is more complicated than the simple model presented here. (Just to mention a few: Uncertainty may be present; demands and discount factors could vary with time; extraction costs might depend on current outflow as well as on how much has been cumulatively taken from a pool. Equation (4) could be

replaced by the seemingly weaker "mixing" condition $0 \leqslant v_i(t) \leqslant 1$, but only at the expense of making the analysis more intricate, and without otherwise altering results. Note that when time units are sufficiently small any fractional extraction policy can be approximated to an arbitrary degree of accuracy by a discrete policy of the form (4) (which rapidly switches back and forth among the pools being "mixed").) Nevertheless, the above formulation might be defended as a rough approximation to the general issue of finding the best pattern for developing a resource from alternative sources or pools. The fact that it is possible to sharply characterize an optimal solution makes problem (1)–(4) a natural preliminary to any more general analysis. And it may even be a reasonable description of some situations.

Depending on the context, the various pools might be mines, pits, wells, deposits, fields, regions, or even resource-based technologies. Actually, with a judicious interpretation of "resource pool," formulation (1)–(4) is sufficiently general to include as special cases many of the standard deterministic operations–research models for such problems as equipment selection and replacement, inventory and production scheduling, or capacity expansion. (In such situations there are typically several *classes* of pools, each of which contains an infinite number of identical members. For equipment problems a "resource pool" is a certain piece of equipment and the "amount extracted" is the length of time it has been in service. With scheduling, inventory, and expansion problems, a pool is a certain strategy-schedule (like ordering inventory or building capacity) which goes up to some expiration or regeneration point (after which costs are interpreted as being infinite); the "amount extracted" is the length of time the given strategy-schedule has been carried out.)

Nondecreasing Costs: A Special Case

Consider now the classical assumption of constant or increasing marginal cost. In symbols,

$$F_i(m) \leqslant F_i(m') \tag{5}$$

for all i, m, m' such that

$$m \leqslant m'.$$

When restriction (5) is imposed, it is easy to characterize an optimal solution to problem (1)–(4). Always extract the next unit from the cheapest source currently available. At time t, fix

$$v_{i^*}(t) = 1$$

for any integer i^* satisfying

$$F_{i^*}(m_{i^*}(t)) = \min_i F_i(m_i(t)). \tag{6}$$

That such a policy is optimal follows directly from applying the standard marginalist argument to a situation with convex cumulative cost functions and positive time discounting.

It is important to realize that the marginalist selection rule (6) does *not* give correct answers in a nonconvex environment where (5) fails to hold. If extraction costs can decline, an optimal policy may require exploiting a source having high initial but low eventual costs. There is no way that a completely myopic algorithm that looks only at incremental costs will pick out this kind of policy. The correct solution concept, if there is one for a problem seemingly as complex as (1)–(4), would have to somehow look ahead at the entire pattern of future costs.

Now we can pose the following basic question. Is there any simple abstraction or generalization of the marginalist principle which works when *no structure whatsoever* is placed on the cost coefficients $\{F_i(m)\}$? The answer turns out to be yes.

The algorithm which is developed in this paper for dealing with an arbitrary cost structure may be of some interest. Situations with a range of decreasing marginal cost are frequently the rule rather than the exception. Substantial set-up costs in overhead, research, development, or the like must usually be incurred just to open up a resource pool. Then there is the pervasive feature of learning by doing or cumulative technological progress, which is likely to lower costs over at least the beginning stages of exploiting a new source. It is true that costs of extracting from a given pool will eventually rise if and when resource deposits start to give out. (Even then, fresh investments in secondary and tertiary recovery methods may cut subsequent operating costs, at least temporarily.) But the evidence suggests that incremental costs are not universally nondecreasing over their entire range.

Solving the General Case

If amount m has already been drawn from pool i, the equivalent stationary cost of taking the next n units in a row from it is the weighted average

$$\Psi_i^n(m) = \sum_{j=0}^{n-1} F_i(m+j)\,\alpha^j \Big/ \sum_{j=0}^{n-1} \alpha^j. \tag{7}$$

Let $\hat{n}$ be the time horizon which minimizes (7),

$$\Psi_i^{\hat{n}}(m) = \min_{n=1,2,\ldots} \Psi_i^n(m). \tag{8}$$

We permit the case $\hat{n} = \infty$. (Strictly speaking, the operation *inf* should replace *min* in (8). Likewise for formulae (6) and (11) if the number of resource pools is infinite.) This could occur, for example, when costs are everywhere nonincreasing.

The *implicit cost* of pool i (when m units have been extracted from it) is defined to be its smallest equivalent stationary cost

$$\Phi_i(m) = \Psi_i^{\hat{n}}(m). \tag{9}$$

The following decision rule completely characterizes an optimal policy in the general case. (That is, it constitutes a necessary and sufficient condition for an optimum.)

Always extract the next resource unit from the pool with lowest implicit cost. Symbolically, in an optimal policy, at each t

$$v_{i^*}(t) = 1 \tag{10}$$

for any integer i^* satisfying

$$\Phi_{i^*}(m_{i^*}(t)) = \min_i \Phi_i(m_i(t)). \tag{11}$$

Converting arbitrary cost streams to stationary equivalents for the purpose of finding the cheapest alternative is an old economist's trick. The optimality of decision rule (10), (11) can in a sense be interpreted as justifying this heuristic procedure under certain conditions.

Note that the implicit cost of a pool reduces to its marginal cost for the special case of nondecreasing costs. This is because when (5) holds, the horizon $\hat{n} = 1$ is always a solution of (8); and then (9) becomes

$$\Phi_i(m) = F_i(m).$$

So the decision rule (11) is indeed a generalization of the marginalist principle (6).

It is not necessary to recalculate implicit costs in each time period. If a pool is not used, its implicit cost does not change. If it is used, its implicit cost must have been lower, or at least no higher, than the implicit cost of any other pool. But then it will remain the lowest implicit cost pool (and hence, will continue to be exploited) for at least a number of periods equal to the equivalent stationary cost minimizing horizon $\hat{n}$ of (8).

This last conclusion can be derived as follows. Split the time interval $[0, \hat{n})$ into two subintervals $[0, h)$ and $[h, \hat{n})$ for h any integer between 1 and $\hat{n} - 1$. The equivalent stationary cost of extracting over the interval $[0, \hat{n})$ is a weighted average of the equivalent stationary costs over its component subintervals,

$$\Psi_i^{\hat{n}}(m) = \left(\frac{1 - \alpha^h}{1 - \alpha^{\hat{n}}}\right) \Psi_i^h(m) + \left(1 - \left[\frac{1 - \alpha^h}{1 - \alpha^{\hat{n}}}\right]\right) \Psi_i^{\hat{n}-h}(m + h). \quad (12)$$

We know from (8) that

$$\Psi_i^{\hat{n}}(m) \leqslant \Psi_i^h(m).$$

It follows that

$$\Psi_i^{\hat{n}-h}(m + h) \leqslant \Psi_i^{\hat{n}}(m).$$

Using (9), the above expression becomes

$$\Psi_i^{\hat{n}-h}(m + h) \leqslant \Phi_i(m). \quad (13)$$

But

$$\Phi_i(m + h) \leqslant \Psi_i^{\hat{n}-h}(m + h).$$

Thus,

$$\Phi_i(m + h) \leqslant \Phi_i(m)$$

for $h = 1, 2, \ldots, \hat{n} - 1$, the result to be proved.

To illustrate the typical form of an optimal policy for exploiting depletable natural resources, consider the following simple example. Suppose there are but three resource pools. Each pool has an initial range of decreasing costs, followed by a final section of increasing costs. Let the pools be ordered so that the first has lower implicit cost than the second, which in its turn is lower than the third. The optimal strategy will be to initially exploit the pool with lowest implicit cost, pool number one. This will be done until the marginal cost of extracting one more unit (in the increasing cost range) becomes greater than the implicit cost of the second pool. At that time pool two will start being exploited, and it will be the exclusive source until its marginal cost in the increasing cost range becomes greater than the marginal cost of pool one. Then pool one or two will alternately be exploited, depending on which is currently the cheaper source at the margin (in a continuous formulation, both would simultaneously be tapped). That is, if pool one has lower incremental cost than two, it will be exploited until its incremental cost exceeds two's, then two will be exploited until its incremental cost exceeds one's, etc. This rising marginal cost phase ends when the implicit cost of the third pool is found to be lower than the incremental costs of the first two pools. Then the

third pool alone is tapped, up to the point where its incremental cost in the increasing cost range becomes greater than the incremental cost of the first or the second pool. From then on all three pools are alternately exploited, that pool being tapped at any given time which has lowest marginal cost.

Note that in a situation where all pools are the same and there is an unlimited collection of them, the optimal policy will be cyclic or recursive. The same conclusion holds if there are several classes of pools, each class containing an infinite number of identical pools (because in an optimal strategy only pools from one class will be tapped). This is why so many of the standard operations research models with renewable options (for example, the simple deterministic models of equipment selection and replacement, inventory and production scheduling, or capacity expansion) end up having a repetitive solution which may be universally characterized as follows. At each decision node, choose the strategy element with lowest equivalent stationary cost.

That such an elementary decision rule as (10), (11) is optimal depends crucially on the simplifying assumptions of the model. There does not seem to be available a sharp characterization of an optimal solution when, for example, demand varies with time or the discount rate is not constant. (About all that might be said of a general character in such cases is that a limiting argument could be used to show the results presented here are valid as an approximation when the stipulated preconditions are close to being met.)

The Basic Monotonicity Property

The correctness of decision rule (10), (11) derives ultimately from an important monotonicity property.

Pick any time t. Suppose that pool i is being exploited throughout the interval $[t, t + k]$ but not in period $t + k$, for k some positive integer. (If $k = \infty$, the monotonicity property is empty. But an important corollary, to be explained later, still holds.) Then pick any time $t' > t + k$. Suppose that pool i' is exploited throughout the interval $[t' - k', t')$ but not in period $t' - k' - 1$, for some positive integer k'. (Naturally, $t + k \leqslant t' - k'$.)

The basic monotonicity property is that with an optimal policy the equivalent stationary cost of exploiting pool i over the interval $[t, t + k)$ must be no greater than the equivelent stationary cost of exploiting i' over $[t' - k', t')$. In symbols,

$$\Psi_i^k(m_i(t)) \leqslant \Psi_{i'}^{k'}(m_{i'}(t' - k')). \tag{14}$$

When $t + k = t' - k'$, this fundamental lemma can be proved by a simple variational argument that reverses the order of pool exploitation. Suppose the original exploitation pattern looks like this:

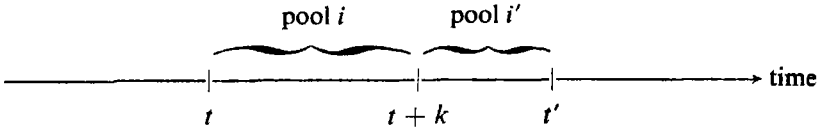


FIGURE 1

If the equivalent stationary cost of exploiting pool i over the interval $[t, t + k)$ is greater than pool i' over $[t' - k', t')$, it will be cheaper to delay incurring the higher cost by changing the exploitation pattern to:

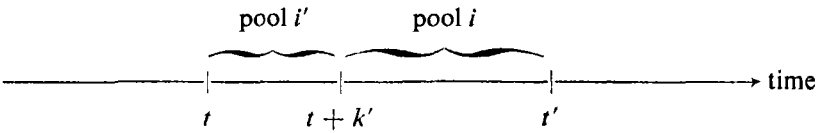


FIGURE 2

More formally, when $t + k = t' - k'$, the present discounted cost over the interval $[t, t')$ of the allegedly optimal policy which exploits pool i in the subinterval $[t, t + k)$ and pool i' in the subinterval $[t' - k', t') = [t + k, t')$ is

$$\alpha^t(1 - \alpha^k)/(1 - \alpha)\, \Psi_i^k(m_i(t)) + \alpha^{t+k}(1 - \alpha^{k'})/(1 - \alpha)\, \Psi_{i'}^{k'}(m_{i'}(t + k)). \tag{15}$$

By comparison, the present discounted cost of the feasible alternative policy which would reverse the order of exploitation of pools i and i' in the interval $[t, t')$, but would otherwise leave the system in the same state at times t and t', is

$$\alpha^t(1 - \alpha^{k'})/(1 - \alpha)\, \Psi_{i'}^{k'}(m_{i'}(t + k)) + \alpha^{t+k'}(1 - \alpha^k)/(1 - \alpha)\, \Psi_i^k(m_i(t)). \tag{16}$$

Because the original policy was optimal, the value of expression (15) must not be higher than (16). The resulting inequality reduces to

$$\Psi_i^k(m_i(t)) \leqslant \Psi_{i'}^{k'}(m_{i'}(t + k)), \tag{17}$$

which is equivalent to (14) for $t + k = t' - k'$.

Suppose now that $t + k < t' - k'$. Divide the time interval $[t + k, t' - k')$ into r contiguous subintervals of the form $[t_{j-1}, t_j)$ with $t_j > t_{j-1}$. The divisions are made so that during each subinterval one distinct pool is being exploited. Let pool $i_j (i_j \neq i_{j-1}$ and $i_j \neq i_{j+1})$ be tapped in the time interval $[t_{j-1}, t_j)$ for $j = 1, 2, \ldots, r$ where $t_0 \equiv t + k$, $t_r \equiv t' - k'$, $i_0 \equiv i$, $i_{r+1} \equiv i'$. Repeatedly applying the appropriate form of result (17) at each of the $r + 1$ transition times $\{t_j\}_{j=0}^{r}$ (where exploitation of one pool ends and of another begins) yields a string of inequalities which can be collapsed into (14).

The basic idea is illustrated in Fig. 3 for the case $r = 2$. From what has previously been shown, the equivalent stationary cost over

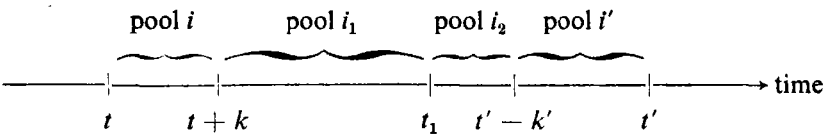


FIGURE 3

the interval $[t, t + k)$ is $\leqslant$ over $[t + k, t_1)$, which is $\leqslant$ over $[t_1, t' - k')$, which is $\leqslant$ over $[t' - k', t')$. This concludes our proof of the basic monotonicity property.

The following corollary to the basic monotonicity property is important.

Let i' be any pool and k' any positive integer. Suppose there is a time T such that pool i' is never tapped after T. Let the equivalent stationary cost of (hypothetically) exploiting i' over the time interval $[T, T + k')$ be

$$\mu = \Psi_{i'}^{k'}(m_{i'}(T)). \tag{18}$$

Suppose pool i is exploited throughout the interval $[T, T + k)$ but not in period $T + k$, for some positive integer k. (We also allow the possibility $k = \infty$. In this case it will be evident that the proof of the corollary still goes through, although there is no opportunity or need to apply the basic monotonicity property. If $k' = \infty$, the statement and proof of the corollary are likewise unaffected.) Let the equivalent stationary cost of exploiting i over the time interval $[T, T + k)$ be

$$\lambda = \Psi_i^k(m_i(T)). \tag{19}$$

In an optimal policy, it must be that

$$\lambda \leqslant \mu. \tag{20}$$

The proof of corollary (20) is as follows. Let the equivalent stationary cost at time T of the optimal policy over the entire interval $[T, \infty)$, during which there is no extraction from i', be β. In other words, if the alledgedly optimal policy costs $f(\theta)$ in period θ,

$$\beta \equiv (1 - \alpha) \sum_{\theta=T}^{\infty} f(\theta)\, \alpha^{\theta - T}. \tag{21}$$

By the basic monotonicity property, λ must be no greater than the equivalent stationary cost over any subinterval of $[T, \infty)$ during which one distinct pool is being exploited. But there is an easy generalization of (12) which says that the equivalent stationary cost of extracting over any interval is a weighted average of the equivalent stationary costs of extracting over each of its component subintervals. It follows that

$$\lambda \leqslant \beta. \tag{22}$$

Consider an alternative policy which exploits pool i' in the interval $[T, T + k')$ while postponing to a starting date of $T + k'$ the allegedly optimal policy which had previously begun at T. In other words, what was policy over $[T, \infty)$ now becomes policy over $[T + k', \infty)$ and pool i' is exploited in the interval $[T, T + k')$. Before time T, the original and alternative strategies coincide.

The cost over $[T, \infty)$ of the alternative variation discounted to time T is

$$\frac{1 - \alpha^{k'}}{1 - \alpha}\, \mu + \frac{\alpha^{k'}}{1 - \alpha}\, \beta.$$

This must be no less than the corresponding discounted cost of the original optimal policy

$$\beta/(1 - \alpha).$$

It follows that

$$\beta \leqslant \mu. \tag{23}$$

Combining (22) and (23) yields the desired corollary (20).

Proof of Optimality

Our proof of the main result is by contradiction. Suppose that the proposed decision rule (10), (11) is nonoptimal. If some other policy is optimal, there must be a time t and a pool $j(\neq i^*)$ such that

$$\Phi_j(m_j(t)) > \Phi_{i^*}(m_{i^*}(t)), \tag{24}$$

and yet,

$$v_j(t) = 1, \tag{25}$$

instead of $v_{i^*}(t) = 1$.

In this allegedly optimal policy, let $t + k$ be the first period after t that some pool other than j is exploited. (The case $k = \infty$ is allowed, which means that pool j alone is exploited after time t.) The equivalent stationary cost of exploiting pool j over the interval $[t, t + k)$ is

$$\gamma = \sum_{h=0}^{k-1} F_j(m_j(t + h))\, \alpha^h \Big/ \sum_{h=0}^{k-1} \alpha^h. \tag{26}$$

From the definition of $\Phi_j(m_j(t))$,

$$\Phi_j(m_j(t)) \leqslant \gamma,$$

which can be combined with (24) to yield

$$\Phi_{i^*}(m_{i^*}(t)) < \gamma. \tag{27}$$

Let n^* be the value of $\hat{n}$ which satisfies (8) for $i = i^*$, $m = m_{i^*}(t)$. If $n^* = \infty$, a direct argument based on (27) and the monotonicity property show that exploiting only pool i^* from time t on has a present discounted cost lower than the proposed policy. In what follows, the case $n^* < \infty$ is treated.

Although pool j is being exploited at time t instead of pool i^* it may be that i^* will be exploited again at some future dates. Suppose, as one case, that at least $m_{i^*}(t) + n^*$ units are eventually extracted from pool i in the allegedly optimal policy with (24), (25) holding. Let t' be the first time when

$$m_{i^*}(t') = m_{i^*}(t) + n^*.$$

Let k' be the positive integer such that pool i^* is exploited throughout the interval $[t' - k', t')$, but not at time $t' - k' - 1$.

For the values $i = i^*$, $m = m_{i^*}(t)$, $\hat{n} = n^*$, $h = n^* - k'$, expression (13) becomes

$$\Psi_{i^*}^{k'}(m_{i^*}(t) + n^* - k') \leqslant \Phi_{i^*}(m_{i^*}(t)). \tag{28}$$

Combining (28) with (27) yields

$$\Psi_{i^*}^{k'}(m_{i^*}(t) + n^* - k') < \gamma. \tag{29}$$

Now inequality (29) is a direct contradiction with the basic monotonicity property (14). A time segment $[t, t + k)$ with *higher* equivalent stationary

cost γ is preceding a segment $[t' - k', t')$ with *lower* equivalent stationary cost $\Psi_{i^*}^{k'}(m_{i^*}(t) + n^* - k')$.

A contradiction (now with the corollary) is also implied if after time t a total of *less* than n^* additional units are to be taken from pool i^* in an allegedly optimal policy. Let T be the time right after the last unit has been taken from pool i^* if this occurs after t, and t otherwise. In the present context let k' be defined as the positive integer satisfying

$$m_{i^*}(T) = \lim_{s \to \infty} m_{i^*}(s) = m_{i^*}(t) + n^* - k'. \tag{30}$$

For k' so defined, (29) continues to hold. Using (30), expression (29) becomes

$$\Psi_{i^*}^{k'}(m_{i^*}(T)) < \gamma. \tag{31}$$

Employing definition (18) for $i' = i^*$, (20) becomes

$$\Psi_{i^*}^{k'}(m_{i^*}(T)) \geqslant \lambda. \tag{32}$$

By the basic monotonicity property and the definition of T,

$$\gamma \leqslant \lambda. \tag{33}$$

Inequalities (32) and (33) can be combined to yield

$$\Psi_{i^*}^{k'}(m_{i^*}(T)) \geqslant \gamma$$

which is a direct contradiction with (31).

This concludes our proof of the form of an optimal policy. (Strictly speaking, we have proved the necessity of (10), (11). That rule specifies a unique selection of i^* for each t (except in the case of ties, for which it is easy to show that how the tie is broken makes no difference to the value of the objective function). Thus, provided an optimum exists, sufficiency of (10), (11) has also been demonstrated.)

Scarcity Price of the Resource

By "efficiency," "shadow," or "scarcity" price of the resource at time t is meant its marginal value at that time, defined in terms of the overall objective (1), discounted to t. (Note that efficiency prices will not necessarily have the "decentralization" property of inducing and supporting optimal decisions unless we are operating in the convex environment of nondecreasing extraction costs.) It turns out that the efficiency

price of the generalized resource can be represented by a simple expression with a direct economic interpretation.

The shadow price of the resource at time t is

$$p_t = (1 - \alpha) \sum_{s=t}^{\infty} F^*(s)\, \alpha^{s-t}, \tag{34}$$

where

$$F^*(s) \equiv F_{i^*(s)}(m_{i^*}(s)) \tag{35}$$

is the extraction cost in period s of an optimal policy.

The above result has a most interesting interpretation. The current shadow price is an exponentially weighted (by the discount rate) average of all future costs. At any time the scarcity price of an exhaustible resource is the equivalent stationary cost of exploitation henceforth. In other words, the efficiency price is exactly equal to that hypothetical constant extraction cost per unit which would yield the same present discounted cost as an optimal policy.

That the shadow price must have the form (34) is easily demonstrated. Suppose that an extra unit of the resource is given as a free gift in period t (recall that the resource could have been measured in arbitrarily small units to begin with). Consider the following proposed policy, which is now feasible. Up to time t, follow exactly the same extraction pattern as the original optimal solution (10), (11). During the time interval $[t, t + 1)$. take advantage of the free gift and extract nothing. Throughout the interval $[s, s + 1)$ for $s \geqslant t + 1$, extract from the pool $i^*(s - 1)$ which in the original optimal solution had been exploited during the previous interval $[s - 1, s)$.

The cost of such a policy discounted to period t is

$$C_t' = \sum_{s=0}^{t-1} F^*(s)\, \alpha^{s-t} + \sum_{t+1}^{\infty} F^*(s - 1)\, \alpha^{s-t}.$$

The cost of the original optimal policy discounted to period t is

$$C_t = \sum_{s=0}^{\infty} F^*(s)\, \alpha^{s-t}.$$

The difference is

$$C_t - C_t' = (1 - \alpha) \sum_{s=t}^{\infty} F^*(s)\, \alpha^{s-t}.$$

Thus, the gain achieved by the gift of an extra unit of the resource in period t is at least p_t, defined by (34). (It would be exactly p_t if the gift

came as an unforseen "surprise" at t, in which case the proposed policy would be optimal. If the date of the gift is known beforehand, a planner may be able to cut present discounted costs by altering the proposed extraction schedule.) Using a similar argument, the loss incurred by taking away one unit of the resource in period $t-1$ is no greater than p_t. The efficiency price of the resource at time t, defined as its undiscounted marginal value, is therefore given by expression (34). (A few mathematical fine points are being glossed over in favor of a cleaner exposition. For example, left- and right-hand-side marginal products need not be equal to each other; the proposed shadow price is merely "trapped between" them. Also, it is implicitly being assumed that time periods are "sufficiently small" to justify the variational arguments being employed.)

As a consequence of (34), the efficiency price of a depletable resource can have a complicated and interesting time profile. It may temporarily decline, corresponding to the development of a source having a significant range of decreasing costs (the decline will be permanent if the decreasing cost stage lasts forever). In the long run, when different pools are exploited the shadow price trends upward because the resource is becoming more expensive as the cheaper sources are exhausted; this increase will of course be monotonic in the classical case where each pool has nondecreasing costs. For operations-research type problems with renewable options, the shadow price exhibits a wave-like recursive pattern.

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Job Search in a Dynamic Economy

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1. INTRODUCTION

The last few years have witnessed an increasing interest in models of information and search with the seminal work of Stigler [24, 25] providing the basic theme. The major motivation for this work is twofold: to understand and guide the microeconomic behavior of individuals in a world of imperfect information, and to use this microeconomic theory as a basis for explaining macroeconomic phenomena like the Phillips curve. Many variants of the original Stigler model have been studied. These include sequential search policies [13, 15], the role of risk aversion [5, 9, 17, 28], adaptation to unknown distributions [20, 26] and a detailed analysis of barriers to search like discrimination and turnover [1, 2, 3, 21].[1] All of these variations retain the basic structure of the standard search model and testify to its enduring importance. It has contributed substantially to economic theory by introducing information costs and search into the highly unrealistic neoclassical models of perfect (costless) information. Nevertheless, the standard search model possesses severe limitations when viewed either as a normative guide to the individual job searcher or as a microeconomic rationale for macroeconomic behavior. The purpose of this paper is to criticize the standard model from both of these perspectives and construct alternative formulations that begin to repair some of the major difficiencies.

The limitations of the standard search methodology are best illustrated by quickly reviewing a stark but representative version. This version is presented in an infinite horizon setting with no discounting. A job offer X_i is received in the ith period of search, where X_i is a random variable with cumulative distribution function F. The distribution function F is known and invariant over time, and the X_i's are mutually independent. The job

[1] For a survey of this literature see [11].

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searcher is assumed to retain the highest job offer, so that the return from stopping after the nth search is given by

$$Y_n = \max(X_1, \ldots, X_n) - nc,$$

where c is the cost per period of search. This cost includes all those out of pocket expenditures, like advertising and transportation, that are incurred each time a job offer is obtained.

The objective is to find a stopping rule that maximizes $E(Y_N)$ where N is the random stopping time, i.e., the random number of job offers before an acceptance.[2] Let R be the expected gain from searching according to the best stopping rule. Consider the first observation X_1. By definition of R, the first wage offer is accepted if it exceeds R and rejected (search continues) if it falls short of R. Clearly then for any wage offer x, the optimal policy has the form

$$\begin{array}{ll} \text{accept job} & \text{if } x \geqslant R, \\ \text{continue search} & \text{if } x < R. \end{array} \tag{i}$$

The expected return from this optimal policy is

$$\text{E}\max(R, X_1) - c.$$

Since R was defined to be the expected return from pursuing the "best" stopping rule, this simple argument shows that the optimal expected return from the optimal stopping rule satisfies

$$R = \text{E}\max(R, X_1) - c. \tag{ii}$$

which is equivalent to

$$c = \int_R^\infty (x - R)\, dF(x) \equiv H(R). \tag{iii}$$

This equation has a simple economic interpretation. The search cost c is, of course, the marginal cost of obtaining another job offer, whereas $H(R)$ is the expected marginal return from one more observation. The critical value R of a job offer is chosen to equate the marginal cost of one more offer with its expected marginal return.

The critical number R is called the *reservation wage*, and any search policy with a form given by (i) is said to possess the *reservation wage property*.

[2] The maximum expected gain is finite provided the expectation of the wage distribution exists. We assume this is the case so the existence of an optimal stopping rule is guaranteed. See [19].

The random variable N, the number of offers until R is exceeded, has a geometric distribution with parameter $p = 1 - F(R)$ and expected value $1/p$. Thus, the expected duration of search, "frictional unemployment," is an increasing function of the reservation wage. The job searcher will never commence search unless the expected return from unemployment is less than R, the expected return from search. This implies that labor force participation is an increasing function of the reservation wage.

The first limitation of this model is the assumption that the wage distribution is static, it is insensitive to the business cycle. Yet this model has been used extensively to "explain" dynamic behavior over the business cycle and to provide "optimal" strategies for job searchers who must contend with the vagaries of the business cycle. Clearly, the state of the economy is in constant flux and is an important determinant of search. In Section 2 we design a search policy that explicitly considers business cycle effects. A search policy that is optimal in a depressed economy is unlikely to be best in a boyuant economy. In particular, we expect the reservation wage to be lower in the former situation. It is precisely behavioral responses like this that generate macroeconomic phenomena. Indeed, most macroeconomic phenomena for which search models were designed manifest themselves only in a dynamic economy.

Our model envisages the economy changing according to a known Markov chain in which the probability of reaching a higher state in one transition is a nondecreasing function of the current state. Each state of the economy is characterized by a known distribution of wages. Assuming that the wage distribution is stochastically increasing in the state of the economy, our analysis demonstrates that for each state there is a reservation wage and this wage increases with the state of the economy. At this point it should be emphasized that ours is not an equilibrium model of search. Only the employee side of the search market is studied. Each searcher has his own reservation wage, and the heterogeneity of labor implies that there is no single wage below which firms cannot hire.

Section 3 presents the implications of the dynamic model for labor market behavior. These behavioral implications are contrasted with those of the static model.

Having introduced the business cycle it is obvious that the searcher may wish to vary his intensity of search as the economy changes. The optimal policy is set forth in Section 4. In contrast to the monotonicity of reservation wages with the state of the economy, it is somewhat surprising that when the individual can vary the rate of search, the optimal search intensity need not increase as the economy improves.

Another major criticism of the standard search model is its misuse in attempting to explain quits and layoffs. For example, the following argu-

ment has been developed to explain the Phillips curve (see [11]). Consider an increase in aggregate demand causing a shift in the demand for a firm's output. This in turn increases the demand for inputs, in general, and labor, in particular. Suppose this increase in demand is viewed as a temporary departure from some long-run trend. Wages rise and (this is the key assumption) searchers, *who are unaware of the increase in aggregate demand*, quickly find offers above R, the reservation wage in effect before the increase. Consequently unemployment declines as the rate of change in the price level increases. (The increase in aggregate demand induces a temporary increase in the rate of inflation.) A temporary decline in aggregate demand sets in motion corresponding forces on the downside. Wages decline and searchers, *being unaware of the decrease in aggregate demand*, do not change R as the wage offer distribution shifts to the left. Thus, unemployment increases as rate of change in inflation diminishes. This loose reasoning implies a negative short-run relationship between the rate of inflation and unemployment. Before questioning the logic of this "theory" consider the following empirical facts. Firms do not adjust wages in response to temporary changes in demand. Wages are sticky, especially on the downside, where employers adjust output by layoffs rather than wage reductions. The second empirical fact concerns the quit behavior during temporary recessions and temporary inflations. Quit rates decrease in recessions and increase in inflations. What does the standard search model "predict?" In recessions wages decline and workers quit because they mistakenly perceive steady wages elsewhere. Alternatively, in inflations workers should be less inclined to quit because they do not perceive that opportunities elsewhere are increasing at the same rate.

Having studied the "empirical implications" of the elementary search model, let us now investigate its theoretical foundations. The elementary search model was designed for a static economy, the wage distribution was assumed to be constant over time. Furthermore, employment constituted an absorbing state in this model. Once an employee accepts a wage offer he remains in that job at that wage until retirement. Similarly, employers were assumed to retain workers until their retirement. In spite of these limitations, the simple search model has been asked to explain a phenomenon that only makes sense in a dynamic economy. This model has also been admonished for not permitting layoffs and making incorrect predictions about quits. Yet the simple search model doesn't know what quits and layoffs are, and therefore, should not be used to make predictions about either. All of this suggests that search theory has been misused. The appropriate search model is one that specifies the search strategy as a function of the business cycle. The model discussed in Section 2 is a

first step in this direction. If, in addition, an explanation of quit and layoff behavior is desired, then the dynamic search model must be modified to include both quits and layoffs.

In Sections 5, 6, and 7 we *begin* to develop a dynamic model that does include, respectively, quits, layoffs, and both quits and layoffs. The model is incomplete in that quits and layoffs are not viewed as jointly determined by both employer and employee. In addition, we assume that wages paid to employed workers do not change with the business cycle. The implicit contract theories of Gordon [8] *et al.* lend some credence to this assumption for certain classes of workers. Clearly, though, the assumption does not strictly hold. Typically, firms do alter wages in response to changing business conditions and if they do not, the precise arrangements underlying the implicit contract should be spelled out in detail. We have not yet resolved this problem, so rather than include an arbitrary adjustment mechanism in our already complex model, we assume the wage rate is constant. We also treat all workers alike and do not impose quit and layoff restrictions as a consequence of implicit employment contracts.

While our model is different from those previously employed in analyzing labor markets, it does include several of the previous models as special cases. We illustrate this in Section 3 for Salop's systematic search model and in Section 5 for Becker's specific human capital model.

2. A Simple Model of Search in a Dynamic Economy[3]

For expositional clarity, we begin with the simplest discrete time form of our model.

Let $\{1, 2,..., K\}$, $K \leqslant +\infty$, be the set of states of the economy, and denote the distribution function of wages associated with state i by F_i. The discrete time discount factor is $\beta \leqslant 1$, and the search cost is $c > 0$. Naturally, we also assume that $\sup \mu_i < \infty$, where μ_i is the mean of F_i.[4] The economy changes according to a discrete time Markov chain, with one-step transition matrix $P = (P_{ij})$, which is independent of the sequence of offers drawn from $\{F_i\}$.[5]

The job offer available is the value of the random draw from last

[3] A similarly structured model has been used by Derman [7] in his study of optimal maintenance policies.

[4] The state space can be the entire real line provided $\lim_{x\to\infty} \mu_x < \infty$, where $\mu_x = \int_0^\infty y dF_x(y)$.

[5] Note this includes as a special case the economy moving according to a nonsymmetric random walk. In the continuous case, this would correspond to Brownian motion with a positive drift.

period's distribution function. If the searcher accepts an offer of x, the process terminates and the searcher is absorved into the employment state for the n periods remaining in his working life. In this context, x can be regarded roughly as the discounted value of all future wages. If the searcher rejects x, he must pay a fixed price c for a draw from the distribution F_i. The economy then moves to a new state j according to P_{ij}. After this transition to state j, the searcher must decide whether to accept or reject the new offer y. The searcher is not allowed to retain rejected offers. The goal is to find a stopping rule which maximizes the β-discounted expected net benefits.

In order to ensure that the larger numbered states are preferable, we make two fundamental assumptions.

ASSUMPTION 1.[6] The distribution functions $\{F_i\}$ are stochastically increasing, that is

$$F_1(t) \geqslant F_2(t) \geqslant \cdots \geqslant F_K(t), \qquad \text{all } t \geqslant 0.$$

ASSUMPTION 2.[7] For any k,

$$\sum_{j=k}^{K} P_{ij} \text{ is nondecreasing in } i;^{8}$$

[6] Assumption 1 implies that $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_K$ where μ_i is the mean of F_i, but, of course, not conversely. Moreover, it implies that $H_{i+1}(R) = \int_R^\infty (x - R)dF_{i+1}(R) \geqslant \int_R^\infty (x - R)dF_i(R) = H_i(R)$ for each R, so that if each state were absorbing and $\beta = 1$, the reservation wage associated with state $i + 1$ would be at least as large as that associated with state i.

[7] Assumption 2 rules out situations in which a high number state is inferior to a low number state. This situation could arise if the transition probability from a high state to a lower state was sufficiently large. Furthermore, define

$$\tilde{\mu}_i^{(n)} = \sum_{j=1}^{K} P_{ij}^{(n)} \mu_j .$$

Then Assumptions 1 and 2 together imply not only that μ_i, the expected value of the immediately forthcoming offer, increases in i, but also that $\tilde{\mu}_i^{(n)}$, the expected value of the $n + 1$st forthcoming offer, increases in i. In fact, the sequence $\langle X_{in} \rangle_{n=1}^{\infty}$ of offers received when starting from state i is stochastically increasing (term by term) in i. That is, $X_{i+1,n} > X_{i,n}$, all n. Thus, in this sense higher states are again preferable to lower states. The higher the state the more "buoyant" the economy.

[8] Assumptions 1 and 2 allow the following "bankruptcy" interpretation of the state of the system: Let i be the assets at the job searcher's disposal, and take X to be a nonnegative random variable (X need not be integer-valued nor even to have countable support since the state space may be uncountable) which represents the *total* return per dollar invested. Thus, if the value of the searcher's assets is i and he continues his search, he spends c on search and invests $i - c$, so that the value of his assets after

The importance of Assumptions 1 and 2 stems from the following facts: If f is a nondecreasing function, then

$$\int_0^\infty f(y)\,dF_i(y) \text{ is nondecreasing in } i \tag{1}$$

and

$$\sum_{j=1}^{K} P_{ij} f(j) \text{ is nondecreasing in } i. \tag{2}$$

We shall say we are in state (i, x) if the economy is in state i and the currently available job offer is x. Let $V_n(i, x)$ denote the maximal β-discounted expected return attainable when n periods remain and we are in state (i, x). Then $V_n(i, x)$ satisfies the recursive equation ($V_0 \equiv 0$)[9]

$$\begin{aligned} V_{n+1}(i, x) &= \max\left\{x, -c + \beta \sum_{j=1}^{K} P_{ij} \int_0^\infty V_n(j, y)\,dF_i(y)\right\} \\ &\equiv \max\{x, R_n(i)\}. \end{aligned} \tag{3}$$

Clearly, $R_n(i)$ is the reservation wage $n+1$ periods remain and the economy is in state i.

If we wish to interpret x as the wage rate rather than the discounted present value of all future wages, then x should be replaced by $x \sum_{j=0}^{n} \beta^j$ as the reward associated with stopping; thus, Eq. (3) becomes

$$V_{n+1}(i, x) = \max\left\{x \sum_{j=0}^{n} \beta^j, R_n(i)\right\}. \tag{3a}$$

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search is given by $j = (i - c)X$. Now, if his assets fall below c he is not permitted any further search (for he is not permitted any reinvestment, this would be palatable if we interpret the search cost c as daily living expenses), and, hence, he will accept the last offer tendered. Assuming that the wage distribution sampled from does not depend upon his assets, except to account for the fact that he must permanently withdraw from the labor market if his assets should fall below c, the model obtains with (F_i) and (P_{ij}) given by $F_K = F_{K-1} = \cdots = F_c$; $F_{c-1} = F_{c-2} = \cdots = F_1 = F_0 =$ point pass at zero and

$$\begin{aligned} P_{ij} &= \text{Prob}\,[X = j/(i - c)], && \text{if } i > c, \\ &= 1 \quad , && \text{if } j = 0 \text{ and } i \leqslant c. \end{aligned}$$

Clearly, Assumptions 1 and 2 are satisfied. (See [5] for an alternative analysis of the effects of asset holdings on optimal search behavior.)

[9] Notice that for a single state economy this reduces, in the infinite horizon case, to the familiar relation

$$c = \int_R^\infty (x - R)dF(x), \qquad \text{with } \beta = 1.$$

Consequently, we can interpret $R_n(i)/\sum_{j=0}^{n}\beta^j$ as the reservation wage rate whereas $R_n(i)$ is merely the minimally acceptable discounted present value of all wages when $n+1$ periods remain and the state of the economy is i.

THEOREM 1. *The optimal return $V_n(i, x)$ is nondecreasing in n, i, and x, so that the reservation wages $R_n(i)$ satisfy*

$$R_n(1) \leqslant R_n(2) \leqslant \cdots \leqslant R_n(K), \tag{4}$$

and

$$R_{n+1}(i) \geqslant R_n(i). \tag{5}$$

If $R_n(i)$ satisfies (3a), then

$$R_{n+1}(i)/\alpha_{n+1} \geqslant R_n(i)/\alpha_n\,, \tag{5a}$$

where $\alpha_n \equiv \sum_{j=0}^{n}\beta^j$.

Proof. Note that $V_1(i, x) \equiv x$, and assume $V_n(i, x)$ increases in both i and in x. Then

$$f(j+1) \equiv \int_0^\infty V_n(j+1, y)\,dF_{i+1}(y) \geqslant \int_0^\infty V_n(j, y)\,dF_{i+1}(y),$$

$$\geqslant \int_0^\infty V_n(j, y)\,dF_i(y) \equiv f(j),$$

by hypothesis and (1), respectively. Applying (1) and (2) to the above implies that $R_n(i+1) \geqslant R_n(i)$, so that $V_{n+1}(i, x)$ increases in both i and in x, establishing (4). Equation (5) is immediate from $V_n(i, x)$ nondecreasing in n.

Suppose (3a) holds. Clearly (5) holds. Now $V_2(i, x)/\alpha_1 \geqslant x\alpha_1/\alpha_1 = V_1(i, x)/\alpha_0$. Assume $V_n(i, x)/\alpha_{n-1} \geqslant V_{n-1}(i, x)/\alpha_{n-2}$ for all i and x, then since (5) holds and $-c/\alpha_n \geqslant -c/\alpha_{n-1}$, we obtain $R_n(i)/\alpha_n \geqslant R_{n-1}(i)/\alpha_{n-1}$ for all i. Therefore,

$$\begin{aligned} V_{n+1}(i, x) &= \alpha_n \max\{x, R_n(i)/\alpha_n\} \\ &\geqslant \alpha_n \max\{x, R_{n-1}(i)/\alpha_{n-1}\} \\ &= (\alpha_n/\alpha_{n-1}) \max\{x\alpha_{n-1}, R_{n-1}(i)\} = (\alpha_n/\alpha_{n-1})\, V_n(i, x). \end{aligned}$$

This completes the induction argument. Q.E.D.

Note too that $R_n(i)$ is a nondecreasing function of $\beta (0 \leqslant \beta \leqslant 1)$.

To treat the infinite horizon problem and obtain analogs of (3) and (4), it suffices to verify that $\langle V_n(K, x)\rangle_{n=1}^{\infty}$ is bounded for each x. To see this, merely note that $V_n(K, x) \leqslant V_n(x)$, where $V_0(x) \equiv 0$

$$V_{n+1}(x) \equiv \max\left\{x, -c + \beta\int_0^\infty V_n(y)\,dF_K(y)\right\},$$

and ($\mu_K \equiv \int_0^\infty [1 - F_K(t)]\,dt$)

$$0 \leqslant V_{n+1}(x) - V_n(x) \leqslant \beta \int_0^\infty [V_n(y) - V_{n-1}(y)]\,dF_K(y)$$

$$\leqslant \cdots \leqslant \beta^n \int_0^\infty [V_1(y) - V_0(y)]\,dF_K(y) = \beta^n \mu_K,$$

so that

$$V_n(x) = \sum_{j=1}^{n} [V_j(x) - V_{j-1}(x)] \leqslant \sum_{j=1}^{n} \beta^{j-1}\mu_K \leqslant \mu_K/(1-\beta).$$

3. Labor Market Interpretation of the Simple Model

Since the reservation wage increases as the economy improves, there will be fewer dropouts in a bouyant economy. For suppose the economy is in state i and the tth individual has reservation wage $R^t(i)$. Suppose further that the return from not participating in the labor market exceeds $R^t(i)$. Then the optimal policy for the tth individual is to drop out. When the economy moves to a higher state j, this decision must be reconsidered, and since $R^t(j) \geqslant R^t(i)$, the tth individual may decide to search for employment. With nonwhites disproportionately represented in the pool of dropouts, we expect them to be disproportionately affected by an improving economy. The empirical literature on this subject is agreeable with this expectation (see [14, 23]).

It has been noted that the cost of search declines in a growing economy.[10] Notice that when the cost of search is state-dependent, with c_i denoting the cost of search when the economy is in state i, Theorem 1 remains true provided

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_k.$$

This, of course, strengthens the negative relationship between economic growth and number of dropouts.

One would expect the length of search, the period of frictional unemployment, to decline as the economy improves. In general, this is not true. However, a simple but rather severe restriction on the Markov transition matrix, viz., $P_{ij} = P_j$, does produce the anticipated result.

More specifically, we wish to inquire as to the behavior of p_i, where p_i is the probability that a search from state i results in a job offer that will be

[10] For example, see Parsons [18] who argues that firms reduce search costs "by increasing advertising, arranging convenient interview times, paying moving expenses, etc."

accepted. To begin, define $q_{ij} = 1 - F_i(R(j))$, where $R(j)$ is the reservation wage (in the infinite horizon problem) for state j. Then q_{ij} is the conditional probability that a job offer resulting from a search from state i will be accepted given that the economy has moved to state j at the time the offer arrives. Hence, p_i satisfies

$$p_i = \sum_{j=1}^{K} P_{ij} q_{ij} .$$

If $P_{ij} = P_j$, then $p_i \leqslant p_{i+1}$. This follows since $q_{ij} \leqslant q_{i+1,j}$ as F_{i+1} is stochastically larger than F_i. This result states roughly that frictional unemployment decreases as the economy improves. More precisely, let Y_i be the number of offers tendered (until one is finally accepted) starting from state i, so that the random variable Y_i is precisely the period of frictional unemployment. Defining $p = \sum_j P_j p_j$ $(= \sum_j P_{ij} p_j)$, we have

$$P(Y_i = 1) = p_i , \; P(Y_i = k) = p(1 - p)^{k-2}(1 - p_i), \qquad k = 2, 3,...,$$

which, coupled with $p_i \leqslant p_{i+1}$, implies[11] that Y_{i+1} is stochastically smaller than Y_i. In other words, the period of frictional unemployment decreases stochastically in the state of the economy.

Recall that in the standard search model the job searcher is assumed to be unaware of the increase (decrease) in the offer distribution and consequently frictional unemployment declines (rises) as the economy improves. Here we make no such assumption about being fooled; the reservation wage increases (decreases) in a precise manner, and yet the behavior of frictional unemployment is, in general, ambiguous. If the standard model is to be consistent, the discouraged workers also will be unaware of changes in the wage distribution, and labor force participation will not vary as the economy changes.

Throughout this analysis we assume that the job searcher knows the wage distributions, the state of the economy, and the Markov chain. We do not address the important adaptive problems that arise when this information is imperfect. It is clear, however, that when the economy is in state j and the searcher believes it to be in i with $i < j$, the reservation wage will be set too low. This will lead to a reduction in frictional unemployment. The converse is true when $j < i$. Underestimation (overestimation) of the number of periods till retirement also reduces (increases) the reservation wage and, hence, frictional unemployment.

[11] Let $\alpha = \sum_{j=2}^{k} p(1 - p)^{j-2} < 1$, so for $k > 1$,

$$\begin{aligned} P(Y_{i+1} \leqslant k) - P(Y_i \leqslant k) &= p_{i+1} + (1 - p_{i+1})\alpha - [p_i + (1 - p_i)\alpha] \\ &= (1 - \alpha)(p_{i+1} - p_i) > 0. \end{aligned}$$

In a recent paper, Salop [22, p. 191] states that "individuals are able to distinguish among firms *ex ante* and they sample *specific firms* in a systematic fashion rather than just sampling the job market in general." There, the reservation wage declines in the face of optimal systematic search. In addition, as the economy moves to inferior states (higher unemployment rates) the probability of a job offer declines and the reservation wage declines. The following version of our model yields similar results.

Assume that in addition to satisfying Assumption 2, $\{P_{ij}\}$ satisfies

$$P_{ij} = \lambda, \qquad \text{all } i. \tag{2*}$$

Equation (2*) implies that the economy spends the same expected time[12] in each state before changing to a new state.

Let F_{ij} be the distribution function for the jth offer received since the last re-entry into state i, $j = 1, 2, \ldots$. In place of Assumption 1, we assume

$$F_{i,j}(t) \geqslant F_{i+1,j}(t), \qquad \text{all } t, \text{ all } j. \tag{1*}$$

Thus, the jth offer received (since the last re-entry) in state i is stochastically smaller than the jth offer received in state $i + 1$, so that the sequence of offers tendered is most preferable (in a probabilistic sense) in the highest numbered states.

Furthermore, since the individual can distinguish among the various firms, we anticipate the result that his systematic search commences with the firm with the "best" offer distribution and proceeds at each step by selecting the firm with the best offer distribution from among those not yet searched. Labeling firms in ascending order according to this anticipated optimal systematic search, the offer distribution $F_{i,j}$ of the jth firm searched from state i is better than $F_{i,j+1}$. We will assume that $F_{i,j}$ is stochastically larger than $F_{i,j+1}$. For simplicity, further assume that

$$F_{ij} = (1 - q_i)\,\delta_0 + q_i F_j, \tag{1*a}$$

where q_i is the probability of receiving a (nonzero) offer resulting from a single search when in state i and δ_0 is the point mass at zero,

$$q_1 \leqslant q_2 \leqslant \cdots \leqslant q_K, \tag{1*b}$$

and

$$F_j(t) \leqslant F_{j+1}(t), \qquad \text{all } t, \text{ all } j. \tag{1*c}$$

Equations (1*a–c) comprise a specific simple parametrization of the stochastically ordered (increasing in the state i and decreasing in the firm

[12] In particular, (2*) implies that the number of job offers received before a change of state is a geometric random variable with parameter λ.

number j) nature within and between the sequences of job offers. (Of course, (1*a–c) imply (1*).) Then, it follows that

(a) the optimal search order is $\langle 1, 2, \ldots, J \rangle$ for each i;

(b) the reservation wage $R_{i,j}$, from state i after j searches, is nondecreasing in i and nonincreasing in j;

(c) the expected duration of frictional unemployment decreases with i provided that $P_{ij} = P_j$ for $j \neq i$.

4. The Discrete Time Model with Variable Search Intensity

Typically the intensity of search is a variable that can be controlled by the job searcher. In general, it will depend on the searcher's preference for leisure, his wealth, and unemployment compensation. We do not incorporate these explicitly into our model, but instead place rather mild restrictions on the search cost function. The model was constructed as follows. Associated with search intensity μ, $0 \leqslant \mu \leqslant \bar{\mu}$, is (i) the search cost $c(\mu)$ and (ii) the probability of receiving a job offer $p(\mu)$, both of which are independent of the state of the economy; furthermore, given that an offer is received, it is independent of the search intensity. Naturally, both $c(\mu)$ and $p(\mu)$ are assumed to be nondecreasing. In addition, we assume that $c(0) = 0 = p(0)$ and $c(\cdot)$ is right continuous. For ease in presentation, we give $p(\cdot)$ the explicit representation $p(\mu) = \mu/(\mu + \lambda)$, where $\lambda > 0$. Since the searcher can have no impact on the economy, we assume that the Markov chain is independent of the search intensity chosen by the searcher.

It is now clear upon reflection that the analog of (3) is given by ($V_0 \equiv 0$)

$$V_{n+1}(i, x) = \max\left\{x, \max_{0 \leqslant \mu \leqslant \bar{\mu}}\left[-c(\mu) + (\beta/(\mu + \lambda))\left(\lambda \sum_{j=1}^{K} P_{ij} V_n(j, 0) + \mu \sum_{j=1}^{K} P_{ij} \int_0^{\infty} V_n(j, y)\, dF_i(y)\right)\right]\right\}$$

$$\equiv \max\{x, R_n(i)\}. \tag{6}$$

We are interested in the behavior of not only the reservation wage $R_n(i)$ but also $\mu_n(i)$, where $\mu_n(i)$ maximizes $R_n(i)$, that is, where $\mu_n(i)$ is the optimal search intensity when $n + 1$ periods remain and the state of the economy is i.

THEOREM 2. *With variable search intensity, $V_n(i, x)$, as given in* (6), *is nondecreasing in n, i, and x, and the reservation wages satisfy* (4) *and* (5).

Proof. To begin, observe that

$$[R_n(i+1) - R_n(i)](\mu + \lambda)/\beta$$

$$\geqslant [-c(\mu_n(i)) + c(\mu_n(i))] + \lambda \sum_{j=1}^{K} (P_{i+1,j} - P_{i,j})\, V_n(j, 0)$$

$$+ \mu_n(i) \left[\sum_{j=1}^{K} P_{i+1,j} \int_0^\infty V_n(j, y)\, dF_{i+1}(y) - \sum_{j=1}^{K} P_{i,j} \int_0^\infty V_n(j, y)\, dF_i(y) \right]$$

$$\geqslant 0.$$

The second term on the R.H.S. is greater than or equal to zero by (2) and the induction hypothesis that $V_n(i, x)$ is nondecreasing in both i and x while the third term exceeds zero by (1), (2) and the induction hypothesis. The first inequality follows from $\mu_n(i)$ being suboptimal from state $i + 1$. Q.E.D.

In light of the fact that the reservation wage $R_n(i)$ increases with the state of the economy, it would come as no surprise if the optimal search intensity μ_n(i) increased with i. In general, this result is not true. In fact, all possible orderings of $\{\mu_n(i)\}$, with the proviso that $\mu_n(K) \geqslant \mu_n(1)$, are possible. Moreover, it is not even necessary that $\mu_n(i)$ be monotone in n. (As will be clear, the added assumption of convexity of $c(\cdot)$ is not relevant in eliminating this seeming pathology.)

EXAMPLE 1. If $P_{ij} = P_j$, for all i, then

$$\mu_n(1) \leqslant \mu_n(2) \leqslant \cdots \leqslant \mu_n(K), \tag{7}$$

and

$$\mu_1(i) \geqslant \mu_2(i) \geqslant \cdots \geqslant \mu_n(i) \geqslant \mu_{n+1}(i) \geqslant \cdots \geqslant \mu(i), \tag{8}$$

where $\mu(i)$ is the optimal search intensity for the infinite horizon problem.

Proof. We can reformulate (6) as follows ($V_0 \equiv 0$ and $V_n(i)$ is to be interpreted as the maximal β-discounted expected return attainable when n periods remain, the state of the economy is i, and no offer is currently available):

$$V_{n+1}(i) = \max_{0 \leqslant \mu \leqslant \bar{\mu}} \left\{ -c(\mu) + (\lambda/(\mu + \lambda))\beta \sum_{j=1}^{K} P_{ij} V_n(j) \right.$$

$$\left. + (\mu/(\mu + \lambda)) \int_0^\infty \max \left[\beta y, \beta \sum P_{ij} V_n(j) \right] dF_i(y) \right\}. \tag{9}$$

In this case, if $P_{ij} = P_j$ for all i, we have, setting $V_n = \beta \sum_{j=1}^{K} P_{ij} V_n(j)$,

$$V_{n+1}(i) = \max_{0 \leqslant \mu \leqslant \beta} \left\{ -c(\mu) + (\lambda/(\mu + \lambda))\, V_n + (\mu/(\mu + \lambda)) \right.$$
$$\left. \times \left[V_n + \int_{V_n/\beta}^{\infty} (\beta y - V_n)\, dF_i(y) \right] \right\}$$
$$= \max_{0 \leqslant \mu \leqslant \beta} \left\{ -c(\mu) + (\mu/(\mu + \lambda)) \int_{V_n/\beta}^{\infty} (\beta y - V_n)\, dF_i(y) \right\} + V_n .$$

It now follows from Eq. (1), with $f(y) = 0$ for $y \leqslant V_n/\beta$ and $f(y) = \beta y - V_n$ for $y > V_n/\beta$, that $\mu_n(i+1) \geqslant \mu_n(i)$, for $i = 1, 2, \ldots, K-1$ and all $n \geqslant 1$.

Finally, note that since $\int_x^{\infty} (y - x)\, dF(y)$ decreases as x increases, V_n nondecreasing in n would imply that

$$c_n \equiv \int_{V_n/\beta}^{\infty} (\beta y - V_n)\, dF_i(y) \geqslant \int_{V_{n+1}/\beta}^{\infty} (\beta y - V_{n+1})\, dF_i(y) = c_{n+1} .$$

Since $c(0) = 0$, V_n is, in fact, nondecreasing in n. Consequently, we can write

$$V_{n+1}(i) = \max_{0 \leqslant \mu \leqslant \beta} \{ -c(\mu) + (\mu/(\mu + \lambda))\, c_{n+1}$$
$$+ (\mu/(\mu + \lambda))(c_n - c_{n+1}) \} + V_n .$$

Thus, the maximand of $V_{n+1}(i)$ is the sum of the maximand of $V_{n+2}(i)$ and and an increasing function of μ. Hence, we can conclude that (8) holds.

Q.E.D.

EXAMPLE 2. $\mu(K) \geqslant \mu(1)$.

Proof. Let $V(i) = \lim_n V_n(i)$, where $V_n(i)$ satisfies (9), so that $V(i)$ is the optimal return for an infinite horizon when the economy is in state i. Then

$$V(i) = \max_{0 \leqslant \mu \leqslant \beta} \left\{ -c(\mu) + (\lambda/(\mu + \lambda))\, \beta\gamma(i) \right.$$
$$\left. + (\mu/(\mu + \lambda))\beta \int_0^{\infty} \max[y, \gamma(i)]\, dF_i(y) \right\}$$
$$= \max_{0 \leqslant \mu \leqslant \beta} \left\{ -c(\mu) + (\mu/(\mu + \lambda))\beta \right.$$
$$\left. \times \left[\int_0^{\infty} \max[y, \gamma(i)]\, dF_i(y) - \gamma(i) \right] \right\} + \beta\gamma(i)$$
$$\equiv \max_{0 \leqslant \mu \leqslant \beta} \{ -c(\mu) + (\mu/(\mu + \lambda))\, \delta(i) \} + \beta\gamma(i),$$

where $\gamma(i) \equiv \sum_{j=1}^{K} P_{ij}V(j)$. Since $V(1) \leqslant V(2) \leqslant \cdots \leqslant V(K)$, it follows that $V(K) \geqslant \gamma(K)$ and $V(1) \leqslant \gamma(1)$. Consequently, we have

$$0 \leqslant V(K) - V(1) \leqslant \beta\gamma(K) - \beta\gamma(1) + (\mu(K)/(\mu(K) + \lambda))[\delta(K) - \delta(1)]$$
$$\leqslant \beta V(K) - \beta V(1) + (\mu(K)/(\mu(K) + \lambda))[\delta(K) - \delta(1)],$$

or, equivalently,

$$\mu(K)[\delta(K) - \delta(1)] \geqslant (1 - \beta)[\mu(K) + \lambda][V(K) - V(1)].$$

If $\mu(K) = 0$, then $V(K) = V(1)$ and, hence, $\delta(K) = \delta(1)$ so that $\mu(K) = \mu(1)$. On the other hand, if $\mu(K) > 0$, then $\delta(K) - \delta(1) \geqslant 0$, in which case we also have $\mu(K) \geqslant \mu(1)$. If $\beta = 1$, then $\mu(K) = 0$ implies $V(K) - V(1) \leqslant \gamma(K) - \gamma(1)$, from which we can conclude that $V(K) = V(1)$. The case $\mu(K) > 0$ with $\beta = 1$ is exactly the same as when $\beta < 1$. (Note that $V_n(i + 1) - V_n(i)$ nondecreasing in n would suffice here to enable us to assert that $\mu_n(K) \geqslant \mu_n(1)$, all $n \geqslant 1$.) Q.E.D.

In presenting the next two examples, it is convenient to introduce and utilize the continuous-time version of the model which is of some interest in its own right. There is essentially no generalization in allowing the Markov chain which governs the economy to have continuous-time parameter. With the introduction of a variable search intensity, however, the continuous-time formulation permits the searcher, in a very natural and convenient way, to receive a number of offers before there is a change in the state of the economy.

In the continuous-time formulation, we maintain the notion of periods. Periods are defined so that one period begins the moment another period ends, and a period ends when one of the following three events occurs: An offer arrives, the state of the economy changes, a fictitious offer of 0 arrives. At the beginning of each period, the decision maker observes the state (i, x); either he accepts the offer x or he selects a search intensity μ, $0 \leqslant \mu \leqslant \bar{\mu}$, which he maintains during the course of that period.[13]

Associated with search intensity μ is the search cost rate (per unit time) $c(\mu)$ and the exponential time, with parameter μ, till a job offffer arrives. In this formulation, we let the time till the economy changes state and the time till an offer arrives be independent exponential random variables with parameters $\lambda_i \equiv \lambda$ and μ, respectively. For convenience in analysis (see [10] for a complete exposition of this device), it is desirable that the length of each period be a random variable independent of both the state

[13] Since the length of the period turns out to be an exponential random variable, there is no loss of optimality in maintaining a single search intensity as opposed to allowing it to be a function of the time since the period began.

of the system and the decision variables. This is achieved by letting the time till a fictitious offer of 0 arrives be an exponential random variable with parameter $\bar{\mu} - \mu$ that is independent of the two previously defined random variables. Consequently, the length of each period is an exponential random variable with parameter $\Lambda = \lambda + \mu + (\bar{\mu} - \mu) = \lambda + \bar{\mu}$. Thus, Λ is the transition rate of the (two-dimensional—state of economy and offer) Markov chain.

Let (i, x) be the state of the system (Markov chain) at the beginning of the period. If the period ends with (i) an offer, say of y, (ii) a change in the state of the economy, say to j, or (iii) a fictitious offer of 0, then the state at the beginning of the next period will be (i, y), $(j, 0)$, or $(i, 0)$, respectively. As before, $F(\cdot)$ is the offer distribution, and P_{ij} is the probability that the economy will change to j from i given that it changes.

Defining $\alpha > 0$ to be the continuous-time discount factor, the expected discounted cost of search during a period when searching at intensity μ is easily seen to be

$$c(\mu)/(\alpha + \Lambda) = c(\mu) \int_0^\infty \left[\int_0^t e^{-\alpha\tau}\, d\tau\right] \Lambda e^{-\Lambda t}\, dt.$$

Similarly, the expected discounted length of a period is given by

$$\Lambda/(\alpha + \Lambda) = \int_0^\infty e^{-\alpha t} \Lambda e^{-\Lambda t}\, dt.$$

Finally, the probabilities that the period ends with an offer, a change in the state of the economy, or a fictitious offer are μ/Λ, λ/Λ, and $(\bar{\mu} - \mu)/\Lambda$, respectively.

Coupling all of these facts, we obtain the continuous time analog of (6) ($V_0 \equiv 0$)

$$V_{n+1}(i, x) = \max\left\{x, (1/(\alpha + \Lambda)) \max_{0 \leqslant \mu \leqslant \bar{\mu}} \left[-c(\mu) + \lambda \sum_{j=1}^{K} P_{ij} V_n(j, 0) + (\Lambda - \lambda - \mu)\, V_n(i, 0) + \mu \int_0^\infty V_n(i, y)\, dF_i(y)\right]\right\}. \quad (10)$$

Before presenting the examples, we note that all of the results in this paper hold for the continuous-time model with the sole exception of Example 1 which requires $n = +\infty$. The proofs are quite similar.

EXAMPLE 3. The maximal search intensity need not be from state K.

Suppose there are three states with F_1 having mass 1 at 0 and F_2 and F_3 having mass 1 at 1. Furthermore, suppose that $c(\mu) = \mu^2$, states 1 and 3

are absorbing, and $P_{21} = 1$. Then for the infinite horizon problem, the return from state 3 when we search at rate μ is

$$(1/\alpha) \cdot (\mu/(\alpha + \mu)) - (1/(\alpha + \mu))\, c(\mu).$$

This yields $\mu(3) = -\alpha + (\alpha^2 + 1)^{1/2}$. Similarly, if search rate μ is employed from state 2, the return

$$(\mu/(\mu + \lambda)) \cdot (1/\alpha) \cdot ((\lambda + \mu)/(\alpha + \lambda + \mu)) - (c(\mu)/(\alpha + \lambda + \mu))$$

is obtained. (The first factor is the probability of eventually getting an offer, the third is the expected discounted time of either receiving an offer or going to state 1, and $1/\alpha$ is the discounted present value of the wage rate 1.) It then follows that

$$\begin{aligned}\mu(2) &= -(\alpha + \lambda) + (\alpha + \lambda)(\alpha^2 + 1)^{1/2}/\alpha \\ &= -\alpha + (\alpha^2 + 1)^{1/2} + \lambda((\alpha^2 + 1)^{1/2}/\alpha - 1) > \mu(3).\end{aligned}$$

Q.E.D.

EXAMPLE 4. The minimal search intensity need not be from state 1.

Suppose that there are three states with F_1 and F_2 having mass 1 at 1 and F_3 having mass 1 at 100. Furthermore, suppose that $c(\mu) = \mu^2$, states 1 and 3 are absorbing, and $P_{21} = P_{23} = \frac{1}{2}$. Then the return from states 1 and 3, respectively, when search intensity μ is employed is

$$V(1) = \frac{1}{\alpha} \cdot \frac{\mu}{\alpha + \mu} - \frac{c(\mu)}{\alpha + \mu} \quad \text{and} \quad V(3) = \frac{100}{\alpha} \cdot \frac{\mu}{\alpha + \mu} - \frac{c(\mu)}{\alpha + \mu},$$

from whence it follows that $\mu(1) = -\alpha + (\alpha^2 + 1)^{1/2}$ and $\mu(3) = -\alpha + (\alpha^2 + 100)^{1/2}$. Similarly, the return from state 2 when rate $\mu(1)$ is used from state 1, $\mu(3)$ from state 3, and μ from state 2 (and, of course, any offer received while in state 2 is accepted) is

$$\begin{aligned}&\frac{\mu}{\lambda + \mu} \cdot \frac{1}{\alpha} \cdot \frac{\lambda + \mu}{\alpha + \lambda + \mu} \\ &\quad + \frac{\lambda/2}{\lambda + \mu}[V(3) + V(1)] \frac{\lambda + \mu}{\alpha + \lambda + \mu} - \frac{c(\mu)}{\alpha + \lambda + \mu},\end{aligned}$$

so that $\mu(2) = -(\alpha + \lambda) + ((\alpha + \lambda)^2 - \eta)^{1/2}$, where $\eta = (\lambda/2)(V(1) + V(3)) - (\alpha + \lambda)/\alpha$. Finally, it can be seen from these formulas that $\mu(1) > \mu(2)$. Q.E.D.

5. The Discrete Time Model with Resignations

We now extend our model, as embodied in Eq. (3), to include the opportunity of quitting in order to seek a better job. We assume that the wages of employed workers do not change over the business cycle. (See the discussion of this point in Section 1.)

At this point, we must interpret the job offer x as the currently available wage *rate*, so that $x/(1-\beta)$ is the β-discounted present value of working forever (i.e., never resigning) at wage rate x. To begin, define $W_n(i, x)$ to be the maximal β-discounted expected return attainable when n periods remain, the state of the economy is i, and x is the currently available wage rate. Then $W_n(i, x)$ satisfies the recursive equations[14] ($W_0 \equiv 0$)

$$W_{n+1}(i, x) = \max\left\{x + \beta \sum_{j=1}^{K} P_{ij} W_n(j, x),\right.$$
$$\left. -c + \beta \sum_{j=1}^{K} P_{ij} \int_0^\infty W_n(j, y)\, dF_i(y)\right\}. \tag{11}$$

Implicit in (11) is the possibility of not searching via accepting the current wage rate x even if $x = 0$. Of course, the wage rate arising from any past job offer that was either not accepted or accepted but subsequently rejected is no longer available.

As before, we can effect an induction argument based on Assumptions 1 and 2 to verify that $W_n(i, x)$ is nondecreasing in i, x, and n, and, hence, $f_n(i, x)$ and $r_n(i)$ are nondecreasing in i, x, and n, where

$$W_{n+1}(i, x) = \max\{x + f_n(i, x); r_n(i)\}, \tag{12}$$

so that for each i, there is a reservation wage and a resignation wage, and the two are, of course, equal. To verify that the reservation wage is nondecreasing in i and in n, it would suffice to verify that $r_n(i) - f_n(i, x)$ is nondecreasing in i and in n, respectively.

The following example reveals, contrary to intuition, it is not necessarily true that introducing the possibility of resignation lowers the reservation wage. (We conjecture that there is a $J \leqslant K$ such that $W(i) \geqslant V(i)$ if $i \geqslant J$, where $W(i)$ and $V(i)$ are the reservation wages from state i with and without the possibility of resignation, respectively.)

[14] The recursive equation for the continuous time model is ($W_0 \equiv 0$)

$$W_{n+1}(i, x) = (1/(\alpha + \lambda + \bar{\mu})) \max\,(x + \bar{\mu} W_n(i, x) + \lambda \textstyle\sum_{j=1}^{K} P_{ij} W_n(j, x);$$
$$-c + \lambda \textstyle\sum_{j=1}^{K} P_{ij} W_n(j, 0) + \bar{\mu} \int_0^\infty W_n(i, y) dF_i(y)).$$

EXAMPLE 5. The reservation wage can increase with the possibility of resignations.

Take $\alpha = \lambda = \mu = c = 1$ and suppose that F_1 has mass 1 at w_1 and F_2 has mass $\frac{1}{2}$ at w_2 and w_3. We choose $w_1 > 2$ so that it is optimal to search while in state 1. We will choose w_i so that $W(2)$, the reservation wage in state 2, exceeds w_2. To begin, note that using this policy we have[15]

$$W(1) = -\frac{c}{\alpha+\lambda+\mu} + \frac{1}{\alpha+\lambda+\mu} \times \left\{\mu\left[\frac{w_1}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda} W(2)\right] + \lambda W(2)\right\}$$

$$= -\frac{c}{\alpha+\lambda+\mu} + \frac{\mu}{\alpha+\lambda+\mu}\frac{w_1}{\alpha+\lambda} + \frac{\lambda}{\alpha+\lambda} W(2),$$

and

$$W(2) = -\frac{c}{\alpha+\lambda+\mu} + \frac{1}{\alpha+\lambda+\mu}\left\{\mu\left[\frac{W(2)}{2} + \frac{w_3}{2\alpha}\right] + \lambda W(1)\right\},$$

so that

$$W(2) = -(2/3) + (w_3/4) + (w_1/12).$$

Setting $w_1 = 8$, $w_2 < 12$, and $w_3 = 48$, we obtain $W(2) = 12 > w_2$, so the reservation wage from state 2 is, indeed, w_3.

When resignation is not permitted, we need consider two policies: accept w_1 and w_3 or accept only w_3.

If we accept w_1 and w_3, we are lead to

$$V(1) = -\frac{c}{\alpha+\lambda+\mu} + \frac{w_1}{\alpha}\frac{\mu}{\alpha+\lambda+\mu} + V(2)\frac{\lambda}{\alpha+\lambda+\mu},$$

and

$$V(2) = -\frac{c}{\alpha+\lambda+\mu} + \frac{1}{\alpha+\lambda+\mu}\left\{\mu\left[\frac{1}{2}V(2) + \frac{1}{2}\frac{w_3}{\alpha}\right] + \lambda V(1)\right\},$$

so that $V(2) = 152/13 < 12$.

If we accept only w_3, then the equation for $V(1)$ becomes $V(1) = -c/(\alpha+\lambda) + \lambda V(2)/(\alpha+\lambda)$ if search is required and $V(1) = \lambda V(2)/(\alpha+\lambda)$ if search is not required. This leads to $V(2) = 45/4$ and $V(2) = 46/4$, respectively.

Consequently, setting $w_2 > 152/13$ (but also $w_2 < 12$), we see that w_2 exceeds $V(2)$, the reservation wage in state 2. Q.E.D.

[15] Here, $W(i)$ is the infinite horizon return starting from state i when an offer is not yet in hand. Similarly for $V(i)$.

At first blush, it would seem evident that the reservation wage is monotone increasing even when the resignation opportunity is included. The following example, however, illustrates the refractory nature of this particular embellishment of the basic model.

EXAMPLE 6.[16] The reservation wages need not be nondecreasing in the state.

Suppose there are three states with $F_1 = F_2 = \frac{1}{2}(\delta_1 + \delta_4)$ and $F_3 = \delta_{100}$, where δ_x is the point mass at x. Let states 1 and 3 absorbing and $P_{22} = p = 1 - P_{23}$ with $p = 1/10$ and take $c < 5/28$ and the discrete time discount factor $\beta = \frac{9}{10}$. We consider the discrete time infinite horizon problem. The sequencing of events within a period is as follows: search, job offer, transition according to $\{P_{ij}\}$, and then collect the wage at the beginning of the next period.

From state 1 there are but two policies to follow: accept both 1 and 4 or accept only 4. The return for these policies satisfy, respectively, the equations

$$W = -c + \beta(\tfrac{1}{2}1 + \tfrac{1}{2}4)/(1 - \beta) = 5 - c$$

and

$$W = -c + \beta\left(\frac{1}{2}\frac{4}{1-\beta} + \frac{1}{2}W\right), \quad \text{i.e.,} \quad W = 5 - c + \frac{1}{2}(2 - c).$$

Thus, the reservation wage from state 1 is 4.

To find the reservation wages associated with the above policies for state 2, we begin by letting X (roughly the discounted value of wages earned while residing in state 2 if an offer of 1 is accepted) satisfy

$$P(X = 1 + \beta + \cdots + \beta^n) = (1 - p)\, p^n.$$

Then $E(X) = 1 + p\beta E(X)$ so that $E(X) = 1/(1 - p\beta)$. The discounted value of the net benefits while in state 2 under the two policies satisfy, respectively,

$$W = -c + \beta\{p(5/2)\, E(X) + (1 - p)\, 0\},$$

and

$$\begin{aligned} W &= -c + \beta\{p[\tfrac{1}{2} \cdot 4E(X) + \tfrac{1}{2}W] + (1 - p)\, 0\} \\ &= -c + 2E(X)\, \beta p + \beta p W/2, \end{aligned}$$

i.e.,

$$W = (2/(2 - \beta p))\{2\beta p E(X) - c\}.$$

As $E(X) = 15/14$, this yields values of $(5/28) - c$ and $(30/29)((4/28) - c)$, so the reservation wage from state 2 is 1. Q.E.D.

[16] Roughly, the counterexample shows that if $F_1 = F_2 = \cdots = F_{K-1} \ll F_K$ and $P_{ij} = 0$ for $j < i$, then we can have the reservation wages decrease ($R(1) > R(2) > \cdots > R(K-1)$) until state K is reached (with $R(K) \gg R(1)$).

Once again, the special case $P_{ij} = P_j$ is tractable. Here, we note that

$$f_n(i, x) = \beta \sum_{j=1}^{K} P_{ij} W_n(j, x) = \beta \sum_{j=1}^{K} P_j W_n(j, x)$$

is independent of i, so that $r_n(i) - f_n(i, x)$ is, indeed, nondecreasing in i. Thus, the conclusion of nondecreasing reservation wages does hold for this special case.

Retaining the assumption that $P_{ij} = P_j$, we can show that the quit rate varies directly with the state of the economy. Consequently, as the state of the economy increases three things happen: Quits rise, the period of frictional unemployment (stochastically) decreases, and labor force participation increases. The net effect of these forces on the unemployment rate is unclear. Whatever the relationship is, it is not the Phillips curve in the usual sense. Here we are postulating real improvements in the economy; that is, we are concerned with the relationship between the change in the real rate of growth and unemployment. The state of the economy is known to everyone, thereby obviating any expectations model.

It has been observed [3] that the probability of resignation decreases the longer a worker has been employed by a given firm (or at least a given industry). The explanation posited rests upon the notion that while employed a worker accumulates specific human capital. The worker's wage reflects this investment; that is, it includes a return on the worker's investment in specific human capital. The important point is that this investment is not transferable. Thus the worker incurs a capital loss when he quits his current job.[17]

To incorporate the wage paid on human capital, let $\epsilon(k) \geq 0$ be the incremental wage paid on human capital to a worker who has been on the job for k periods. Define $W_n(i, x, k)$ to be the maximal β-discounted expected return attainable when n periods remain, i is the state of the economy, x is the current available wage rate, and k is the number of periods the searcher has been employed at his current job ($k = 0$ means that he is currently unemployed). Clearly, $W_n(i, x, k)$ satisfies ($W_0 = 0$)

$$\begin{aligned} W_{n+1}(i, x, k) &= \max \Bigg\{ x + \beta \sum_{j=1}^{K} P_{ij} W_n(j, x + \epsilon(k), k + 1); \\ &\qquad - c + \beta \sum_{j=1}^{K} P_{ij} \int W_n(j, y, 0)\, dF_i(y) \Bigg\} \\ &\equiv \max\{x + w_n(i, x, k); r_n(i)\}. \end{aligned} \tag{13}$$

[17] We assume here that the worker was the sole investor in this firm specific capital. In practice, firms and workers tend to share in this investment. Consequently, firms are less inclined to fire. See [18].

Defining $R_{n+1}(i, k)$ to be the unique value of x for which $x = r_n(i) - w_n(i, x, k)$, we can conclude that $R_n(i, k)$ is the *resignation wage* when n periods remain, i is the state of the economy, and k is the number of periods the searcher has been employed. (Thus, $R_n(i, 0)$ is the reservation wage.) To see that $R_{n+1}(i, k)$ exists and is uniquely defined, it suffices (see Fig. 1) to show that $w_n(i, x, k)$ is a nondecreasing function of x. Once again, a straightforward induction argument utilizing Assumptions 1 and 2 verifies that $W_n(i, n, k)$, and, hence, $w_n(i, x, k)$, are nondecreasing in i and x. Moreover, if $\epsilon(k)$ is a nondecreasing [nonincreasing] function of k, then so is $W_n(j, x, k)$. Consequently, the resignation wage $R_n(i, k)$ is a nonincreasing [nondecreasing] function of k if $\epsilon(k)$ is a nondecreasing [nonincreasing] function of k. (Note too that if $P_{ij} = P_j$, $R_n(i, k)$ is nondecreasing in i since $w_n(i, x, k)$ is independent of i.)

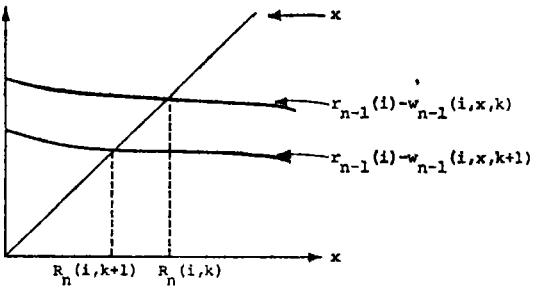


FIGURE 1

Presumably, $\epsilon(k)$ increases with k, or equivalently, specific human capital increases, so our model predicts the anticipated result that the resignation wage decreases with time on the job. That is, the longer the time on the job, the more reluctant the worker is to resign (equivelently, the smaller the wage needed to induce him to remain on the job).

6. THE DISCRETE TIME MODEL WITH FIRING

We now admit the possibility of being fired. This is incorporated by assuming that the probability of being fired is f_i, where i is the state of the economy. Note that f_i does not depend on the current wage. Letting $(i, 0)$ and $(i, 0^-)$ be the state of the system when the economy is in state i and

the worker is not employed or employed at wage rate 0, respectively, we obtain the recursive equations[18] ($U_0 \equiv 0$)

$$U_{n+1}(i, x) = \max \Big\{ x g_{n+1}(i) + \beta \sum_{j=1}^{K} P_{ij}[f_i U_n(j, 0) + (1 - f_i)\, U_n(j)]; \\ - c + \beta \sum_{j=1}^{K} P_{ij} \int_0^{\infty} U_n(j, y)\, dF_i(y) \Big\}, \tag{14}$$

where

$$U_{n+1}(i) = \beta \sum_{j=1}^{K} P_{ij}[f_i U_n(j, 0) + (1 - f_i)\, U_n(j)], \tag{15}$$

and

$$U_{n+1}(i, 0^-) = \beta \sum_{j=1}^{K} P_{ij}[f_i U_n(j, 0) + (1 - f_i)\, U_n(j, 0^-)], \tag{16}$$

and $g_{n+1}(i)$ is the expected discounted time employed when $n + 1$ periods remain and the initial state is i. As given in (14), (15), and (16), our formulation permits the searcher to work at the wage 0 in order to avoid the search cost; however, search cannot be resumed until he is fired (from this job with wage rate 0). Clearly, there is a reservation wage for each state of the economy.

7. The Discrete Time Model with Resignations and Firing

We now combine the two possibilities of quits and layoffs. As before, let f_i be the firing probability associated with state i and let $Z_n(i, x)$ denote the maximal β-discounted expected return attainable when n periods remain,

[18] The recursive equation for the continuous time model is ($U_0 \equiv 0$)

$$U_{n+1}(i, x) = (1/(\alpha + \Lambda)) \max \big(x \cdot g_{n+1}(i) + \lambda \textstyle\sum P_{ij} U_n(j) \\ + f_i U_n(i, 0) + (\Lambda + \lambda - f_i) U_n(i); \\ - c + \lambda \textstyle\sum P_{ij} U_n(j, 0) \\ + (\Lambda - \lambda - \mu) U_n(i, 0) + \mu \textstyle\int U_n(i, y) dF_i(y) \big),$$

where

$$U_{n+1}(i) = (1/(\alpha + \Lambda))\, (\lambda \textstyle\sum P_{ij} U_n(j) + f_i U_n(i, 0) + (\Lambda - \lambda - f_i) U_n(i)),$$

$\Lambda = \lambda + \mu + \max(f_i)$, and f_i, λ, μ are the parameters of the exponential time till being fired, change of state, and arrival of an offer, respectively, and $g_n(i)/(\alpha + \Lambda)$ is the expected discounted time employed when n periods remain and the initial state is i.

the state of the economy is i, and x is the current available wage rate. Then $Z_n(i, x)$ satisfies[19] ($Z_0 \equiv 0$)

$$Z_{n+1}(i, x) = \max \left\{ x + \beta \sum_{j=1}^{K} P_{ij}[f_i Z_n(j, 0) + (1 - f_i) Z_n(j, x)]; \right.$$
$$\left. - c + \beta \sum_{j=1}^{K} P_{ij} \int_0^\infty Z_n(j, y)\, dF_i(y) \right\}$$
$$\equiv \max\{z_n(i, x); \rho_n(i)\}. \tag{17}$$

As before, the existence of a reservation wage (that also equals the resignation wage) can easily be established inductively by showing that $Z_n(j, x)$ is nondecreasing in x.

8. Conclusion

This paper has constructed a model of job search in a changing economy. As expected, the reservation wage increased as the economy moved to improved states. However, only under fairly stringent assumptions on the stochastic process were we able to say anything definite about the duration of search unemployment in relation to the changing economy. Similarly, it was shown that the optimal search intensity need not increase with the state of the economy unless the same restrictions were imposed. Quits and layoffs were also considered in this dynamic setting, and the existence of reservation and resignation wages was established; however, the further assumption that $P_{ij} = P_j$ was required to ensure that the reservation wage is monotone increasing in the state of the economy.

There are several directions in which these models can be modified. First, the treatment of specific human capital can be extended to the model with both resignations and layoffs with the relative contributions of employer and employee to the investment being explicitly considered. This should yield testable hypotheses with respect to quit and layoff behavior over the business cycle. Second, we have emphasized the difficulties associated with our models of quits and layoffs. Presumably, firms will be adjusting the wages of employees as the economy changes. We did not attempt to model this adjustment process, but merely assumed that

[19] The recursive equation for the continuous time model is ($Z_0 \equiv 0$, $\Lambda = \lambda + \mu + \max(f_i)$)

$$Z_{n+1}(i, x) = (1/(\alpha + \Lambda)) \max(x + \lambda \sum P_{ij} Z_n(j, x) + f_i Z_n(i, 0) + (\Lambda - \lambda - f_i) Z_n(i, x);$$
$$-c + \lambda \sum P_{ij} Z_n(j, 0) + (\Lambda - \lambda - \mu) Z_n(i, 0) + \mu \int Z_n(i, y) dF_i(y)).$$

wages were constant. Clearly, additional work is required here. Third, the interaction of optimal employer and employee policies gives rise to equilibrium wage distributions. (See [4, 12, 16, 27] for analyses of equilibrium wage distributions in a static setting.) Analysis of an endogenous wage distribution in a dynamic setting strikes us as an extremely important topic for additional research. It is precisely models like these which will provide a rigorous foundation for the study of macroeconomic phenomena. Finally, we have assumed throughout that job searchers know the stochastic structure of the process. In practice, they will know neither the wage distributions nor the Markov transition probabilities. The reformulation of these models to include adaptation is another area for future research.

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Integrability Implications for Locally Constant Demand Elasticities

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1. INTRODUCTION

In this paper I uncover some new, surprising, and strong implications of utility maximization for the behavior of the elasticities of demand of an individual consumer. Over a fixed arbitrary open subset of price–income space:

(i) All nonunitary income elasticities of an individual's demands that are constant must be equal to one another (Theorem 1);

(ii) a good having constant unitary income elasticity must have a zero elasticity with respect to the price of any other good with a constant nonunitary income elasticity (Theorem 2);

(iii) all goods whose own and cross-price elasticities are independent of income must have constant income elasticities (Theorem 3);

(iv) if, in addition, these goods have constant elasticities (some nonzero) with respect to each others prices, then a long list follows of simple and powerful restrictions on all their demand elasticities (Theorem 4);

(v) finally, the only demand system with all price elasticities constant is the classical system with unitary income elasticities, zero cross-price elasticities, and own price elasticities of -1 (Corollary 6).

These theorems seem to have escaped the notice of 50 years of demand theory. I conjecture that the reason is that these are results on the second derivatives of neoclassical demand functions, while our standard comparative statics methodology has only been well developed to analyze first derivatives. In this paper, a completely novel approach to the comparative statics of demand is taken. It is based entirely on the income compensation function, pioneered by McKenzie [5] and Hurwicz and Uzawa [3].

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The results reported here may find a variety of uses. They can provide guidance to theorists who incorporate individuals' demand functions into their market models. The theorems may find empirical application in the important and promising area of analysis of the economic behavior of individual households, through the use of micro-data-files.[1] Moreover, the new analytic techniques applied here may be found useful for investigating other second-derivative comparative statics properties of maximizing behavior.

2. Basic Tools

In this section, I briefly lay out the tools utilized below to prove the theorems. I work with a standard neoclassical price-taking, nonsatiated consumer who faces prices $p = (p_1, p_2, \ldots, p_n)$ and income m, and who maximizes utility $U(x_1, \ldots, x_n)$ with the twice-differentiable demand functions $X^1(p, m), \ldots, X^n(p, m)$. These, of course, satisfy the Slutsky integrability conditions. An indirect utility function can be defined by

$$\mathscr{L}(p, m) \equiv U[X^1(p, m), \ldots, X^n(p, m)].$$

Any ordinal transformation of $\mathscr{L}$ can also serve as an indirect utility function. It is well known that the demand functions of a consumer are related to any of his indirect utility functions $\mathscr{L}(p, m)$ by Roy's identity (see [4] for a clear exposition)

$$X^k(p, m) = -\mathscr{L}_k(p, m)/\mathscr{L}_m(p, m).^2 \tag{1}$$

The income compensation function μ, can be implicitly defined by

$$\mathscr{L}(p', \mu(p' \mid p, m)) \equiv \mathscr{L}(p, m). \tag{2}$$

$\mu(p' \mid p, m)$ is the income necessary with prices p' to enable the optimizing consumer to reach the same level of utility as that obtained with prices p and income m. Trivially,

$$\mu(p \mid p, m) = m. \tag{3}$$

[1] In the recent past, the theorems would have been regarded as useful for the analysis of market demand (see [1, 2, 6], for example). However, Sonnenschein has demonstrated [7] that any continuous, homogeneous, and balanced set of market demand functions can arise from the utility maximizing behaviors of a collection of consumers.

[2] Throughout, subscripts will denote partial derivatives, with i serving as a proxy for p_i.

The partial derivatives of $\mu(p' \mid p, m)$ with respect to p_i' can be calculated from (2), in light of (1), to be

$$\partial\mu(p' \mid p, m)/\partial p_i' = X^i(p', \mu(p' \mid p, m)). \tag{4}$$

Now, in (2), hold p' constant and think of p and m as variables. While the right-hand side is the indirect utility function, the left-hand side is a strictly monotone increasing function of μ, since $\mathscr{L}$ is strictly increasing in income. Hence, $\mu(p' \mid p, m)$ is an ordinal transformation of $\mathscr{L}(p, m)$ which will also serve as an indirect utility function.

In fact, Hurwicz and Uzawa have shown in their modern classic [3] that given a set of demand functions $X^i(p, m)$ that satisfy the Slutsky integrability conditions and some technical smoothness requirements, the function μ is uniquely determined by the system of partial differential equations (4) with boundary condition (3). Further, they have shown that with the definition

$$l(p, m) \equiv \mu(p' \mid p, m), \tag{5}$$

for any fixed p', l serves as a neoclassical, demand generating, indirect utility function.

One of the marvels of this theorem is that it contains the instructions, (3) and (4), for constructing an indirect utility function from a set of demand functions. This technique is central to the proof of Theorems 1 and 2.

3. Constant Income Elasticities

Theorem 1 states that all nonunitary income elasticities which are constant over some open subset T of price–income space must be equal to one another. Each constant income elasticity imposes a strong restriction on the way income enters the indirect utility function. The proof shows that the restrictions are only consistent with each other if the elasticities are the same.

Theorem 1. *If $X^i(p, m)$, $i = 1,\ldots, n$, are twice-differentiable, Slutsky-integrable demand functions with $X^i > 0$ and*

$$\frac{\partial X^j(p, m)}{\partial m}\frac{m}{X^j(p, m)} \equiv \eta_j \neq 1 \qquad j = 1, 2,\ldots, r < n \tag{6}$$

for $(p, m) \in T \subseteq R_+^{n+1}$, T open, then

$$\eta_1 = \eta_2 = \cdots \eta_r .$$

Proof. It clearly suffices to prove the theorem for $r = 2$. With this case established, for $r > 2$ simply take the r goods sequentially, two at a time, to get the more general result.

Equation (6) can be viewed as a differential equation in X^j and m, with the vector p held fixed. It's solution is easily seen to have the form

$$X^j(p, m) \equiv F^j(p)\, m^{\eta_j}. \tag{7}$$

Now, let us construct the indirect utility function,

$$l(p, m) \equiv \mu(p_1', p_2', p_3', \ldots, p_n' \mid p, m).$$

It is convenient to proceed by stages, beginning with

$$\mu(p_1', p_2, p_3, \ldots, p_n \mid p, m).$$

From (4) and (7) we know that

$$\begin{aligned} \partial\mu(t, p_2, p_3, \ldots, p_n \mid p, m)/\partial t &= X^1(t, p_2, \ldots, p_n, \mu) \\ &\equiv F^1(t, p_2, \ldots, p_n)\, \mu^{\eta_1} \end{aligned} \tag{8}$$

This is an easily solved differential equation in μ and t, with all else held constant. Since $\eta_1 \neq 1$, (8) can be rewritten as

$$\partial(\mu^{1-\eta_1})/\partial t = (1 - \eta_1)\, F^1(t, p_2, \ldots, p_n).$$

Integrating between $t = p_1'$ and $t = p_1$, and recalling (3) yields[3]

$$\begin{aligned} \mu(p_1', p_2, \ldots, p_n \mid p, m)^{1-\eta_1} - m^{1-\eta_1} &= (1 - \eta_1) \int_{p_1}^{p_1'} F^1(t, p_2, \ldots, p_n)\, dt \\ &\equiv G^1(p). \end{aligned}$$

This equation serves to define $G^1(p)$, in which the argument p_1', held fixed throughout, is suppressed. Rearranging,

$$\mu(p_1', p_2, \ldots, p_n \mid p, m) = [G^1(p) + m^{1-\eta_1}]^{1/1-\eta_1}. \tag{9}$$

Again using (4) and (7),

$$\begin{aligned} \partial\mu(p_1', t, p_3, \ldots, p_n \mid p, m)/\partial t &= X^2(p_1', t, p_3, \ldots, p_n, \mu) \\ &\equiv F^2(p_1', t, p_3, \ldots, p_n)\, \mu^{\eta_2}. \end{aligned}$$

[3] Calculations of this sort are useful for computing consumer's surplus. See [8, 9].

Dividing through by μ^{η_2} and integrating between $t = p_2'$ and $t = p_2$ gives

$$\mu(p_1', p_2', p_3, \ldots, p_n \mid p, m)^{1-\eta_2} - \mu(p_1', p_2, p_3, \ldots, p_n \mid p, m)^{1-\eta_2}$$
$$= (1 - \eta_2) \int_{p_2}^{p_2'} F^2(p_1', t, p_3, \ldots, p_n)\, dt \equiv G^2(p_2, \ldots, p_n).$$

Substituting from (9),

$$\mu(p_1', p_2', p_3, \ldots, p_n \mid p, m)$$
$$= [G^2(p_2, \ldots, p_n) + [G^1(p) + m^{1-\eta_1}]^{(1-\eta_2)/(1-\eta_1)}]^{1/1-\eta_2}. \tag{10}$$

Now, $\mu(p_1', p_2', q_3, \ldots, q_n \mid p, m)$, as a function of $q_3, \ldots, q_n$, can conceptually be derived as the solution to the system of partial differential equations

$$\partial\mu(p_1', p_2', q_3, \ldots, q_n \mid p, m)/\partial q_i = X^i(p_1', p_2', q_3, \ldots, q_n, \mu)$$

with the boundary condition (10). Note that p_1, p_2, and m enter the solution only through (10). Thus, we can write

$$I(p, m) \equiv \mu(p_1', \ldots, p_n' \mid p, m)$$
$$\equiv \Phi[[G^2(p_2, \ldots, p_n) + [G^1(p) + m^{1-\eta_1}]^{(1-\eta_2)/(1-\eta_1)}]^{1/1-\eta_2}, p_3, \ldots, p_n]. \tag{11}$$

The functional form (11) has embodied in it the assumptions made on η_1 and η_2. We try to recover the demand functions X^1 and X^2 by means of Roy's identity (1). For notational convenience, define

$$c = (1 - \eta_2)/(1 - \eta_1), \qquad a = 1 - \eta_1.$$

Since $\eta_1 \neq 1$ and $\eta_2 \neq 1$, $c \neq 0$ and $a \neq 0$. Mechanical differentiation and division yield

$$X^1(p, m) \equiv -G_1^1 m^{1-a}/a,$$

which has the proper form, (7).

However, from (11), (1), and (7), we have

$$F^2(p)\, m^{\eta_2} \equiv X^2(p, m) \equiv -(G_2^1 m^{1-a}/a) - (G_2^2 m^{1-a}/ca[G^1 + m^a]^{c-1}).$$

We can derive the necessity of $c = 1$, or $\eta_1 = \eta_2$, from this identity. Divide through by the nonzero expression $F^2(p)\, m^{1-a}$, and rearrange to get

$$[G^1 + m^a]^{1-c}(G_2^2/caF^2) \equiv -(G_2^1/aF^2) - m^{\eta_2+a-1}.$$

Differentiating this identity with respect to m and then with respect to p_1, and factoring gives

$$(1-c)\,am^{a-1}[G^1+m^a]^{-c}\left[\frac{\partial[G_2{}^2/caF^2]}{\partial p_1}-\frac{cG_2{}^2G_1{}^1}{caF^2[G^1+m^a]}\right]\equiv 0.$$

By their definitions, $G_2{}^2\neq 0$, $G_1{}^1\neq 0$, $c\neq 0$, and $a\neq 0$. Then, since $\partial[G_2{}^2/caF^2]/\partial p_1$ is independent of m, and $cG_2{}^2G_1{}^1/caF^2[G^1+m^a]$ is dependent on m, their difference cannot be identically zero. Hence, it follows that $(1-c)\,am^{a-1}[G^1+m^a]^{-c}\equiv 0$, $c=1$,

$$l(p,m)=\Phi[(G(p)+m^{1-\eta})^{1/1-\eta},p_3,\ldots,p_n],$$

and $\eta_1=\eta_2$. Q.E.D.

COROLLARY 1. *If the income elasticities of all goods purchased are constant over T, then they are all unitary.*

Proof. Let $(mX_m{}^i)/X^i=\eta_i$. Denote $R_i=p_iX^i$. The budget constraint is $\sum_i R_i=m$. Differentiating with respect to m gives

$$\sum_i R_i\eta_i=m,$$

and

$$\sum_i R_i(\eta_i-1)=0.$$

By Theorem 1, either $\eta_i=1$ or $\eta_i=\eta$. Let $S=\{i\mid \eta_i=\eta\}$. Then, $\sum_{i\notin S}R_i(\eta_i-1)+\sum_{i\in S}R_i(\eta_i-1)=(\eta-1)\sum_{i\in S}R_i\equiv 0$. Thus, either $\eta=1$, or $R_i=0$ for all $i\in S$, and the income elasticities of all goods which are ever purchased are unitary. Q.E.D.

This would be an easy result if the set T were the whole price–income space. Then, if all elasticities were not 1, at least one elasticity would have to be greater than 1. The demand for that good would violate the budget constraint for suitably high income. However, Corollary 1 proves the result over the arbitrary set T which need not include such high levels of income.

4. A CONSTANT UNITARY INCOME ELASTICITY

Theorem 1 left open the possibility that two constant income elasticities of demand could be unequal, if one of them were unitary. This is indeed the case, but only if the demand for the good with unit income elasticity is insensitive to the price of the other good in question.

THEOREM 2. *If $X^i(p, m)$, $i = 1,..., n$, are twice-differentiable, Slutsky-integrable, demand functions with*

$$\frac{\partial X^1(p, m)}{\partial m} \frac{m}{X^1(p, m)} \equiv 1, \quad X^1 \neq 0, \quad \frac{\partial X^2(p, m)}{\partial m} \frac{m}{X^2(p, m)} \equiv \eta_2 \neq 1,$$

and $X^2 \neq 0$ for $(p, m) \in T \subseteq R^{n+1}_+$, T open, then $\partial X^1(p, m)/\partial p_2 \equiv 0$ over T.

Proof. We proceed through the same sequence of steps as in the proof of Theorem 1. Referring back to (8), since $\eta_1 = 1$, we have

$$\partial \ln \mu(t)/\partial t = F^1(t, p_2,..., p_n)$$

and

$$\mu(p_1', p_2,..., p_n) = m\left[\exp \int_{p_1}^{p_1'} F^1(t, p_2,..., p_n)\, dt\right] \equiv mG^1(p). \quad (12)$$

As before, using (4), (7), and now (12), we have

$$\begin{aligned}\mu(p_1', p_2', p_3,..., p_n \mid p, m)^{1-\eta_2} - [mG^1(p)]^{1-\eta_2} \\ \equiv (1 - \eta_2) \int_{p_2}^{p_2'} F^2(p_1', t, p_3,..., p_n)\, dt \\ \equiv G^2(p_2,..., p_n)\end{aligned}$$

and

$$\mu(p_1', p_2', p_3,..., p_n \mid p, m) \equiv [G^2(p_2,..., p_n) + (mG^1(p))^{1-\eta_2}]^{1/1-\eta_2}.$$

By the same argument used previously, we can write

$$\begin{aligned}I(p, m) &\equiv \mu(p_1',..., p_n' \mid p, m) \\ &\equiv \Phi[[G^2(p_2,..., p_n) + (mG^1(p))^{1-\eta_2}]^{1/1-\eta_2}, p_3,..., p_n]. \quad (13)\end{aligned}$$

Now, applying (1) to (13) yields $X^1(p, m) \equiv -mG_1^1/G^1$, which agrees with (7), $\eta_1 = 1$. Equations (1), (13), and (7) also give

$$F^2(p)\, m^{\eta_2} \equiv X^2(p, m) \equiv \frac{G_2^2 + m^{1-\eta_2}(1 - \eta_2)\, G_2^1 (G^1)^{-\eta_2}}{(\eta_2 - 1)\, m^{-\eta_2}(G^1)^{1-\eta_2}}.$$

Simple cross-multiplication yields the identity

$$m^{1-\eta_2}(1 - \eta_2)(G^1)^{-\eta_2} G_2^1 \equiv F^2(\eta_2 - 1)(G^1)^{1-\eta_2} - G_2^2.$$

Since $G^1 \neq 0$ and, by hypothesis, $\eta_2 \neq 1$, the left-hand side is a function of m, while the right-hand side is not. Then, $m^{1-\eta_2}(1 - \eta_2)(G^1)^{-\eta_2} G_2^1 \equiv 0$

and $G_2^1 \equiv 0$. From (12) it is clear that it must be the case that $F_2^1 \equiv 0$, and so, from (7), $X_2^1 \equiv 0$. Q.E.D.

Now, in the statement of Theorem 1, we can exchange for the restriction that $\eta_j \neq 1$, a condition on the cross-price elasticities of demand.

COROLLARY 2. *If* $(\partial X^j(p, m)/\partial m)\, m/X^j(p, m) \equiv \eta_j$, $j = 1,..., r$, *and if for each* $j \leqslant r$ *there is a* $k \leqslant r$ *such that* $\epsilon_k^j \not\equiv 0$ (where $\epsilon_k^j(p, m) \equiv (\partial X^j(p, m)/\partial p_k)\, p_k/X^j(p, m)$), *for* $(p,m) \in T \subseteq R_+^{n+1}$, T *open, then* $\eta_1 = \eta_2 = \cdots = \eta_r$.

Proof. This follows immediately from Theorems 1 and 2.

5. INCOME-INDEPENDENT PRICE ELASTICITIES

Theorem 3 shows that if all the price elasticities of a good are independent of income, then the good's income elasticity is constant. The result is probably known and is proven with standard methods. However, in light of Theorems 1 and 2, it takes on new significance. Corollaries 3, 4, and 5 state the strong restrictions on income elasticities of integrable demands which follow from the common assumption of income-independent price elasticities.

THEOREM 3. *Let*

$$\epsilon_i^1(p, m) \equiv \frac{\partial X^1(p, m)}{\partial p_i} \frac{p_i}{X^1(p, m)}$$

and

$$\eta_1(p, m) \equiv \frac{\partial X^1(p, m)}{\partial m} \frac{m}{X^1(p, m)}.$$

If $\partial \epsilon_i^1(p, m)/\partial m \equiv 0$, *for* $i = 1,..., n$, $(p, m) \in T \subseteq R_+^{n+1}$, T *open, then* $\eta_1(p, m) \equiv \eta_1$ (*constant*) *over* T.

Proof.

$$\partial \epsilon_i^1/\partial m = p_i[X^1 X_{im}^1 - X_i^1 X_m^1]/(X^1)^2 \equiv 0 \equiv X^1 X_{im}^1 - X_i^1 X_m^1.$$

Then

$$\partial \eta_1/\partial p_i = m[X^1 X_{mi}^1 - X_m^1 X_i^1]/(X^1)^2 \equiv 0.$$

Thus, the income elasticity η_1 is independent of all prices, and is, at most, a function $\eta_1(m)$, of income.

Now, the demand function $X^1(p, m)$ is homogeneous of degree zero. Hence, we have the Euler equation

$$\sum_i X_i^1 p_i + X_m^1 m = 0.$$

Dividing by X^1 gives

$$\eta_1(m) = -\sum_i \epsilon_i^1(p).$$

Since the right-hand side is independent of m, η_1 is constant. Q.E.D.

COROLLARY 3. *If* $\partial\epsilon_i^j(p, m)/\partial m \equiv 0$ *over* T *for* $j = 1, 2,..., r$ *and* $i = 1, 2,..., n$, *then for each* j, *either* $\eta_j = 1$ *or* $\eta_j = \eta$.

COROLLARY 4. *If* $\partial\epsilon_i^j(p, m)/\partial m \equiv 0$ *for* $j = 1, 2,..., r$ *and* $i = 1, 2,..., n$, *and if for each* $j \leqslant r$ *there is a* $k \leqslant r$ *with* $\epsilon_k^j \not\equiv 0$ *over* T, *then* $\eta_1 = \eta_2 = \cdots = \eta_r$.

Proof. This follows from Corollaries 2 and 3.

COROLLARY 5. *If* $\partial\epsilon_i^j(p, m)/\partial m \equiv 0$ *over* T *for* $i, j = 1, 2,..., n$, *then* $\eta_i = 1$ *over* T, $i = 1, 2,..., n$.

Proof. This follows immediately from Theorem 3 and Corollary 1.

6. CONSTANT PRICE ELASTICITIES

In this section I attempt to derive the strongest sets of integrability implications for systems of demand elasticities. Theorem 4 postulates that all the elasticities of the first r goods with respect to the first r prices are constant and that all the other price elasticities of these goods are independent of income. A detailed list of restrictions on the price and income elasticities of these goods follows. The striking Corollary 6 proves that the only integrable demand system with all price elasticities constant is the classical Marshallian system, derivable from a Cobb–Douglas utility function. Here, all income elasticities are 1, all own price elasticities are -1, and all cross-price elasticities are 0. All income shares are constant.

THEOREM 4. Suppose $\partial\epsilon_i^j(p, m)/\partial m \equiv 0$ for $j = 1,..., r$ *and* $i = 1, 2,..., n$, *and that* $\epsilon_k^j(p, m) \equiv \epsilon_k^j$ *for* $j, k \leqslant r$, *for* $(p, m) \in T \subseteq R_+^{n+1}$, T *open. Then, for* $j, k \leqslant r, j \neq k$:

(a) *If* $\epsilon_k^j = 0$ *and* $\epsilon_j^k \neq 0$, *then* $\eta_j(p, m) \equiv 1$, $\eta_k(p, m) \equiv \eta \neq 1$, $\epsilon_j^j = -1$, *and* $\epsilon_l^j = 0$ *for* $l \neq j$, *over* T;

(b) *if* $\epsilon_k^j \neq 0$ *and* $\epsilon_j^k \neq 0$, *then* $\eta_j(p, m) \equiv \eta_k(p, m) \equiv \eta$, $\epsilon_j^j = \epsilon_j^k - 1$, $\epsilon_k^k = \epsilon_k^j - 1$, $\epsilon_k^j \equiv [X^k p_k / X^j p_j]\, \epsilon_j^k$, *and* $\epsilon_l^j(p, m) \equiv \epsilon_l^k(p, m)$ *for* $l \neq j, k$, *over* T;

(c) *if* $\epsilon_k^j = \epsilon_j^k = 0$, *then* $\eta_j(p, m) \equiv \eta_k(p, m)$ *over* T.

Proof. We know from Corollary 3 that either $\eta_j(p, m) \equiv 1$ or

$$\eta_j(p, m) \equiv \eta.$$

(a) By Slutsky symmetry, $X_k^j + X^k X_m^j \equiv X_j^k + X^j X_m^k$. Since $\epsilon_k^j = 0$, this implies that $X^j X^k(\eta_j - \eta_k) \equiv m X_j^k \neq 0$, and that $\eta_j \neq \eta_k$. So, Theorem 2 tells us that $\eta_j = 1$ and $\eta_k = \eta \neq 1$. Further rearrangement of the Slutsky equation gives $p_j X^j \equiv m\epsilon_j^k/(1 - \eta)$. Differentiation with respect to p_j and p_l yields the conclusion.

(b) Corollary 2 implies $\eta_j = \eta_k = \eta$. Then $X_k^j + X^k X_m^j \equiv X_j^k + X^j X_m^k$ implies that $X_k^j = X_j^k$, $\epsilon_k^j = [p_k X^k / p_j X^j]\, \epsilon_j^k \neq 0$, and $p_k X^k / p_j X^j = \epsilon_k^j / \epsilon_j^k = \text{constant}$. Differentiation of this last identity gives the conclusions.

(c) This follows simply from Slutsky symmetry. Q.E.D.

COROLLARY 6. *Suppose* $X^j > 0$ *and* $\epsilon_k^j(p, m) = \epsilon_k^j$ *for all* $j, k \leq n$, $(p, m) \in T \subseteq R_+^{n+1}$, T *open. Then* $\eta_k(p, m) = 1$, $\epsilon_k^k = -1$, *and* $\epsilon_k^j = 0$ *for all* $k \leq n, j \neq k$.

Proof. Differentiate the budget constraint, $\sum_j R_j = m$, with respect to p_k to get $\sum_j R_j \epsilon_k^j + R_k \equiv 0$. By Corollary 5, $\eta_k(p, m) = 1$ for all k, ruling out case (a) of Theorem 4. By Theorem 4, then, either $\epsilon_k^j = 0$ or $\epsilon_k^j = 1 + \epsilon_k^k$, for $j \neq k$. Letting $S = \{i \mid i \neq k, \epsilon_k^i \neq 0\}$,

$$0 \equiv \sum_j R_j \epsilon_k^j + R_k = \sum_{j \neq k} R_j \epsilon_k^j + (1 + \epsilon_k^k) R_k$$

$$= (1 + \epsilon_k^k)\left(R_k + \sum_{i \in S} R_i\right).$$

Thus, $\epsilon_k^k = -1$, for all k, and $\epsilon_k^j = 0$, $j \neq k$, whether $j \notin S$ or $j \in S$. Q.E.D.

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General Equilibrium with Money; Indeterminacy of Price Level and Efficiency

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1. INTRODUCTION

There have been many attempts to introduce money into a general equilibrium framework in a meaningful way. This problem has been around ever since Walras wrote his famous book [16]. However, most serious attempts at a rigorous formulation are of fairly recent vintage. Following Hahn's contribution on a sequence of markets [4] and a number of Starr's works and Sontheimer's attempt to formulate a theory of money in an Arrow–Debreu economy (the most important in the current context are Starr [14] and Sontheimer [13]), several economists have formulated a monetary economy in a sequence of markets.

The most important and therefore the most frequently discussed problems are those of existence and of efficiency. The current paper reviews the existing contributions and attempts to reconcile some of the problems arising from this review.

For the most part, we will be concerned with the work of Heller [7] on existence and Hahn [5] and Starrett [15] on efficiency.

2. PROPERTIES OF MONEY

Let us first review the paper by Heller [7]. His model is essentially that of Kurz's [9], i.e., an economy with individualized transaction technology of finite time horizon. He introduced fiat money in this model (therefore, future money or bonds, too) as a store of value as well as a means of payment. Money is completely durable and possesses no intrinsic

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value. Therefore in order to support positive price of money in each sequential market, one needs a terminal condition on money so that somebody within or outside the economy is willing to hold money at the end of the horizon. In this model, this function is performed by the government; once the horizon is reached, the government collects from each household exactly the amount of money with which the household started. This peculiar terminal condition gives rise to a strange (but not surprising) conclusion that there are multiple equilibria, one corresponding to any terminal price of money. This must be the case roughly because, in Heller's terms, there is no wealth effect even if the price of money (or general price level) changes. Or, in other words, each household must end up with the same money stock as what he had at the outset; a special kind of Say's law which sets one price free in the equilibrium.

While his proof of the existence of a monetary equilibrium is very neat, his model has several problems of its own. We have been taught that money has three basic roles to play; a unit of account, a means of payment, and a store of value. In his paper, the first two roles are not treated in their full scope. His price system allows the price of money to change every period. Therefore, the unit of account property of money exists, if at all, among prices of the same period.

He also attempts to treat money as a means of payment. However, his model allows us to spend money which is acquired in the same period. This assumption amounts to the usual budget constraint which states that purchases are less than sales unless the price of money is zero. Hence, money becomes indistinguishable from other commodities aside from the property of complete durability and of no intrinsic utility. In other words, any commodity would serve as money in his model as long as it is durable and has a positive price; it need not be accepted as a means of payment.[1]

The last point is concerned with the individualized transaction technology. Economies which use fiat money are usually in the stage of relatively developed market structures. If bilateral trades or market posts are dominant ways to make transactions, money used in such economies is likely to be commodity money. Therefore, it seems that it is more desirable to construct a model with fiat money where households make transactions with retailers (or stores) rather than with each other.

Starrett's [15] and Hahn's [5] papers are based on Hahn's [4] model, which seems to be a standard model with transaction cost. Trades are carried out only through the trading specialists with the expense of resources as a transaction cost. All the trades are barter (commodity

[1] For example, paper wrappers of chocolate would serve the purpose as far as the government collects them.

with commodity) and money does not enter explicitely. Transaction costs give rise to reopening of markets and, therefore, consumers must make a plan of decisions for each period. Hahn [4] defines an equilibrium as sequences of prices and consumers' plans such that consumer's plans are optimal under the sequence of prices and the plans clear the market period. Therefore, this is a kind of an equilibrium with a self-fulfilling expectation or an equilibrium with plans, prices, and price expectations [12]. With a suitable set of assumptions, Hahn showed that there is an equilibrium and that the equilibrium may very well be inefficient.

Starrett [15] and Hahn [5] are concerned with inefficiency rather than existence of equilibrium. Starrett points out that the principal cause of inefficiency lies in the fact that a consumer must equate his expenditure for purchases with income from sales in each period. Thus it is natural to introduce some mechanism which links different periods. Starrett, eventually, shows that if there is a free asset (money) which he can borrow or carry over to the future without cost, the efficiency property of an equilibrium will be restored.

Hahn considered essentially the same model as Starrett, except he extends the model to cases where there is an outside money and where bond transactions are costly. He assumed, as Heller and others do, that all the outside money will be collected by the government in the form of a tax. Assuming that all the consumers start with positive amount of outside money endowment, he showed that an efficient equilibrium exists and that an equilibrium is efficient if all the bond markets are inactive.

Our economy has the same property as Heller's in some respects. It is a finite horizon model with a similar tax scheme at the end of the economy. However, different from his, our model contains firms as well as households. Also, money is really a means of payment in the sense that "money buys goods and goods buy money; but goods do not buy goods" [1], and also in the sense that "the total value of goods demanded cannot in any circumstances exceed the amount of money held by the transactor at the outset of the period" [1]. In short, we eliminate the possibility of buying goods with money which one obtains by selling goods in the same period. We also give a completely different proof for the existence of an equilibrium which gives us more insight into the mechanism of the model.

3. The Economy

There are T time periods denoted by t or τ, H households denoted by h, and I firms denoted by i. There are J commodities for consumption in each period denoted by j, money, and I shares corresponding to each firm.

We shall use the following notation.

$x_\tau^\alpha(t) \in R_+^J$ = vector of the commodity purchases at t for delivery at τ by $\alpha(\alpha = h$ or $i)$.
$y_\tau^\alpha(t) \in R_+^J$ = vector of commodity sales at t for delivery at τ by α.
$x_\tau^{\alpha e}(t) \in R_+^I$ = vector of share purchases at t for delivery at τ by α.[2]
$y_\tau^{\alpha e}(t) \in R_+^I$ = vector of share sales at t for delivery at τ by α.
$p_\tau^B(t) \in R_+^J$ = vector of monetary prices of commodities for purchase by households (or sales by firms) at t for delivery at τ.
$p_\tau^S(t) \in R_+^J$ = vector of prices of commodities for sales by households (or purchase by firms) at t for delivery at τ.
$p_\tau^{eB}(t) \in R_+^I$ = vector of share prices for purchase by households at t for delivery at τ.
$p_\tau^{eS}(t) \in R_+^I$ = vector of share prices for sales by households at t for delivery at τ.
$z(t) = (z_t(t), z_{t+1}(t),..., z_T(t))$ for $z = x^\alpha, y^\alpha, x^{\alpha e}, p^B, p^S, p^{eB}, p^{eS}$.
$c^h(t) \in R_+^J$ = vector of consumption at t by h.
$s^h(t) \in R_+^J$ = vector of storage output at t for h.
$s_0^h(t) \in R_+^J$ = vector of storage input at t by h.
$e^h(t) \in R_+^I$ = vector of share holding at the beginning of t by h.
$w^h(t) \in R_+^J$ = vector of commodity endowment at t for h.
$\pi^i(t) \in R_+$ = profit of ith firm at t.
$\pi(t) \in R_+^I$ = vector of profits at t.
$z = (z(1), z(2),..., z(T))$ for $z = x^\alpha, y^\alpha, x^{\alpha e}, y^{\alpha e}, p^B, p^S, p^{eB}, p^{eS}, c^h, s^h, s_0^h, e^h, w^h, \pi^i, \pi$.
$x_\tau^{\alpha b}(t) \in R_+$ = purchase of bond at t which matures at τ by α.[3]
$y_\tau^{\alpha b}(t) \in R_+$ = sales of bond at t which matures at τ.
$p_\tau^{bB}(t) \in R_+$ = price of bond for purchase by household at t which matures at τ.
$p_\tau^{bS}(t) \in R_+$ = price of bond for sales by household at t which matures at τ.
$z(t) = (z_{t+1}(t), z_{t+2}(t),..., z_T(t))$
$z = (z(1), z(2),..., z(T' - 1))$ for $z = x^{\alpha b}, y^{\alpha b}, p^{bB}, p^{bS}$.
$\bar{e}^h \in R_+^I$ = vector of initial share holding by h.
$\bar{m}^h \in R_+$ = initial cash balance by h.

For a household, h, the problem is the following. Assume its con-

[2] A unit of share of each firm is normalized so that the total existing share in the economy adds up to unity. Namely, $\sum_{h=1}^{H} \bar{e}^{hi} = 1$ for all i.

[3] A unit of bond is a contract to deliver a unit of money at its maturing date.

sumption possibility set is a nonnegative orthant of the relevant Euclidean space.

$$c^h(t) = w^h(t) + \sum_{\tau=1}^{t} [x_t^h(\tau) - y_t^h(\tau)] + s^h(t) - s_0^h(t) \geqslant 0 \text{ [4]} \quad \text{for all } t. \tag{1}$$

For initial endowment, we assume

(E.1) $w^h(t) > 0$ for all t and $\sum_{h=1}^{H} w^h(t) \gg 0$ for all t.

(E.2) $s^h(1) = 0$ for all h.

Let S_t^h be h's storage possibility set to store commodities from $t-1$ to t, then

$$(s_0^h(t-1), s^h(t)) \in S_t^h \qquad \text{for all } t. \tag{2}$$

S_t^h is assumed to satisfy

(S.1) S_t^h is closed, convex and $(0, 0) \in S_t^h$ for all h and t.

(S.2) If $(s_0^h(t-1), s^h(t)) \in S_t^h$ and if $\hat{s}_0^h(t-1) \geqslant s_0^h(t-1)$, $\hat{s}^h(t) \leqslant s^h(t)$ then $(\hat{s}_0^h(t-1), \hat{s}^h(t)) \in S_t^h$ for all h and t.

In order to write the budget constraint, we define

$$\begin{aligned} M^h(t) = {} & p^S(t)\, y^h(t) + p^{eS}(t)\, y^{he}(t) + p^{bS}(t)\, y^{hb}(t) - p^B(t)\, x^h(t) \\ & - p^{eB}(t)\, x^{he}(t) - p^{bB}(t)\, x^{hb}(t) + \sum_{\tau=1}^{t-1} [x_t^{hb}(\tau) - y_t^{hb}(\tau)] \\ & + \pi(t)\, e^h(t). \end{aligned}$$

Namely, $M^h(t)$ is h's net cash receipt within period t. Using $M^h(t)$, the budget constraint is

$$\bar{m}^h + \sum_{\tau=1}^{t-1} M^h(\tau) \geqslant 0 \qquad \text{for all } t \neq T, \tag{3}$$

$$\bar{m}^h + \sum_{t=1}^{T} M^h(t) \geqslant \bar{m}^h. \tag{4}$$

Equation (4) follows because, following Heller, we assume that the government collects $\bar{m}^h$ units of money from h at the end of the economy. For $\bar{m}^h$, we assume

(E.3) $\bar{m}^h > 0$ for all h.

[4] We use the following notations. $X > y$ if $x_i \geqslant y_i$ for all i, $x > y$ if $x \geqslant y$ but $x \neq y$, $x \gg y$ if $x_i > y_i$ for all i.

However, even if h satisfies all budget constraints ((3) and (4)), it may not be able to purchase goods at t unless it has enough cash balance. Therefore (an expenditure constraint),

$$\bar{m}^h + \sum_{\tau=1}^{t} M^h(\tau) \geqslant p^B(t)\, x^{hB}(t) + p^{eB}(t)\, x^{he}(t) + p^{bB}(t)\, x^{hb}(t). \quad (5)$$

Finally, since shares are stocks, its holding can never become negative.

$$e^h(t) = \bar{e}^h + \sum_{\tau=1}^{t-1} \sum_{\nu=1}^{\tau} [x_\tau^{he}(\nu) - y_\tau^{he}(\nu)] \geqslant 0, \quad (6)$$

where we assume

(E.4) $\bar{e}^h \geqslant 0$ and $\sum_{h=1}^{H} \bar{e}^{hi} = 1$ for all i.

Let us now define household h's action $a^h \equiv (c^h, x^h, y^h, x^{he}, y^{he}, x^{hb}, y^{hb}, s^h, s_0{}^h, e^h)$. h's budget correspondence, B^h, is defined as a mapping from the price space into the action space if $\pi(t)$ is also a mapping from the price space into $R_+{}^I$. Since this is eventually the case which will be shown later,

$$B^h(p) = \{a^h \geqslant 0 \mid (1)\text{–}(6) \text{ are simultaneously satisfied}\},$$

where $p \equiv (p^B, p^S, p^{eB}, p^{eS}, p^{bB}, p^{bS})$.

Each household h has preference ordering, $\succcurlyeq_h$, on the consumption possibility set. We assume:

(P.1) $\succcurlyeq_h$ is a semistrictly convex and closed partial preorder.

(P.2) $\succcurlyeq_h$ is monotone.

The demand correspondence, $\xi^h(p)$, is defined as usual.

$$\xi^h(p) = \{a^h \in B^h(p) \mid c^h \succcurlyeq \hat{c}^h \text{for all } \hat{a}^h \in B^h(p)\}.$$

We shall assume that all firms try to maximize their instantaneous profits under their own technology. We can, of course, assume alternatively that firms would try to maximize their given utility functions defined over profit stream.[5] We employ this rather strong assumption about firms' behavior because we will later attempt to compare our results with that of Hahn's. Since firms are also required not to place demands which exceed the value of money holding, they must be allowed to own initial money holding $\bar{m}^i \geqslant 0$. Therefore, we assume

(E.5) $\bar{m}_i > 0$ for all i.

[5] For example, see [12]. In the present context within a monetary economy, see also [10, Chap. II].

Since all profits are paid out in the same period, firms will always carry over $\bar{m}^i$ units of money into the next period. At the end, this money holding would, again, be collected by the government as tax. Therefore, for firm i,

$$(x^i(t), y^i(t), x^{ie}(t), y^{ie}(t), x^{ib}(t), y^{ib}(t)) \in T_t^i, \tag{7}$$

where T_t^i is firm i's transaction technology which is assumed to satisfy the following:

(T.1) $x^i(t) > y^i(t)$, $x^{ie}(t) = y^{ie}(t)$ and $x^{ib}(t) = y^{ib}(t)$ if $(x^i(t), y^i(t), x^{ie}(t), y^{ie}(t), x^{ib}(t), y^{ib}(t)) \in T_t^i$ unless all entries are zero's.

(T.2) T_t^i is closed and convex and $(0, 0, 0, 0, 0, 0) \in T_t^i$.

(T.3) If $(x^i(t), y^i(t), x^{ie}(t), y^{ie}(t), x^{ib}(t), y^{ib}(t)) \in T_t^i$, $\hat{x}^i(t) \leqslant x^i(t)$, $\hat{y}^i(t) \leqslant y^i(t)$, $\hat{x}^i(t) - \hat{y}^i(t) \geqslant x^i(t) - y^i(t)$, $\hat{x}^{ie}(t) = \hat{y}^{ie}(t) \leqslant y^{ie}(t)$, and $\hat{x}^{ib}(t) = \hat{y}^{ib}(t) \leqslant y^{ib}(t)$, then $(\hat{x}^i(t), \hat{y}^i(t), \hat{x}^{ie}(t), \hat{y}^{ie}(t), \hat{x}^{ib}(t), \hat{y}^{ib}(t)) \in T_t^i$.

Assumption (T.1) needs some clarification. The first part of (T.1) implies that transaction is costly. Namely, $x^i(t) - y^i(t)$ is the transaction cost necessary to make transactions. We also exclude any possibility of carrying inventories in either the form of storage or the form of future contracts. Thus at any moment, sales of any goods must be met by purchase of the same good in the same period. Thus the latter two equalities in (T.1) follow. Assumption (T.2) is standard and (T.3) is our version of free disposal with transaction cost.

Firms also have to back all purchases by money held in their hands.

$$\bar{m}^i \geqslant p^S(t)\, x^i(t) + p^{eS}(t)\, x^{ie}(t) + p^{bS}(t)\, x^{ib}(t). \tag{8}$$

Let

$$(x^i(t), y^i(t), x^{ie}(t), y^{ie}(t), x^{ib}(t), y^{ib}(t)) \equiv b^i(t),$$

and

$$(x^i, y^i, x^{ie}, y^{ie}, x^{ib}, y^{ib}) \equiv b^i.$$

Then firm i's feasible action correspondence, $F_t^i(p)$, at t is defined as

$$F_t^i(p) = \{b^i(t) \geqslant 0 \mid b(t) \in T_t^i \text{ and satisfies (8)}\}.$$

Profit of i at t is

$$\pi^i(t) = \max_{b^i(t) \in F_t^i(p)} \{p^B(t)\, y^i(t) + p^{eB}(t)\, y^{ie}(t) + p^{bB}(t)\, y^{ib}(t) - p^S(t)\, x^i(t) - p^{eS}(t)\, x^{ie}(t) - p^{bS}(t)\, x^{ib}(t)\}.$$

Firm i's supply correspondence, $\eta^i(p)$, is defined using $\pi^i(t)$.

$$\eta^i(p) = \{b^i \geqslant 0 \mid b^i(t) \in F_t^i(p) \text{ and } p^B(t)\, y^i(t) + p^{eB}(t)\, y^{ie}(t) + p^{bB}(t) y^{ib}(t) - p^S(t)\, x^i(t) - p^{eS}(t)\, x^{ie}(t) - p^{bS}(t)\, x^{ib}(t) = \pi^i(t) \text{ for all } t\}.$$

An equilibrium of the economy is a triple, $\{\langle a^h\rangle_{h=1}^H, \langle b^i\rangle_{i=1}^I, p\}$ such that

(D.1) $a^h \in \xi^h(p)$ for all h,

(D.2) $b^i \in \eta^i(p)$ for all i,

(D.3) $\sum_{h=1}^H x^h \leqslant \sum_{i=1}^I y^i$, $\sum_{i=1}^I x^i \leqslant \sum_{h=1}^H y^h$,

(D.4) $\sum_{h=1}^H x^{he} \leqslant \sum_{i=1}^I y^{ie}$, $\sum_{i=1}^I x^{ie} \leqslant \sum_{h=1}^H y^{he}$,

(D.5) $\sum_{h=1}^H x^{hb} \leqslant \sum_{i=1}^I y^{ib}$, $\sum_{i=1}^I x^{ib} \leqslant \sum_{h=1}^H y^{ib}$.

Our definition of an equilibrium corresponds to Heller's equilibrium with positive price of money since all prices are monetary prices; in other words, the price of money ($p_{0t}(t)$ in Heller) is set to be unity always. This formulation makes Heller's assumption (S) (substitutability between consumption in any period and consumption at T) redundant.[6] That is, this assumption is unnecessary if we formulate a monetary economy properly.

4. Existence

Since an existence proof for a similar model is already given in [7], I shall only sketch our proof, here. The price system in which we shall work is a slightly different from the usual one.

$$P_\mu = \left\{ p \geqslant 0 \;\middle|\; \sum_{t=1}^T \sum_{\tau=t}^T \left[\sum_{j=1}^J (p_{j\tau}^B(t) + p_{j\tau}^S(t)) + \sum_{i=1}^I (p_{i\tau}^{eB}(t) + p_{i\tau}^{eS}(t)) \right] = \mu \text{ for some } \mu > 0 \right\}.$$

We restrict commodity and share prices to be in usual simplex which sum up to an arbitrary number, μ, because of our extra degree of freedom.

[6] To see this, note that the assumption S is assumed in [7] in order to exclude the situation where the price of money relative to other goods at t, $P_{0t}(t)$, is zero while the same price at T, $P_{0T}(T)$, is positive. In our formulation, however, the accounting price of money is set to be unity for all t and the accounting price of other goods must be in the simplex. Namely, the relative price of money can never be zero by construction. In short, Heller's assumption S is necessary because he worked with an inappropriate price system.

There is no restriction on bond prices, because of a technical reason which will become apparent later. In fact, $[1/P_\tau^{bS}(t)] - 1$ and $[1/P_\tau^{bB}(t)] - 1$ being borrowing and lending rates of interest for households, it is obvious that in equilibrium bond prices cannot exceed unity, for otherwise interest rate would become negative.

From (T.2), $\pi^i(t) \geqslant 0$ for all i and t. Since $\bar{m}^h > 0$ for all h, $B^h(p)$ is continuous and therefore $\xi^h(p)$ is upper semicontinuous. After suitably compactifing the economy (which is possible by virtue of (S.1), (S.2), and (T.1)–(T.3), correspondences $\xi^h(p)$ and $\eta^i(p)$ after compactification are nonempty, convex, compact-valued upper semicontinuous correspondences (by convexity of S_j^i, T_j^i and $\succcurlyeq_h$).

Define

$$P_\mu^n = \{p \in P_\mu \mid |p_\tau^{bB}(t)| \leqslant n, |p_\tau^{bS}(t)| \leqslant n\},$$

where n is a natural number. If p is restricted in P_μ^n, which is convex and compact, one can define a convex and compact valued upper semicontinuous correspondence, Γ_μ^n, such that

$$\Gamma_\mu^n(\langle a^h\rangle_{h=1}^H, \langle b^i\rangle_{i=1}^I) = \left\{ p \in P^n \;\middle|\; \begin{array}{l} \sum_{h=1}^H (p^B x^h - p^S y^h + p^{eB} x^{he} - p^{eS} y^{he} + p^{bB} x^{hb} - p^{bS} y^{hb}) \\ + \sum_{i=1}^I (p^S x^i - p^B y^i + p^{eS} x^{ie} - p^{eB} y^{ie} + p^{bS} x^{ib} - p^{bB} y^{ib}) \\ \text{is maximized} \end{array} \right\}.$$

Then the correspondence, Φ_μ^n, such that

$$\Phi_\mu^n[p, (\langle a^h\rangle_{h=1}^H, \langle b^i\rangle_{i=1}^I)] = \Gamma^n(\langle a^n\rangle_{h=1}^H, \langle b^i\rangle_{i=1}^I) \times \bigtimes_{h=1}^H \xi^h(p) \times \bigtimes_{i=1}^I \eta(p)$$

satisfies all the conditions for the Kakutani's fixed point theorem. Moreover, its fixed point satisfies all the conditions for an equilibrium except for (D.5). If $(\langle a_n^{h*}\rangle_{h=1}^H, \langle b_n^{i*}\rangle_{i=1}^I, p_n{}^*)$ is a fixed point, using the definition of Γ_μ^n, the following condition must be satisfied.

$$\sum_{h=1}^H [p^B x_n^{h*} - p^S y_n^{h*} + p^{eB} x_n^{he*} - p^{eS} y_n^{he*} + (p^{bB} - u) x_n^{hb*} - (p^{bS} - u) y_n^{hb*}]$$

$$+\sum_{i=1}^{I}[p^{S}x_n^{i*}-p^{B}y_n^{i*}+p^{eS}x_n^{ie*}-p^{eB}y_n^{ie*}$$

$$+(p^{bS}-u)\,x_n^{ib*}-(p^{bB}-u)\,y_n^{ib*}]\leqslant 0$$

for all $p\in P_\mu{}^n$ where u is a unit vector whose dimension is equal to that of p^{bB} and p^{bS}).

By suitably choosing $p\in P_\mu{}^n$, one can establish (D.3), (D.4), and

$$\sum_{h=1}^{H}x_{n\tau}^{hb*}(t)-\sum_{i=1}^{I}y_{n\tau}^{ib*}(t)\leqslant\beta/(n-1),$$

$$\sum_{i=1}^{I}x_{n\tau}^{ib*}(t)-\sum_{h=1}^{H}y_{n\tau}^{hb*}(t)\leqslant\beta/(n-1),$$

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where β is a constant independent of n.

Hence, if we consider a sequence, $(\langle a_h^{h*}\rangle_{h=1}^{H},\langle b_n^{i*}\rangle_{i=1}^{I},p_n{}^*)$, it has a limit for $\langle a_h^{h*}\rangle_{h=1}^{H}$, $\langle b_n^{i*}\rangle_{i=1}^{I}$ which satisfies (D.3)–(D.5). Similarly, $\lim_{n\to\infty}p_n{}^*$ would exist if the range of p_n^{bS*} and p_n^{bB*} were bounded, for the range of all other prices is bounded.

Suppose $p_{n\tau}^{bS*}(t)$ were not bounded. For sufficiently large n, $\sum_{h=1}^{H}y_{n\tau}^{hb*}(t)$ would always be at the boundary of the compactified economy since the borrowing rate for household is negative. On the other hand by (8), $\sum_{i=1}^{I}x_{n\tau}^{ib*}(t)$ would be very small (namely, $\sum_{i=1}^{I}x_{n\tau}^{ib*}(t)<\sum_{h=1}^{H}y_n^{hb*}(t)$). However, this would be a contradiction in view of the definition of $\Gamma_\mu{}^n$.

Next, suppose $p_{n\tau}^{bS*}(t)$ were bounded but not $p_{n\tau}^{bB*}(t)$. Then $\sum_{h=1}^{H}x_{n\tau}^{hb*}(t)=0$ for sufficiently large n. However, $\sum_{i=1}^{I}y_{n\tau}^{ib*}(t)>0$ because the profit a firm attains by selling one unit of bonds (and, at the same time, buying one unit of bonds because of (T.1)) is the difference between $p_{n\tau}^{bB*}(t)-p_{n\tau}^{bS*}(t)$ and the necessary transaction cost. Transaction cost in terms of commodities would not change when n varies, and therefore the cost in terms of dollar would be uniformly bounded due to the bound on commodity prices. Hence for sufficiently large n, the profit would become positive and $\sum_{i=1}^{I}y_{n\tau}^{ib}(t)>0$. Clearly, this would be again a contradiction.

Therefore an equilibrium exists. Moreover, we have shown the existence of an equilibrium for any price system, P_μ. In other words, for any commodity and share price level, there always exists a monetary equilibrium. It is clear, therefore, that the price of money is indeterminate *not* with respect to bond prices *but* with respect to real (commodity) prices.

THEOREM 1. *There exists an equilibrium for any level of prices of commodities and shares with respect to money.*

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37329

As Heller has observed, this particular indeterminacy is caused by our special assumption of collecting money at the end. The most critical is the condition that the initial cash balance and the tax at the end must be the same for each household. Therefore, we are treating the span of the economy from the beginning to the end of a monetization. Otherwise, the above assumption would not make sense, for once money is introduced, households start trading money as well.

A more realistic picture may be obtained by introducing banks which issue money at the beginning of a monetization (but not necessarily at the beginning of the time span under consideration) and collect the money at the end of the horizon. If this is the case, the bank as a firm must issue money in exchange for investment by stockowners. In other words, (outside) money is a title to bank profits. This original investment (or capital) must be held by the banks either in the form of capital or retained profit thoughout the monetization. As monetization ends, the retained profit will be returned to the stockholders in exchange for the money. After formulating the model properly, one can prove that the price of money is uniquely determined except for the case in which the time span of the economy is the same as that of the monetization. In this case, there is no wealth effect and the price level becomes indeterminate; more generally, if money were already in the hands of the public, there would be a wealth effect. Once the monetary economy starts functioning, the price of money must be determined historically. (For more detail, see [10, Chap. II].)

5. Efficiency

Hahn [4, 5], Heller [7], Kurz [9], and Starrett [15] noticed that an equilibrium in a sequence of markets may be inefficient because each household is facing a sequence of budget constraints instead of one budget constraint. Hahn and Starrett also realized that if there is any way to alter the economy so that each household can borrow and lend freely (or borrow from himself) then inefficiency disappears.

To introduce money into the economy is, in essence, to reduce the constraints. In particular when all households and all firms are endowed with a sufficient amount of money, inefficiency will be eliminated. If the above happens, i.e., every agent has enough initial wealth to guarantee an efficient equilibrium, then there is no need for any agent to borrow money (sell bonds). Therefore in an efficient equilibrium all bond markets must be inactive. (See [5] for a more rigorous discussion.)

In our economy, there is an additional source of inefficiency. All agents are faced not only with a sequence of *budget* constraints but also with a sequence of *expenditure* constraints. If in any period, an agent fails to

accumulate sufficient amount of money, he cannot purchase planned (or notional) demand for goods. However, again if he has a sufficient amount of initial money holding, he is able to preserve sufficient cash balance in every period to back his notional demand. The above result is a particular case of Hahn's argument on Grandmont and Younes' [3] assumption when $k = 0$ (see [5 pp. 459–460]).

Therefore, applying Hahn's result in our economy, one can easily prove

THEOREM 2. *There exists $\bar{\mu}$ such that for all $\mu \leqslant \bar{\mu}$, there exists an efficient equilibrium with $p \in P_\mu$. In an efficient equilibrium, all bond markets are closed (or inactive).*

So far, we have used a word "efficient" without its definition. Efficiency in the above sense corresponds to the usual Pareto efficiency. In other words, a state of the economy is called efficient if a state is maximal for the Pareto ordering among the set of states which satisfy storage and transaction technologies (i.e., which satisfy (1), (2), (6), (7), (D.3)–(D.5)). However, we are dealing with an intertemporal allocation. In an intertemporal model with a decentralized information, all economic agents must satisfy a sequence of budget constraints. In other words as in the case of the second best, we have an additional institutional constraint as well as technical constraints. Thus, it would be more appropriate to define efficiency in terms of all these constraints.

Let

$$F = \{\langle a^h\rangle_{h=1}^H, \langle b^i\rangle_{i=1}^I \mid \text{there exists } p \text{ in some price system such that } a^h \in B^h(p) \text{ for all } h,\ b^i \in F^i(p) \text{ for all } i, \text{ and (D.3)–(D.5) are satisfied}\}.$$

Then a state of the economy, $(\langle a^h\rangle_{h=1}^H, \langle b^i\rangle_{i=1}^I)$, is defined to be *institutionally efficient* if $\langle a^h\rangle_{h=1}^H, \langle b^i\rangle_{i=1}^I$ is maximal for the Pareto ordering in F with respect to some price system.

Note that a Pareto efficient equilibrium allocation is always institutionally efficient. Therefore, if we allow the price system, P_μ, to vary for any $\mu < 0$, the following corollary holds.

COROLLARY. *There exists an institutionally efficient equilibrium if price level, μ, is arbitrary.*

In other words, with a fixed price level, μ, there may exist an institutionally efficient equilibrium which is not Pareto efficient, but by varying the price level we may in fact achieve a Pareto efficient equilibrium which

is institutionally feasible. This result is fundamentally different from that in an economy without money since an equilibrium in an economy without money may be institutionally efficient, for any price system, but not Pareto efficient.

6. Conclusion

As I (and Heller and Hahn) have stressed many times, most of our peculiar conclusions come from our particular terminal condition. One immediate query, therefore, would be whether we ultimately need such an assumption. The answer in "no." Starr [14], Kurz [8], Okuno [10, Chap II] used different conditions to support an equilibrium. Also, more importantly, we can find an equilibrium without any arbitrary assumptions when an economy is of infinite horizon. (Interested readers should refer to [6, 11].

Also, if expectation formation is not of the self-fulfilling type, there may exist a monetary equilibrium. For example, if one consumer thought that the economy does not end at T, and at $T + 1$ (which exists only in his imagination) the price of money is positive, we would find a monetary equilibrium with all the money collected by him at T. More realistic models of this nature have been studied extensively under the heading of temporary equilibrium (see, for example, [2]). In all of the above cases, there is no multiplicity of equilibrium and, consequently, the efficiency argument above is no longer pertinent.

Nonetheless, it seems that the above efficiency result is important in understanding how money improves the liquidity of economic agents and how different liquidity positions relate to the level of economic welfare. An interesting related topic is, therefore, how assets with different liquidity affect the economy. I will leave this problem for another occasion.

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Unequal Inequalities. I

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INTRODUCTION

Many people consider the reduction of economic inequalites as a basic aim of society. Such ideas are, however, largely nonoperational, sterile, and even meaningless, as long as what is called inequality is not stated with precision. This is so because, as well appear below, different measures of inequality give widely different, and even opposite, results. Such policy which diminishes some apparently reasonable measure increases other ones. And I can take any country and prove that in some period (whatever it is) inequality has increased or decreased in it, or any two countries and prove that inequality is higher in the one or in the other, by choosing different inequality measures, all of which would probably seem good and valuable at first sight. Furthermore, this embarrassing situation happens not only when we consider the complex social, or socioeconomic, or economic position of persons, or even their consumption–labor bundles over time or even timeless, but even in the simplest case where the variable is a unique magnitude (and, even, a quantity) per person, such as its income or its wealth.[1]

It thus seems essential to appraise the economic and, if we dare say, ethical, implications of the inequality measures, and to build measures embodying the economic and ethical properties we feel inequality means. Several economists, among them Pigou [1], Dalton [2, 3], Taussig [4], Cannan [5], and Loria [6], have stated their views about whether some well-defined changes in incomes increase or decrease their inequality; these are implicit properties of inequality measures or indices. Dalton also added more specific but less valid structures for them (see [7, Sect. VI]). A number of other properties and their relations were introduced in

[1] I wish to thank Professor A. Atkinson for having provided the motivation for writing down these ideas, invaluable linguistic corrections, very valid (if not met) criticism, and important ideas which will be mentioned in the text.

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[7, Sects. VI and VII],[2] along with some explicit inequality indices. Atkinson ([8]) then presented anew some of these relations, singled out one of these indices, and initiated the empirical application of it, which was followed in a number of other studies. This will be the starting point of this paper, Part II of which will appear in the next issue of this journal.

In Part I, we first consider two specific measures labeled "rightist" and "leftist," their opposite and common properties (Sect. I), and we derive them axiomatically from sets of basic properties (Sect. II). They happen to be extreme cases of a more general, "centrist," class of measures (Sect. III). We then discuss the question of "per person" versus "per pound" inequality (Sect. IV), and we find how these measures vary with their parameters and with equal absolute and relative variations in all incomes (Sect. V). Part II will then consider broader measures and further properties. Inequality invariance under these two latter kinds of income variations brings out the difference between the rightist and leftist measures; but if we abandon some property common to both ("independence"), these invariances will be reconciled by a general class of measures, among which are the standard deviation and coefficient of variation (Sect. VI). Pigou and Dalton's "principle of transfers" (i.e., a transfer of one pound from a richer person to a poorer one diminishes inequality) will be extended into a "principle of diminishing transfers" (Sect. VII), and relations between inequality measures and Lorenz and "concentration" curves will be analyzed (Sect. VIII). We will then turn to the important question of how inequality is affected by addition of incomes (or taking from them, or growth in them), and to a law of "diminishing returns in equality" (Sect. IX). And the lumping together of different populations also implies a specific relation between inequality measures (Sect. X). Section XI will finally present the relations between the most general properties of inequality measures (and of the "social welfare functions" they imply).

The properties and results are presented here for income distributions. But almost all of them interestingly apply to distributions of other things. Most notable is their immediate interpretation in terms of risk and uncertainty analysis, which we do not state explicitly because it is thoroughly straightforward and would be cumbersome (furthermore, a part of this translation appeared in [9]). The application to the com-

[2] Unfortunately, a systematic misprint in the beginning of Section VI of this paper makes its reading difficult: The preference relations $\succ$ and $\succcurlyeq$ have everywhere been written as $>$ and $\geqslant$, and they sometimes appear in the same definitions and theorems as "larger than." This paper was a contribution to a conference (I.E.A., Biarritz, 1966), and the author was not presented with the galley proofs of the publication to check.

parison of growth paths (distribution of dated magnitudes) is also fruitful. And the results can of course also be applied to other interpersonal distributions (wealth, consumption, etc.). The next step consists in passing from unidimensional to multidimensional (or multivariate) distributions; this is done in [10], which also reviews the history of the literature about the various economic applications of the ideas of "Schur-concavity," "rectifiance," "isophily," "concentrations preferences," "averages preferences," "stochastic dominance," "transfers principle," etc.

I. Rightist vs Leftist Measures of Inequality

I.a. *Two Conceptions of Inequality*

Inspired by a recent paper by Atkinson [8], a rapidly increasing number of studies use the index of inequality

$$I_\epsilon = 1 - \left[\sum (x_i/\bar{x})^{1-\epsilon} f(x_i)\right]^{1/(1-\epsilon)},$$

or

$$I_\epsilon = 1 - \prod (x_i/\bar{x})^{f(x_i)},$$

which is its limit when ϵ tends to zero ($\prod$ being the product sign), to measure inequalities in the distribution of income. x_i is then an income level, $f(x_i)$ is the proportion of persons whose income has this level ($\sum f(x_i) = 1$), $\bar{x}$ is average income ($\bar{x} = \sum x_i f(x_i)$), ϵ is a nonnegative coefficient (if $\epsilon = 0$, $I = 0$, if ϵ tends to infinity, I tends to $1 - (\underline{x}/\bar{x})$, where $\underline{x}$ is the smallest of the x_i's); $I_\epsilon = 0$ when there is no inequality (all x_i's are equal and equal to $\bar{x}$), and only in this case if $\epsilon > 0$. Professor Atkinson himself provides in the same paper an extremely interesting application of this index to an intercountries comparison of income distribution inequalities.[3] Jakobsson and Normann have used this index to analyze the redistributive effects of tax policies in Sweden [11]. A very important work by Bruno and Habib uses this index to evaluate the effect of income tax-transfer schemes on income inequality, and apply it to data of Israel [12].[4] Another very important work by Bruno uses it to measure the effects of education policies on income inequality, with, again, numerical application with data from this country [13]. This index

[3] It should be recalled that Professor Atkinson also discusses properties of other measures of inequality in this article (among them what will be called I_t below).

[4] Their qualitative results remain the same, however, for the other measures of inequality discussed below.

is also being used for figures of Canada and of various other European countries. It thus is timely to recall the social implications of this measure and its alternatives.

In May 1968 in France, radical students triggered a student upheaveal which induced a workers' general strike. All this was ended by the Grenelle agreements which decreed a 13% increase in all payrolls. Thus, laborers earning 80 pounds a month received 10 pounds more, whereas executives who already earned 800 pounds a month received 100 pounds more. The Radicals felt bitter and cheated; in their view, this widely increased incomes inequality. But this would have left unchanged an inequality index I_r computed according to the above formula.

This is so because this formula implies that, if all incomes x_i are multiplied by the same number, I_r does not change. In other countries (I have been quoted examples from England and The Netherlands), trade unions are more clever and often insist on equal absolute, rather than relative, increases in remuneration, so as to avoid the above effect. And I have found many people who feel that it is an equal absolute increase in all incomes which does not augment inequality,[5] whereas an equiproportional increase makes income distribution less equal or more unequal—and these were people of moderate views. Now, the index

$$I_l = (1/\alpha) \log \sum e^{\alpha \cdot (\bar{x} - x_i)} f(x_i)$$

does not change when all incomes x_i are increased by the same amount. $\bar{x}$ and $f(x_i)$ have the same meaning as previously, α is a nonnegative parameter ($I_l = 0$ if $\alpha = 0$, and when α tends to infinity I_l tends to $\bar{x} - \underline{x}$, i.e., the gap between average and minimum income); $I_l = 0$ when there is no inequality (all x_i's are equal and equal to $\bar{x}$), and only in this case if $\alpha > 0$.

When all incomes x_i are multiplied by the same number, whereas I_r does not change, I_l is multplied by this number. Therefore, if we study variations of I_l over time in an inflationary country, we must call x_i the real incomes, discounted for inflation; or if we make international comparisons of I_l, we must use the correct exchange rates. This need not be done if we use I_r. But these problems are exactly the same ones which are traditionally encountered in the comparisons of national or *per capita*

[5] The topic was an equal increase in all incomes rather than an equal decrease in them. But it is the first point which is relevant in our progressive societies. Anyway, all that is said here is that it is no less legitimate to attach the inequality between two incomes to their difference than to their ratio. One view must not be judged from the other's prejudice. The term "leftist" used below must not be taken too literally: the measure corresponds to this view of society in very important cases, not in all imaginable ones.

incomes ($\bar{x}$), and they can be given the same traditional solutions. Anyway, convenience could not be an alibi for endorsing injustice.

These two measures are reconciled, in the sense that they are equal to each other for all income distributions, if and only if ϵ and α are zero, which entails $I_r = I_l = 0$ for all distributions, that is, when no inequality is felt (this is also a necessary and sufficient condition for each measure to remain the same for all distributions). When ϵ and α tend to infinity, both measures classify distributions with equal $\bar{x}$ according to $\underline{x}$. Of course, when all x_i's are equal, these two measures are equal whatever ϵ and α since they are zero. Otherwise, the very properties mentioned (I_r invariant when all incomes are multiplied by the same number, I_l invariant when the same amount is added to all incomes) show that these measures, and more significantly their variations over time or societies (nations, regions, professions, etc.), will differ widely.

The economic literature is, of course, relatively rich in opinions about the effects of equal or equiproportional variations in incomes on the inequality of their distribution. They roughly tend to support Abba Lerner's contention that economic science tends to shift its servents to the right. For instance, Taussig [4, p. 485] feels that a variation of all incomes in the same proportion does not change inequality; this is I_r's property. Loria [6, p. 369], Cannan [5, p. 137], and Dalton [2, 3] feel that an equal addition to all incomes decreases inequality; I_l of course does not satisfy this condition, whereas it will be shown in Section V below that I_r does. For Dalton [2, 3] again, an increase of all incomes in the same proportion increases inequality; I_r of course does not satisfy this property, whereas it will be shown in Section V below that I_l does. From this we see that Dalton would have liked neither I_r nor I_l. But the "centrist" measures of inequality presented in Section IV below might suit his taste, since Section V will show that they satisfy both his requirements. Finally, [7, Sects. VI and VII] presents a systematic exposition of properties of inequality measures (there called "unjustice") and of their implications; in fact, a large part of the present article is only an elaborate exposition of some of its points.

I.b. *Differences and Similarities between These Two Measures*

The two initially mentioned properties may be called "inequality invariance under equiproportional, resp. equal absolute, variations" of incomes. The first one also says that I_r is an "intensive" magnitude in the physicists' sense.[6] Measures I_r and I_l differ from these viewpoints, but they also have in common several interesting properties, whereas

[6] This property was thus called "intensive justice" in [7].

there exist still other significiant properties according to which these measures differ and which give an interest to one or the other. We shall now mention the properties in these two categories; that they hold will be obvious for some but not for others. In the latter case, the proofs will be delayed to the following sections. Also, this presentation will exclude the trivial cases $\epsilon = 0$ and $\alpha = 0$,

Among the properties I_r and I_t have in common, they are zero when all incomes are equal and positive otherwise (ϵ and α being positive). In both these measures, too, incomes are distinguished only by their level, and not in any other way by their recipients; these measures in this sense respect principles of "equal treatment of equals," or "horizontal equity," or "impartiality" (the label we shall keep); in other words, a permutation of incomes between income recipients does not affect the inequality of the distribution, that is, the inequality measure is a symmetric function of incomes.

Both measures also satisfy a fundamental series of equivalent properties. One is that a transfer of one pound from a richer person to a poorer one decreases inequality; this is Dalton's "principle of transfers" ([7]'s "rectifiance," also previously mentioned by Pigou in [1]). Another one is that inequality is lower when the Lorenz curve is everywhere higher, for two distributions of the same total income (or, more generally, [7]'s "isophily," which also applies when the latter condition does not hold and consists in comparing concentration curves, i.e., the sums of the jth smallest incomes for all j's; cf. Part II, Sect. VIII). Other properties equivalent to these two are presented in [7, 8, 10]. In mathematical terms, all this just means that I_r and I_t are Schur-convex functions of the set of individual incomes.[7]

[7] We say that an inequality measure function $I(x)$ is *rectifiant* when $x_i < x_j$ and $0 < h \leqslant (x_j - x_i)/2$ imply that the replacement of x_i by $x_i + h$ and of x_j by $x_j - h$ decreases I, for all pairs i, j and all x's. This is strict rectifiance; for weak rectifiance, I would not be increased by this transfer. If I is symmetric, the condition on h can equivalently be replaced by $0 < h < x_j - x_i$. If I is differentiable, weak and strict rectifiance are respectively equivalent to $x_i < x_j$ implying $\partial I/\partial x_i \leqslant \partial I/\partial x_j$, and $\partial I/\partial x_i < \partial I/\partial x_j$ almost everywhere. Taking an h tending to zero shows the necessity, applying to successive small variations in h shows the sufficiency. Rectifiance plus symmetry is equivalent to $I(Bx) \leqslant I(x)$ for all x and all bistochastic matrix B (i.e., $B = [b_{ij}]$, $b_{ij} \geqslant 0$, $\sum_i b_{ij} = \sum_j b_{ij} = 1$, for all i, j's) for the weak form, and to a similar relation with strict inequality if Bx's coordinates are not a permutation of x's for the strict one.

These properties were used in [7 Sects. VI and VII] in justice theory and in [10, p. 110–119] in risk and uncertainty theory and portfolio choice analysis (x_i then being wealth in eventuality i with a definition of the i's such that they all have the same probability of occurrence), both for I and for evaluation functions $V(x)$ with a reversal of the inequality signs—these properties of these two functions are equivalent when

A refinement of this "transfers principle" is that the decrease in inequality obtained by a one pound transfer to an income lower by a given amount is larger when these two incomes are smaller. Both I_r and I_l satisfy this "principle of diminishing transfers" (cf. Part II, Sect. VII).

Both measures also satisfy another of Dalton's principles, the "principle of proportionate additions to persons," i.e., the inequality does not change when the numbers of income recipients in each income class vary in the same proportion (hence, total population also varies in this proportion). For instance, if society is exactly duplicated, the numbers of persons in each income class and in total being doubled, inequality in the income distribution remains the same. The reason is that the numbers of persons only enter in I_r and I_l through the $f(x_i)$'s which are their *ratios* to total population.

But, on the other hand, I_r and I_l differ for the following interesting properties, although we shall find that they still have common points in some of these questions.

One property refers to the addition of incomes of several kinds (the "composition" of their distribution, in statistical jargon). For instance, how does income inequality relate to inequalities in earned and unearned incomes? Or, how does the inequality in income increments from the last to the present year make income inequality vary? Or, how does fiscal inequality transform before tax into after tax income inequalities? Or, how does the inequality in government transfers affect the inequality in incomes? Or, how do the inequalities in "private income" and in the individuals' benefits from government activities combine to determine the inequality in the distribution of a more general income concept encompassing both? Generally, if an income distribution is a sum ("composition") of several distributions, what relation is there between its inequality and its components'? The property, which holds for I_r, is the following: *Inequality in a sum of distributions is lower than the sum of these distributions' inequalities* weighted by total or average incomes in each, except when these distributions are proportional (in which case all the inequality measures mentioned are equal). If we consider another index

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the relations between V and I are the ones presented below (for the risk theory case, I would be a risk-premium or an insurance premium). The equivalence between rectifiance and symmetry on the one hand and the property mentioned on the other hand was first proved by Ostrowski in the differentiability case ([14, Theorem VII]). Rectifiance plus symmetry is Schur-convexity of I or Schur-concavity of V, since functions satisfying the equivalent property for the concavity case and positive variables were introduced by I. Schur in his generalization of Hadamard, Julia, and Parodi's inequalities derived from positive Hermitian forms [15, 16, 17] (Schur also showed the necessity of the marginal rectifiance condition).

$\bar{x}I_r$ or $n\bar{x}I_r$ (n being the total number of income recipients), the property says that inequality in the sum is lower than the (unweighted) sum of inequalities, except when the constituting distributions are proportional, in which case inequality in the sum is the sum of inequalities (cf. Part II. Sect. IX). This property may be called "nonincreasing inequality under additions of distributions," or "subadditivity" of the inequality measure. It might be an appropriate description because one may feel that the composition of differently unequal distributions evens out the inequality in some sense and in some degree. No such property holds for I_l.

There is, on the other hand, another, somehow related, property, which I_l has and I_r has not, although I_r has it if we consider only distributions of the same total amount, and $\bar{x}I_r$ and $n\bar{x}I_r$ have it. It appears when we consider several additions of the same distribution to another given one (for instance a sequence of income increments), and more generally what is "added" could now include some decrease (a "negative addition") for some (or all) incomes. The property then says that *inequality increments are larger and larger* (*or decrements smaller and smaller*). There is a limiting exception which we already know: When added incomes are the same for all individuals, I_l does not change (also, when they are proportional to the initial distribution, the increments in $\bar{x}I_r$ and $n\bar{x}I_r$ are constant). For I_l, if in particular we take as the initial distribution the one in which all incomes are zero (the "null distribution"), this property shows that *a doubling of all incomes more than doubles inequality* if it is not zero (and a similar property for a multiplication by any positive number λ). The same property can be expressed on operations of proportional bridging of gaps between distributions: If we bridge half the gap between two distributions, for all incomes, the resulting inequality is less than the (arithmetic) average of the two respective inequalities. The exception for I_l then is the case when the gap is the same for all incomes: The two initial distributions and the halfway one all have the same inequality (for $\bar{x}I_r$ and $n\bar{x}I_r$, the exception is the case when the two distributions are proportional: inequality of halfway is halfway inequality). In particular, bridging half the gap from an unequal distribution to an equal one diminishes inequality by more than half (the equal one could for instance be a completely equalizing redistribution of the initial one). These properties constitute a "law of increasing average and marginal inequality," or of "diminishing returns in equality." Mathematically, they are a case of convexity of the inequality measure as a function of all incomes (cf. Part II, Sect. IX).

A consequence of these properties is that, for I_l, for $\bar{x}I_r$, and also for I_r when we consider distributions of same total or average income, if we bridge part of the gap between two distributions, in the same pro-

portion for all incomes (for instance each income is the arithmetic average of what it is in these two distributions), the inequality is not larger than the largest one for the two initial distributions (this is a property of "quasi-convexity"; cf. Part II, Sect. IX).

Instead of adding persons' incomes, we may add persons. That is, we may consider lumping together several populations, or, conversely, partitioning a society into several subpopulations. How does total inequality relate to the inequalities of the various subpopulations? How does European inequality relate to inequalities in the various European countries? This question requires an extension of Dalton's "principle of proportionate additions to persons," which applies to the special case when all constituting populations are identical except for the multiplication of the number of people in each income class and in total by some number. It must generally be untrue that if all constituting populations present the same degree of inequality, the global population also shows this degree of inequality. Since if each of the constituting populations has a perfectly equal income distribution, but personal incomes differ from one to the other, the distribution will not be equal in the aggregated society. In brief, inequality in the global population will have two sources: inequalities *within* the constituting populations, and inequalities *between* them. From this viewpoint, I_r and I_l are more similar than different. For each of them, total inequality is not lower than a weighted sum of the subpopulations' inequalities, but the weights are the number of persons for I_l and total incomes for I_r. For both, if the subpopulations present the same degree of inequality, the aggregated population also has this degree if and ony if the subpopulations also have the same average income (cf. Part II, Sect. X).

We have noted earlier that both I_l and I_r have the same relation to the disposition of Lorenz curves for distributions of the same total amount. But when the total amounts of the distributions differ, so do these measures in this respect. With I_r as the measure, a distribution whose Lorenz curve is nowhere under and somewhere above another distribution's curve has a smaller inequality, whatever their total or average incomes. This is not so for I_l. What can be said for this measure is that it is lower if both the Lorenz curve is nowhere under and somewhere above, and total or average income is not larger (cf. Part II, Sect. VIII).

Finally, the formulas show that we cannot write negative incomes (x_i) into I_r, whereas this poses no problem in I_l. Now, business failures make negative incomes a reality.

I.c. *Similar Properties of Other Inequality Measures*

One then naturally wonders how the other measures of inequality fare in front of these properties. We shall see this in detail for the "centrist"

measures presented below, which are intermediate cases between I_r and I_1 and contain them as special limiting cases. As for the more conventional measures, an interesting result is that the standard deviation of incomes $\sigma = [\sum (x_i - \bar{x})^2 \cdot f(x_i)]^{1/2}$, and its division by average income which is the coefficient of variation $\sigma/\bar{x}$, satisfy almost all properties in the following sense (some of these properties are obvious, and the other ones will be proved in the following sections).

σ and $\sigma/\bar{x}$ are zero at equality and positive elsewhere. They satisfy the "impartiality" or symmetry property. The multiplication of all incomes by the same number does not change $\sigma/\bar{x}$. The addition of the same amount to all incomes does not change σ. σ and $\sigma/\bar{x}$ satisfy Dalton's "principle of transfers" and its equivalent properties, such as lower value for distributions of the same amount with uniformly higher Lorenz curves. Furthermore, for $\sigma/\bar{x}$ this latter property holds even if the distributions do not have the same total income. However, their decrease for a one pound transfer to an income smaller by a given amount is proportional to this amount and thus independent of the income levels. Both satisfy Dalton's "principle of proportionate additions to persons." The σ of a sum (composition) of nonproportional unequal distributions with $n > 2$ is lower than the sum of their σ's. σ, and $\sigma/\bar{x}$ given $\bar{x}$, satisfy the "diminishing returns properties" because they are convex functions of the set of incomes: Successive identical variations in incomes give these measures increasing increments, or decreasing decreases, except, for σ, when they are proportional to the initial distribution or when the variation is the same for all incomes; the measure for halfway between two distributions neither proportional nor with the same difference for all incomes is less than the average of their measures; to bridge half the gap from an unequal distribution to an equal (nonnull) one diminishes the measure by more than a half. Furthermore, the values of σ and $\sigma/\bar{x}$ for a union of populations are higher than the sums of their values for these populations, respectively weighted by the numbers of persons and total incomes, except when all these populations have the same σ's and $\bar{x}$'s and $\sigma/\bar{x}$'s, in which case the union's corresponding values are also the same. Finally, negative incomes can be perfectly included in the computations of σ and $\sigma/\bar{x}$ (we assume positive average $\bar{x}$, however).

II. An Axiomatic of These Two Measures

To really see what these measures imply, it is necessary to build an axiomatic of them, i.e., to find for each one a set of properties which are equivalent to its adoption. If these properties are small in number and as

intuitive as possible, they will display the implicit assumptions made by this choice.

We shall use the following notations. i will now be the index of an income recipient unit (called a "person"), rather than of an income class (but the difference is only formal). There are n i's. x_i is i's income. x is the n-vector of the x_i's. $\bar{x} = (1/n) \sum x_i$ is the average income. e is the n-vector each coordinate of which is 1. λ and μ are scalars (real numbers). A measure or index of inequality is a function $I(x)$. We consider, for this function, the following properties.

II.a. *Properties*

(1) When all incomes are equal, $I = 0$. This is a natural requirement for a measure of inequality. It can be written as $I(\lambda e) = 0$ for all admissible λ's.

(2) When incomes are unequal, $I > 0$. This also is a natural requirement for I. At the limit, we would have $I \geqslant 0$.

(3) "Impartiality": I is left unchanged by a permutation of the x_i's, i.e., it is a symmetrical function of the x_i's. This property is unavoidable as long as income recipients are not distinguished by anything else but their income. It is akin to the old "equal treatment of equals" and "horizontal equity" principles of public finance.

(4) "Transfers principle" (Pigou, Dalton, etc.): The transfer of a pound from a richer person to a poorer one decreases inequality. More precisely, this is I's "rectifiance": $(\partial I/\partial x_i) - (\partial I/\partial x_j)(x_i - x_j) > 0$ if $x_i \neq x_j$. At the limit, this transfer could have no effect and we would write the inequality as $\geqslant$ for all x_i and x_j's (that is, rectifiance can be strict or weak).

(5) The addition (or subtraction) of the same amount to all incomes does not change I. That is, $I(x + \mu e) = I(x)$ for all admissible μ's and x's.

(6) "Intensive inequality": The multiplication of all incomes by the same scalar (an equiproportional variation in all incomes) does not change I. That is, $I(\lambda x) = I(x)$ for all admissible λ's and x's.

(7) $[\partial(\bar{x} - I)/\partial x_i]/[\partial(\bar{x} - I)/\partial x_j]$ does not depend upon x_k for all k's either i or j, all i's and j's, and all x's.

(8) $\{\partial[(1 - I)\bar{x}]/\partial x_i\}/\{\partial[(1 - I)\bar{x}]/\partial x_j\}$ does not depend upon x_k for all k's either i or j, all i's and j's, and all x's.

To evaluate (7) and (8), we observe that $\bar{x} - I$ or $(1 - I)\bar{x}$ can be considered as indices of social welfare, with two different concepts of

inequality measurement, one which is in some sense absolute, and the other one which is in some sense relative. Those two concepts will be discussed in detail in Section IV below. Presently, it suffices to remark that in the first case inequality is the number of pounds sterling which must be deducted from average income in order to obtain a measure of global welfare which takes into account both the average per capita income and the inequality in distribution, whereas in the second case inequality is the proportion of average income which must be deducted from it in the same intent. It may be said that in these two concepts inequality respectively cuts down or scales down average income to obtain a measure of social welfare. Then (7) and (8) say: If we think that one pound more to a person who earns 90 pounds per month increases social welfare as much as one pound and a half more to a person who earns 80 pounds per month, this opinion does not depend upon the incomes of other people; and this must hold for all pairs of incomes and all distributions. This property may be labeled "welfare independence." Well-known results in economics show that it is equivalent to saying that there exists a function of this social index which can be written as a sum of functions of each of the x_i's. In other words, social welfare is of course an ordinal concept, but we consider here that its ordinal index (i.e., defined up to an increasing transformation) has two interesting specifications: one is an absolute specification, measured in pounds sterling, which is $\bar{x} - I$ or $(1 - I)\bar{x}$, and the other one is a cardinal specification (defined up to a linear transformation) which is a sum of functions each of only one x_i.

II.b. *Result*

Now, the result[8] is:

(1°) (a) Properties (1), (3), (5), and (7) hold altogether if and only if I is of the form

$$I_t = (1/\alpha) \log \left[(1/n) \sum e^{\alpha \cdot (\bar{x} - x_i)}\right].$$

(b) These properties, plus (2) or (4) (which can thus replace (3)), hold altogether if and only if, furthermore, $\alpha > 0$ ($\alpha \geqslant 0$ if we choose the weak form of (2) or (4)).[9]

[8] It is a straightforward application of [7, theorems 13–17] (where the proofs are not reported) or of the equivalent results in the theory of choice under uncertainty (more specifically, of portfolio choice since the random variable is a unidimensional quantity) first presented (at least for (1°)) in [10, p. 128].

[9] Sections IX and XI in Part II will show that conditions (2) or (4) can also be replaced

(2°) (a) Properties (1), (3), (6), and (8) hold altogether if and only if I is of the form

$$I_r = 1 - \left[(1/n) \sum (x_i/\bar{x})^{1-\epsilon}\right]^{1/(1-\epsilon)}$$

or

$$I_r = 1 - \prod (x_i/\bar{x})^{1/n}.$$

(b) These properties, plus (2) or (4) (which can thus replace (3)), hold altogether if and only if, furthermore, $\epsilon > 0$ ($\epsilon \geqslant 0$ if we choose the weak form of (2) or (4)).[10]

II.c. *Proofs*

Here, as in the rest of this article, the reader who does not feel like going through a bit of mathematics can skip the proofs or the mathematical remarks he does not feel at home with. These are not however to be considered as an "appendix" (and eventually materially reported in one), because their development generally shows significant and interesting economic and logical properties.[11]

We first need a preliminary remark. We shall meet functions of the x_i's of the form $\varphi^{-1}[(1/n) \sum \varphi(x_i)]$ where φ is a function of one variable. This is the special case of the "generalized mean" of the x_i's with function φ where all the weights $1/n$ are equal. Now, we shall want to use theorems established for the general form $\varphi^{-1}[\sum q_i \varphi(x_i)]$ where the q_i's are any weights ($q_i \geqslant 0$ for all i's and $\sum q_i = 1$) and about properties valid for all q_i's. Such a property is also valid for the special case where all q_i's are equal and thus are $1/n$. But the important point is that the reverse is also true if the property for equal weights is valid for all n's and all admissible x_i's. This is so because one can consider m (let us say) i's with equal x_i's

in (1°)(b) and (2°)(b) by any of four other properties which are *convexity, quasi-convexity, constant-sum convexity, constant-sum quasi-convexity of* I_1 *or of* $\bar{x}I_r$, or *constant-sum convexity, constant-sum quasi-convexity of* I_r. Thus, with the properties of (1°)(a) or (2°)(a), all these properties are equivalent to each other and to $\epsilon > 0$ or $\alpha > 0$ (>0 for the weak forms); in particular, the transfers principle or merely nonnegativity (or positivity out of equality) then implies the convexities mentioned.

[10] See Footnote 9.

[11] This warning to the less mathematically oriented readers must be supplemented by another one to the more mathematically oriented ones. We want to keep the mathematical apparatus to the lowest possible level, and to this end we do not mention explicitly some of the properties' conditions when this gap can be easily and straightforwardly filled in by the reader; this often happens in particular for the domains of variation of the variables.

and lump them in the equal weights case so as to form a term $(m/n)\,\varphi(x_i)$, and m/n can be made to approach any q_i as close as we want to, by increasing the number n if necessary. We note that this m/n is nothing but the $f(x_i)$ of Section 1.

We now prove the result (1°) (a). Let us call here $\bar{x} = \bar{x} - I$. This $\bar{x}$ is thus a function of x, as are $\bar{x}$ and I. Property (7) shows that it can be written as a function of a sum of functions of only one x_i each. And $\bar{x}$ is a symmetrical function of the x_i's since $\bar{x}$ and I are (property (3)). These functions of each x_i must thus be the same functions. We can then write

$$\bar{x} = F\left[(1/n) \sum \varphi(x_i)\right].$$

When all x_i's are equal, they are equal to $\bar{x}$, and so is $\bar{x}$ from property (1). Then,

$$\bar{x} = F[\varphi(\bar{x})]$$

for all $\bar{x}$'s, which means that $F = \varphi^{-1}$, the inverse function of φ. $\bar{x}$ is thus of the form

$$\bar{x} = \varphi^{-1}\left[(1/n) \sum \varphi(x_i)\right].$$

From a well-known theorem (see for instance [18, Theorem 83]) and the above preliminary remark, two functions φ give the same value to $\bar{x}$ for all x's in this expression if and only if they are in a linear relation. We thus choose to replace $\varphi(y)$ (where y is the current variable) by $\varphi(y) - \varphi(0)$, so that, with this new φ, $\varphi(0) = 0$. Since adding the same constant μ to all x_i's transforms $\bar{x}$ into $\bar{x} + \mu$, property (5) means that it transforms $\bar{x}$ into $\bar{x} + \mu$, that is,

$$\bar{x} = \varphi^{-1}[(1/n)\,\varphi(x_i + \mu)] - \mu.$$

If we consider the new function defined by $\psi(y) = \varphi(y + \mu)$, this expression is

$$\bar{x} = \psi^{-1}\left[(1/n) \sum \psi(x_i)\right],$$

which shows that the functions φ and ψ are in a linear relation, which can be written as

$$\psi(y) = \varphi(y + \mu) = a(\mu) \cdot \varphi(y) + b(\mu),$$

where a and b, constant in y, are functions of μ, and $a(\mu) \neq 0$. Then, $y = 0$ gives $\varphi(\mu) = b(\mu)$, so that

$$\varphi(y + \mu) = a(\mu) \cdot \varphi(y) + \varphi(\mu).$$

Interchanging y and μ gives

$$\varphi(y + \mu) = a(y) \cdot \varphi(\mu) + \varphi(y),$$

and the equality of the right-hand sides can be written as

$$[a(y) - 1]/\varphi(y) = [a(\mu) - 1]/\varphi(\mu),$$

which shows that this expression is a constant c, so that $a(y) = c\varphi(y) + 1$. Then

$$\varphi(y + \mu) = c \cdot \varphi(y) \cdot \varphi(\mu) + \varphi(y) + \varphi(\mu).$$

If $c = 0$, this expression reduces to

$$\varphi(y + \mu) = \varphi(y) + \varphi(\mu),$$

which implies that $\varphi(y) = k \cdot y$ where k is a constant. If $c \neq 0$, writing $\chi(y) = c\varphi(y) + 1$ shows that

$$\chi(y + \mu) = \chi(y) \cdot \chi(\mu),$$

the general solution of which is $\chi(y) = e^{-\alpha y}$ where α is a constant, and the corresponding φ is $\varphi(y) = (e^{-\alpha y} - 1)/c$ which is equivalent to $e^{-\alpha y}$ to compute $\bar{x}$:

$$\bar{x} = -(1/\alpha) \log \left[(1/n) \sum e^{-\alpha x_i}\right],$$

and I is

$$I_i = \bar{x} - \tilde{x} = (1/\alpha) \log \left[(1/n) \sum e^{-\alpha \cdot (x_i - \tilde{x})}\right],$$

where we see that the linear case is the limit of this expression when α tends to zero.

We now prove (1°) (b). $I_i = 0$ for all x's if and only if $\alpha = 0$. For $\alpha \neq 0$, $e^{-\alpha y}$ is a convex function of y. From Jensen's inequality [19] this is equivalent to $e^{-\alpha \tilde{x}} < (1/n) \sum e^{-\alpha x_i}$ for all x_i's not all equal. Thus, $\bar{x} > \tilde{x}$ out of equality if and only if $\alpha > 0$. Besides, I is Schur-convex if and only if $\bar{x} = \bar{x} - I$ is Schur-concave (the transfer does not change $\bar{x}$). And $e^{-\alpha \bar{x}}(\partial \bar{x}/\partial x_i) = (1/n)\, e^{-\alpha x_i}$ shows that the derivatives $\partial \bar{x}/\partial x_i$ have their magnitude classified in order inverse to the x_i's, which is $\bar{x}$'s strict Schur-concavity if and only if $\alpha > 0$.

Let us now prove (2°) (a). We now call $\bar{\bar{x}} = (1 - I)\bar{x}$. This $\bar{\bar{x}}$ thus is a function of x, as are $\bar{x}$ and I. Property (8) shows that it can be written as a function of a sum of functions of only one x_i each. And $\bar{\bar{x}}$ is a symmetrical function of the x_i's since $\bar{x}$ and I are (property (3)). These functions of each x_i must thus be the same functions. We can then write

$$\bar{\bar{x}} = F\left[(1/n)\sum \varphi(x_i)\right].$$

As previously shown, property (1) then implies that $F = \varphi^{-1}$ and thus

$$\bar{\bar{x}} = \varphi^{-1}\left[(1/n)\sum \varphi(x_i)\right].$$

Since multiplying all x_i's by the same scalar λ multiplies $\bar{x}$ by λ, property (6) and the present definition of $\bar{\bar{x}}$ shows that it also multiplies $\bar{\bar{x}}$ by λ. But this property, the structure of $\bar{x}$ as a generalized mean with equal weights, and the preliminary remark, give the result from a well-known theorem (see, for instance, [18, Theorem 84]): φ can be written as a power or logarithmic function $\varphi(y) = y^{1-\epsilon}$ or $\varphi(y) = \log y$. Then,

$$\bar{\bar{x}} = \left[(1/n)\sum x_i^{1-\epsilon}\right]^{1/(1-\epsilon)}$$

or

$$\bar{\bar{x}} = \left[\prod x_i\right]^{1/n},$$

and I is

$$I_r = 1 - \left[(1/n)\sum (x_i/\bar{x})^{1-\epsilon}\right]^{1/(1-\epsilon)}$$

or

$$I_r = 1 - \left[\prod x_i/\bar{x}\right]^{1/n},$$

which is the limiting case of the previous one when ϵ tends to one.

We finally Prove (2°) (b). $\bar{x} = \bar{\bar{x}}$ and $I_r = 0$ for all x_i's if and only if $\epsilon = 0$. For $\epsilon = 1$, $\bar{\bar{x}} < \bar{x}$ if x_i's are not all equal from the well-known relation between arithmetic and geometric means. Using Jensen's inequality for x_i's not all equal, we see that: If $\epsilon < 0$, the function $y^{1-\epsilon}$ is strictly convex increasing, $\bar{\bar{x}}^{1-\epsilon} > \bar{x}^{1-\epsilon}$, $\bar{\bar{x}} > \bar{x}$; if $0 < \epsilon < 1$, $y^{1-\epsilon}$ is strictly concave increasing, $\bar{\bar{x}}^{1-\epsilon} < \bar{x}^{1-\epsilon}$, $\bar{\bar{x}} < \bar{x}$; if $\epsilon > 1$, $y^{1-\epsilon}$ is strictly convex decreasing, $\bar{\bar{x}}^{1-\epsilon} > \bar{x}^{1-\epsilon}$, $\bar{\bar{x}} < \bar{x}$. Thus, $I_r > 0$ out of equality if and only if $\epsilon > 0$. Besides, I is Schur-convex if and only if $\bar{\bar{x}} = (1 - I)\bar{x}$

is Schur-concave (the transfer does not change $\bar{x}$). And $\bar{x}^{-\epsilon}(\partial\bar{x}/\partial x_i) = (1/n)\, x_i^{-\epsilon}$ shows that the derivatives $\partial\bar{x}/\partial x_i$ have their magnitudes classified in order inverse to the x_i's, which is $\bar{x}$'s strict Schur-concavity, if and only if $\epsilon > 0$.

II.d. *Relations between the Basic Properties of Inequality Measures*

The basic properties of inequality measures mentioned are not independent of each other. Although the most general properties of inequality measures will be discussed in more detail in Part II, let us mention these relations here and give a sketch of the proof of those which will not be proved under their present form. The following relations hold between these properties, with respective correspondence between weak and strict forms of the properties when this is relevant (when this distinction is meaningful for the hypotheses only, the implication is true for either form), and with the eventually required assumptions about differentiability and domain of variation.

(i) Zero at equality (1) and impartiality (3) and rectifiance (4) imply nonnegativity or positivity out of equality (2).

(ii) Independence ((7) or (8)) and zero at equality (1) and nonnegativity or positivity out of equality (2) imply impartiality (3) and rectifiance (4).

(iii) Independence ((7) or (8)) and rectifiance (4) imply impartiality (3).

(iv) Independence ((7) or (8)) and rectifiance (4) and zero at equality (1) imply nonnegativity or positivity out of equality (2).

(v) Given independence ((7) or (8)) and zero at equality (1), nonnegativity or positivity out of equality (2) and rectifiance (4) are equivalent and they imply impartiality (3).

(v) is a synthesis of (ii) and (iv). Relation (iii) shows that, with independence, rectifiance suffices to define Schur-convexity (rectifiance plus impartiality). Relation (i) will be proved in Part II, Section XIa.

We remark that properties (1), (2), (3), and (4) can be expressed equivalently on an inequality measure $I(x)$ or on $\bar{x}$. $I(x)$ or on $I(x)/\bar{x}$ (with $\bar{x} > 0$). Then, independence means that one of these functions can be written as $I(x) = x - \Phi[\sum \varphi^1(x_i)]$. Independence plus impartiality means that it can be written as $I(x) = \bar{x} - \Phi[\sum \varphi(x_i)]$. Independence plus impartiality plus zero at equality means that it can be written as $I(x) = \bar{x} - \varphi^{-1}[(1/n) \sum \varphi(x_i)]$.

This latter form is thus implied both by the hypotheses of (ii) plus impartiality (3), and by the hypotheses of (iv) if (iii) is true. Rectifiance

then means that φ is concave if it is increasing and convex if it is decreasing, and Jensen's inequality shows that these structures are equivalent to nonnegativity (weak forms) or positivity out of equality (strict forms) of I.

It remains to show that independence plus either (1) and (2), or (4), implies impartiality (3). We consider two neighboring distributions of the same $\bar{x} = \xi$. In one, all x_k are equal to ξ. The second one differs from it only by $x_i = \xi + \epsilon$ and $x_j = \xi - \epsilon$ with $\epsilon > 0$ and tending to zero. We choose a Φ increasing at the first point. To pass from the second situation to the first one is both to pass from inequality to equality and to make a transfer from a richer to a poorer person. Thus, if either (1) and (2), or (4), holds, it must not decrease $\varphi^i(x_i) + \varphi^j(x_j)$:

$$\varphi^i(\xi + \epsilon) + \varphi^j(\xi - \epsilon) \leqslant \varphi^i(\xi) + \varphi^j(\xi).$$

If φ^i and φ^j are differentiable at ξ this implies $\varphi^{i\prime}(\xi) - \varphi^{j\prime}(\xi) \leqslant 0$. Reversing the roles of i and j similarly implies $\varphi^{j\prime}(\xi) - \varphi^{i\prime}(\xi) \leqslant 0$. Therefore, $\varphi^{i\prime}(\xi) = \varphi^{j\prime}(\xi)$. Integrating and letting j go from 1 to n shows that $\varphi^k(y) = \varphi(y) + c_k$ for $k = 1,\ldots, n$. And changing the function Φ into $\Psi(z) = \Phi(z + \sum c_i)$ shows I's impartiality.

*

We thus have necessary and sufficient conditions for an inequality measure to be I_r or I_l. But there exist both intermediate measures, between these two, and other measures which synthesize most of their important properties.

III. Centrist Measures of Inequality

III.a. *The Most Specific Common Generalization*

Many people feel that an equal augmentation in all incomes decreases inequality, whereas an equiproportional increase in all incomes increases it. Dalton, for one, was of this opinion [2, 3]. Neither I_l nor I_r suits these "centrists" as a measure of inequality. They will thus feel comforted by the existence of a class of measures which has the property they require, is the closest extension of both I_r and I_l, and contains them as special, limit, cases. By "closest extension" of I_r and I_l we mean that all the properties—suitably defined—of these two measures are retained, except the criticized invariances for equal or equiproportional variations in all incomes. In particular, we retain both the general properties of inequality measures—value zero for equal distributions (1), positivity otherwise (2), impartiality-symmetry (3)—and the "welfare independence" property which is more specific to I_r and I_l; they happen to also possess the Schur-

convexity (and thus "transfers principle") and even convexity properties, as will be shown further. Both I_r and I_l are one-parameter families of functions (the parameters being ϵ and α). A generalization of both must therefore be a several-parameter family. And the form of inequality measure which is a necessary and sufficient condition for all the properties mentioned to hold will turn out to be a family of functions with only two parameters.

To see at best how these measures are intermediaries between I_r and I_l, it is enlightening to cast a geometrical glance in the n-dimensional vector space of the income distributions x. To multiply the vector x by the scalar λ is a "blowing-up" of x from the origin (if $\lambda > 1$; it is a "blowing-down" if $\lambda < 1$); this leaves I_r unchanged, and it thus multiplies $\bar{x}I_r$ by λ. To add μ to each x_i, i.e., the vector μe to x, is a shift in the direction of the vector e; this leaves I_l unchanged. This shift can be considered as a blowing-up from a point which is at infinity in the direction of vector e (or rather $-e$ if we consider oriented directions and if we want an addition, i.e., $\mu > 0$, to be the limit case of a blowing-up, i.e., $\lambda > 1$, and a subtraction, i.e., $\mu < 0$, to be the limit case of a blowing-down, i.e., $\lambda < 1$). These two transformations are both special cases of a blowing-up from a point in the x vector space. More precisely, they are special cases of the case when this point is on the "generalized bisector" which is the straight line Δ passing through the origin and bearing vector e. But if we consider a property only based upon a blowing-up operation, this latter, specific position will certainly be imposed by the symmetry property. If we call X the point, or n dimensional vector, which is the center of this blowing-up, this operation transforms vector $x - X$ into vector $\lambda \cdot (x - X)$, and therefore vector x into vector $\lambda \cdot (x - X) + X$. The property will say-that this transforms the inequality measure I into λI. More precisely, the property is

$$(9) \quad I[\lambda \cdot (x - X) + X] = \lambda \cdot I(x)$$

for some X and all admissible x's and λ's.

Let us first check that this includes the two properties studied above as special cases. When $X = 0$ (the origin), (9) is $I(\lambda x) = \lambda \cdot I(x)$, which is a property that the measure $I = \bar{x}I_r$ has. When X goes to infinity in the direction of vector $-e$, let us write $X = -\xi e$ where ξ is a scalar which tends to infinity; then $\lambda \cdot (x - X) + X = \lambda x + (\lambda - 1)\xi e$; given any number μ, we choose a $\lambda = 1 + \mu/\xi$; then $(\lambda - 1)\xi = \mu$, and when ξ tends to infinity λ tends to 1; at the limit, $\lambda \cdot (x - X) + X = x + \mu e$, and (9) becomes $I(x + \mu e) = I(x)$, i.e., property (5).

The result then is[12]:

[12] A similar structure was introduced in the analysis of choice under uncertainty for portfolio selection theory in [9, p. 129].

(a) Properties (1), (3), (7), and (9) hold altogether if and only if I is of the form

$$I_c = \bar{x} + \xi - \left[(1/n)\sum (x_i + \xi)^{1-\epsilon}\right]^{1/(1-\epsilon)}$$

or

$$I_c = \bar{x} + \xi - \prod (x_i + \xi)^{1/n},$$

where ϵ and ξ are numbers.

(b) These properties plus (2) or (4) (which can thus replace (3)) hold altogether if and only if, furthermore, $\epsilon > 0$ ($\epsilon \geqslant 0$ if we choose the weak form of (2) or (4)).[13]

III.b. *Proof and Differential Characterization*

This result is deduced from the result for I_r (Sect. II.b, 2°) by a mere change in variables from x to $x - X$ following a redefinition of I into $\bar{x}I$. More specifically, this last change transforms property (8) into property (7) and property (6) into $I(\lambda x) = \lambda \cdot I(x)$ for all admissible λ's and x's. A change of variables from x into $x - X$ transforms the latter relation into $I[\lambda \cdot (x - X)] = \lambda \cdot I(x - X)$. A new change in the definition of I from $I(x)$ into $I(x + X)$ transforms it into property (9). Equations (9) and (3) imply that X is on Δ if I is not a function of $\bar{x}$ only (since the family of x-space manifolds defined by $I(x) =$ constant then has as centers of homotheticity all points derived from X by permutations of its coordinates, and it can have only one). Besides, when x is on Δ (i.e., all x_i's are equal), $I(x) = 0$ from (1); (9) then gives $I[\lambda x + (1 - \lambda)X] = 0$, which (1) and (2) (in strict form) show to hold if and only if $\lambda x + (1 - \lambda)X$ is on Δ, which is equivalent to saying that X is on Δ, i.e., that all its coordinates are equal. Call, then, $-\xi$ the coordinates of X. Then, $x - X$ is on Δ if and only if x is on Δ, permutations of the x_i's and of the coordinates $x_i + \xi$ of $x - X$ are equivalent, the x_i's and the $x_i + \xi$'s are classified in the same order, so that properties (1), (2), (3), and (4) are conserved in the transformations of functions and variables mentioned. The results found for I_r thus imply the ones mentioned for I_c.

The second form of I_c is of course the limit of the first one when ϵ tends to zero. Besides, these forms are defined only for $x_i + \xi \geqslant 0$ for

[13] The remarks which were presented for I_r and I_l should be repeated here. With (7) and (1), (2) and (4) imply each other and imply (3). In result (b), (2) or (4) could be replaced by either convexity, or quasi-convexity, or constant-sum convexity, or constant-sum quasi-convexity, of I.. Properties of (a) thus make these properties equivalent to each other, and equivalent to $\epsilon > 0$ ($\epsilon \geqslant 0$ for the weak forms); in particular, the transfers principle or merely nonnegativity (or positivity out of equality) then implies these convexities.

all i's. (We shall remark below that $\epsilon \geqslant 1$ and $x_i + \xi = 0$ for one i at least imply $I_c = \bar{x} + \xi$.) If $\xi \geqslant 0$, this is implied by $x_i \geqslant 0$ for all i's. But if $\xi < 0$, this condition restricts the domain of variation of x to $x_i \geqslant -\xi$ for all i's. However, we shall see in Section V below that this latter case implies that if all x_i's are increased in the same proportion, $I_c/\bar{x}$ decreases; such a measure would thus be in this sense "more rightist" than I_r rather than centrist.

Evidently, if we redefine the index i so as to represent an income class, and if $f(x_i)$ is the proportion of persons who have income x_i, I_c is rewritten as

$$I_c = \bar{x} + \xi - \left[\sum (x_i + \xi)^{1-\epsilon} f(x_i)\right]^{1/(1-\epsilon)}$$

or

$$I_c = \bar{x} + \xi - \prod (x_i + \xi)^{f(x_i)}.$$

I_c is a two-parameter measure (ϵ and ξ). When $\epsilon = 0$, $I_c = 0$ whatever x (and ξ). When ξ tends to infinity, whereas ϵ remains finite, I_c tends to 0, whatever x (and ϵ). When ϵ tends to infinity whereas ξ remains finite, I_c tends to $\bar{x} - \underline{x}$, i.e., the difference between average and minimum income.

We may now check that $\bar{x}I_r$ and I_l are special cases of I_c. I_c is obviously $\bar{x}I_r$ when $\xi = 0$. And we shall show that I_c tends to I_l when both ϵ and ξ tend to infinity whereas their ratio ϵ/ξ tends to a finite value which is α. In fact, this limit can be found by the following casual remark.

We first notice that $\bar{x}I_r$, I_l, and I_c are all three of the form $\bar{x} - \tilde{x}$ where $\tilde{x}$ has the form $\tilde{x} = \varphi^{-1}[(1/n)\sum \varphi(x_i)]$ where φ is a function and φ^{-1} is its inverse function: Writing y for the current variable, $\varphi(y)$ is respectively $y^{1-\epsilon}$ or $\log y$ for $\bar{x}I_r$, $e^{-\alpha y}$ for I_l, and $(y + \xi)^{1-\epsilon}$ or $\log(y + \xi)$ for I_c. A remark of Section (II.c) above shows that the class of functions φ which give the same $\tilde{x}$ for all x's is of the form $a\varphi + b$ if φ is one of them and a and b are constants. This constitutes a two-parameter (a and b) family of functions φ, and the φ considered are twice differentiable. This family is thus characterized by its second-order differential equation which contains neither a nor b (they will be determined by the integration variables). Since φ'/φ'' does not depend upon the transformation of φ into a $\varphi + b$, this equation can be written as $\varphi'/\varphi'' = h(y)$ where h is some function, unless it is $\varphi'' = 0$, i.e., $\varphi = a'y + b'$ (a' and b' are constants) which is the φ for I_r or I_c with $\epsilon = 0$ or for I_l with α very close to zero. We find $h = -(1/\epsilon)\,y$ for $\varphi = y^{1-\epsilon}$ and $h = -y$ for $\varphi = \log y$, $h = -(1/\alpha)$ for $\varphi = e^{-\alpha y}$, $h = -(1/\epsilon)(y + \xi)$ for $\varphi = (y + \xi)^{1-\epsilon}$ and $h = -(y + \xi)$ for $\varphi = \log(y + \xi)$. Clearly, $-(1/\epsilon)(y + \xi)$ becomes $-(1/\epsilon)\,y$ when $\xi = 0$, and it becomes $-1/\alpha$ when both ϵ and ξ tend to infinity with ϵ/ξ tending to α.

This remark has shown us in passing that "welfare independence" and $I = 0$ at equality, which were seen to imply the forms $\bar{x} - \bar{\bar{x}}$ and the "mean" form for $\bar{\bar{x}}$, and φ'/φ'' of general linear form, characterize the measures I_ϵ, with the special cases of φ'/φ'' homogeneous linear for I_r and constant for I_l.[14] [15]

IV. Inequality Per Pound or Inequality Per Person?

The above remarks show that I_r and I_l can be said to differ for two reasons: They are not based upon the same function φ, and they are not derived in the same way from it since $I_l = \bar{x} - \bar{\bar{x}}$ and $I_r = (\bar{x} - \bar{\bar{x}})/\bar{x} = 1 - (\bar{\bar{x}}/\bar{x})$.

If $\sum \varphi(x_i)$ were a "social evaluation ("welfare") function," $\bar{\bar{x}}$ defined by $n\varphi(\bar{\bar{x}}) = \sum \varphi(x_i)$ would be the "equal equivalent" income ([7, Sect. VI]). Generally, if $V(x)$ is an ordinal (i.e., $V(x)$ can be replaced by any $F[V(x)]$ where F is any increasing function) "social evaluation function," the "equal equivalent" income is $\bar{\bar{x}}$ defined by $V(\bar{\bar{x}}, \bar{\bar{x}}, \ldots, \bar{\bar{x}}) = V(x_1, x_2, \ldots, x_n)$ [7, Sect. VI]. It is uniquely defined if, for all x's, $\partial V/\partial x_i \geqq 0$ ("nonmalevolence") for all i's and $\partial V/\partial x_i > 0$ ("benevolence") for at least one i. In other words, the "equal equivalent" income $\bar{\bar{x}}$ is the income level such that, if all persons had the same income at this level, society's welfare would be considered as as good (or bad) as it effectively is. $\bar{\bar{x}}$ is a function of x and a functional (function of function) of $V(\ )$. It is a special specification of the ordinal index V.[16]

[14] The consideration of general linear φ'/φ'' and its integration were first introduced into the theory of choice under uncertainty and in portfolio theory in [10, p. 129] as the generalization of the cases of proportional and constant φ'/φ''. This structure of φ'/φ'' was then used by H. Leland in his work in dynamic portfolio analysis and then by Borch, Mossin and Hagen in their analysis of financial market efficiency. The decomposability of portfolio choice into choice between a riskless asset and a risky portfolio and choice of the latter's composition, which is associated to this structure of the utility function, belongs more generally to utility functions in the contingent incomes (the x_i's) having the structure which will be called $-\xi e$-homogenity or e-translatedness in Part II, Sect. XI.

[15] On practical grounds, if we choose to use a centrist measure of inequality to compare income distributions, apart from the question of the choice of ϵ, which arises also for I_r (as that of α for I_l), that of the choice of ξ is raised. If we compare distributions with the same average $\bar{x}$, the choice $\xi = \bar{x}$ seems reasonable. If not, a ξ which is the average of averages weighted by populations (i.e., the average income for a population which is the gathering of the compared ones) may also be suggested as suitable.

[16] As will be discussed below, "impartiality" (symmetry of V and the "transfers principle" are altogether characteristic of Schur-concavity of a differentiable V, and they imply $\bar{\bar{x}} \prec x$ ($\prec$ out of equality for strict form) (cf. Part II, Sect. XI). $\bar{\bar{x}}$ can then be defined as the smallest average income which allows the same welfare level V as x.

More precisely, $\bar{\bar{x}}$ is in some sense average welfare per person measured in pounds sterling. $\bar{x}$ is average income per person. $\bar{\bar{x}}$ differs from $\bar{x}$ because all x_i's are not equal. Therefore, $\bar{x} - \bar{\bar{x}}$ can be taken as a "per person" pound measure of inequality. We could also use the "per pound" pound measure of inequality $(\bar{x} - \bar{\bar{x}})/\bar{x}$ (we could also have divided by $\bar{\bar{x}}$ rather then $\bar{x}$, replacing a per "income pound" measure by a per "equal welfare pound" measure). Total pound measure of inequality $n \cdot (\bar{x} - \bar{\bar{x}})$ may also be significant. We shall call $I^a = \bar{x} - \bar{\bar{x}}$ and $I^r = (\bar{x} - \bar{\bar{x}})/\bar{x} = 1 - (\bar{\bar{x}}/\bar{x})$ (a and r stand for "absolute" and "relative," but if I^r, expressed in pounds per pound, is, as implied, a pure dimensionless number, I^a is absolute only in pounds since it is otherwise expressed per person).

I_r is a per pound I^r, and I_l and I_c are per person I^a's. We can thus also consider a per person rightist measure $I_r{}^a = \bar{x} I_r$ and per pound leftist and centrist measures $I_l{}^r = I_l/\bar{x}$ and $I_c{}^r = I_c{}^r/\bar{x}$.

All measures which are not "per pound" raise the problem of measuring real, comparable, "pounds," which was mentioned earlier about I_l. The solutions are still the same, and also the same as the traditional ones for any comparison of incomes.

We must, in addition, present a remark about the domain of variation of the x_i's. The functions used may imply some restrictions on it. For I_l, no such restriction exists. But we of course define I_r only for $x_i \geqslant 0$, and $I_c{}^a$ and $I_c{}^r$ only for $x_i \geqslant -\xi$, for all i's. When x_i's are income or wealth, there exists negative x_i's: business failures make negative incomes, and a net indebtedness is a negative wealth. I_r would thus be unacceptable to measure the inequality in such magnitudes. I_l, on the other hand, can take all these cases into account, as can I_c if we choose ξ large enough. It may however be that one might neglect incomes or wealths which are negative or smaller that $-\xi$ ($\xi > 0$). But this certainly cannot be done for other interesting uses of inequality measures. (For income or wealth, however, $\bar{x} > 0$ is certainly the relevant case, and we thus shall assume it.)

V. Variations of Inequality with Its Parameters and with Equal Proportional and Absolute Variations of Incomes

V.a. *General Properties*

We now have six inequality measures: the rightist, leftist, and centrist ones, and in each case *a* per person ("absolute") and per pound ("relative") measure. These indices depend upon three parameters, ϵ, α and ξ, and, of course, upon the income distribution. It is interesting to know how they vary with these parameters, and with the two kinds of variations in incomes which we have considered: equal proportional or absolute increase

(or decrease) in all incomes. These two questions turn out to be closely related, and they must therefore be studied together.

We shall deal separately, at the end of this section, with three kinds of special cases: (1°) $\xi < 0$, which will turn out to give a measure which not really "centrist," (2°) $\epsilon \geqslant 1$ and one $x_i + \xi$ is zero (including one x_i zero for $\xi = 0$), (3°) the measures take a zero value. The following results hold for the other cases.

The inequality measures vary as ϵ or α and inversely with ξ. An equal absolute increase (resp., decrease) in all incomes diminishes (resp., increases) all measures except I_l, which remains unchanged. An equiproportional increase (resp., decrease) in all incomes increases (resp., diminishes) all measures except I_r which remains unchanged, and I_r^a's variations are proportional, and I_l and I_c's more than proportional, to the incomes'.

We now prove these propositions. Each inequality index will be written as a function of its parameters and of x. λ and μ are numbers ($\lambda > 0$). When x is transformed into λx or $x + \mu e$, $\bar{x}$ is transformed into $\lambda\bar{x}$ or $\bar{x} + \mu$.

We first observe that:

$$
\begin{aligned}
I_r(\epsilon, \lambda x) &= I_r(\epsilon, x), \\
I_r^a(\epsilon, \lambda x) &= \lambda \cdot I_r^a(\epsilon, x), \\
I_l^r(\alpha, \lambda x) &= I_l^r(\alpha\lambda, x), \\
I_l(\alpha, \lambda x) &= \lambda \cdot I_l(\alpha\lambda, x), \\
I_c(\epsilon, \xi, x + \mu e) &= I_c(\epsilon, \xi + \mu, x), \\
I_l(\alpha, x + \mu e) &= I_l(\alpha, x).
\end{aligned}
$$

V.b. *Demonstrations*

This last property shows that I_l^r varies inversely with μ (i.e., in the opposite direction).

How does I_l vary with α? $I_l = \bar{x} - \tilde{x}$, and $\tilde{x}$ only depends upon α. But $\tilde{x} = -(1/\alpha)\log((1/n)\sum e^{-\alpha x_i})$ is of the form $\varphi^{-1}[(1/n)\sum \varphi(x_i)]$ with $\varphi(y) = e^{-\alpha y}$, i.e., it is a "generalized mean" with this function. As α' is another α, we call $\psi(y) = e^{-\alpha' y}$. The function $\psi\varphi^{-1} = y^{\alpha'/\alpha}$ is convex or concave according as $\alpha' \gtrless \alpha$. And φ and ψ are strictly monotonic and ψ is decreasing (since $\alpha' > 0$). Hence, applying the demonstration of [18, Theorem 83] shows that $\tilde{x}$ is smaller, α the larger. I_l thus varies in the same direction as α.

$I_l^r = I_l/\bar{x}$ also varies as α. And since to multiply α or x by λ gives the same I_l^r, I_l^r varies as λ. I_l thus also varies as λ, and more than proportionally.

As a function of ϵ, I_c varies in the direction opposite to that of

$((1/n)\sum \eta_i^{1-\epsilon})^{1/(1-\epsilon)}$ where $\eta_i = x_i + \xi$. A well-known result (for instance, [18, Theorem 16]) says that this "mean" of the η_i's varies as $1 - \epsilon$, i.e., in the direction opposite to that of ϵ. I_c then varies as ϵ. And so do $I_c^r = I_c/\bar{x}$ and their special cases for $\xi = 0$, I_r^a and I_r.

How does I_c vary with its second parameter, ξ? We have

$$dI_c/d\xi = 1 - \left((1/n)\sum \eta_i^{1-\epsilon}\right)^{\epsilon/(1-\epsilon)} (1/n)\sum \eta_i^{-\epsilon}.$$

This expression has the sign of

$$\left((1/n)\sum \eta_i^{-\epsilon}\right)^{-1} - \left((1/n)\sum \eta_i^{1-\epsilon}\right)^{\epsilon/(1-\epsilon)},$$

which has the sign of

$$\left((1/n)\sum \eta_i^{-\epsilon}\right)^{-1/\epsilon} - \left((1/n)\sum \eta_i^{1-\epsilon}\right)^{1/(1-\epsilon)}.$$

These two terms are "means" which differ only by their exponents $-\epsilon$ and $1 - \epsilon$. From the above-mentioned result, they compare as these exponents. Therefore, $dI_c/d\xi < 0$.

$I_c^r = I_c/\bar{x}$ also varies inversely with ξ.

A variation in ξ, and an equal variation in all x_i's, have the same effect on I_c. I_c thus varies inversely with an equal variation in all x_i's. So a fortiori does $I_c^r = I_c/\bar{x}$, and so do their special cases for $\xi = 0$, I_r^a and I_r.

To see the effect on I_c^r of an equiproportional variation in all x_i's, we replace a multiplication of all x_i's by λ (> 0) by a multiplication of all x_i's and ξ by λ followed by the subtraction of $(\lambda - 1)\xi$ from ξ. The first operation leaves I_c^r unchanged. The second one increases or decreases it according as $(\lambda - 1)\xi \gtrless 0$. Thus, for $\xi > 0$, I_c^r varies as an equiproportional variation in all x_i's, and for $\xi < 0$ it varies inversely with it. For $\xi > 0$, $I_c = \bar{x}I_c^r$ a fortiori varies as λ, and more than proportionally.

The variation of I_c with an equiproportional variation in all x_i's when $\xi < 0$ remains the only effect the direction of which is not a priori determined. In fact, the two extreme cases would be $\xi = 0$ where $I_c = I_r^a$ varies proportionally to λ, and $\xi \to -\infty$, which transforms an equiproportional increase (resp., decrease) in all incomes into an equal absolute increase (resp., decrease) in all incomes, which we have seen to decrease (resp., increase) I_c. Thus, for $\xi < 0$, an equiproportional variation in all x_i's sometimes increases, and sometimes decreases, I_c. It all depends upon where x, ξ and ϵ stand. However, the case $\xi < 0$ is not the most interesting, since, as we have just seen, the inequality measure I_c^r then decreases when all x_i's are increased in the same pro-

portion, and this happens only in this case. It thus hardly deserves its adjective of "centrist" and is rather "far right."

Finally, we notice that the monotonicity of the variations for ϵ and α proves that all per person ("absolute") measures are between 0 and $\bar{x} - \underline{x}$, and all per pound ("relative") measures are between 0 and $1 - (\underline{x}/\bar{x})$.

V.c. *Special Cases*

The special cases when the inequality measure is zero could come either from $\epsilon = 0$ for I_r, I_r^a, I_c, or I_c^r, or from $\alpha = 0$ for I_l or I_l^r, or from the equality of all x_i's for all these measures. In all these cases, an equal absolute or relative variation in all incomes does not change the value of the measure, which remains zero: If $\epsilon = 0$ or $\alpha = 0$ the index is always zero, and if all the x_i's are equal the variation keeps them equal and the measure remains zero. Furthermore, if $\epsilon = 0$ in I_c or I_c^r, a variation in the parameter ξ does not change the measure, which remains zero.

A last category of special cases remains to be considered.

$\bar{x} - I_l = \tilde{x}$ is zero for $\alpha > 0$ if and only if $x_i = 0$ for all i's. Similarly, $\bar{x} - I_r^a = \tilde{x}$ is zero for $\epsilon < 1$ if and only if $x_i = 0$ for all i's. But for $\epsilon \geqslant 1$, $\bar{x} - I_r^a = \tilde{x}$ is zero if and only if one x_i (at least) is zero. This is ethically meaningful: When ϵ is high enough, one zero income suffices to bring the equal equivalent income down to zero and thus the inequality measure I_r up to 1 (if not all x_i's are zero), which is its maximum value ($\underline{x} = 0$). Then, for $\epsilon \geqslant 1$ and one x_i zero, I_r^a and I_r do not depend upon ϵ (but their variations for equiproportional variations, or equal increases, in all x_i's are as the general case).

More generally, for $\epsilon \geqslant 1$ and $x_i + \xi = 0$ for one i (at least), $[(1/n)\sum(x_i + \xi)^{1-\epsilon}]^{1/(1-\epsilon)} = 0$ and $I_c = \bar{x} + \xi$. Note that $x_i + \xi = 0$ cannot happen for $\xi > 0$, which is the most interesting case of $\xi \neq 0$. We thus consider $\xi < 0$. This I_c does not depend upon ϵ, and neither does $I_c^r = I_c/\bar{x}$. It satisfies $dI_c/d\xi = 1 > 0$, and I_c^r is also an increasing function of ξ. It is increased or decreased by amount μ when all the x_i's are, and $I_c^r = 1 + (\xi/\bar{x})$ also varies in this direction. It varies as an equiproportional variation in all x_i's, more than proportionnally, and I_c^r also varies in this direction.

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Trader-Commodity Parity Theorems*

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I. INTRODUCTION

Most economists know that given a number of alternative techniques for producing m goods, no more than m techniques need be employed to insure efficient production. Similarly, the optimal solution to a linear programming problem involving m resource constraints involves no more than m control variables assuming positive values. This paper establishes a similar theorem: a Pareto improvement in an exchange economy with m commodities requires the active participation of no more than m traders. If, in a world characterized by positive information and transactions costs, the achievement of Pareto improvements is inversely related to the number of individuals whose cooperation is required, then our result suggests that Pareto efficiency is more likely than one might otherwise suspect.

This paper complements the recent works of Rader [7], Feldman [1], and Madden [5]. Both Rader and Feldman examine special situations in which Pareto improvements, if possible at all, can be obtained through bilateral trades. To establish this result Rader requires the existence of a "broker"—a trader that can deal in all goods—while Feldman requires the existence of a "partial money"—a commodity desired and possessed in positive quantities by all traders—together with an assumption that the utility functions of traders are continuous through first partial derivatives. Madden demonstrates that under weaker assumptions, including nonsatiation, Pareto efficiency for all groups of m traders implies overall Pareto efficiency. Jennergren *et al.* in an independent effort delete the nonsatiation assumption and find that a Pareto-improving exchange

* This paper represents a synthesis of [4] and [4'] and has benefited from comments by Paul Madden and Alan Feldman.

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requires no more than $m + 1$ members. Another related paper is [3], which analyzes a trading process with trading group size $m + 1$ for one-dimensional utility function maximization.

The discussion by Madden is concerned solely with Pareto-improving trades: trades which might not move some participating trader from his indifference curve. In the present work, we are concerned additionally with trades which are mutually advantageous: trades in which all participants achieve strictly preferred commodity bundles. We find, under a simple free disposal assumption of the type suggested by Madden, that if a mutually advantageous trade is possible, it is possible among m or fewer traders. The development of this result makes the role of the underlying assumptions especially transparent and also emphasizes the result's inherently geometric nature. This strengthened theorem can then be applied to a trading process described in [9] to show that even if barter groups are restricted to consist of m traders ($m < n$), the limit points of the process are Pareto optimal.

II. Preliminaries

Consider an n-trader, m-commodity economy. The *existing endowment* of the ith trader, w_i, is an element from the nonnegative orthant of m-dimensional Euclidean space, R_+^m. It is assumed without loss of generality that[1]

$$\sum_N w_i > 0, \qquad N := \{1, 2, \ldots, n\}.$$

An *allocation* x is an element from

$$A := \left\{x = (x_1, x_2, \ldots, x_n) \in R_+^{mn} \;\middle|\; \sum_N x_i = \sum_N w_i\right\}.$$

For each trader there is a relation, D_i, defined on R_+^m and read "as desirable as." We assume that

(A.1) D_i is complete, reflexive and transitive.

From D_i can be defined relations for "strict preference," P_i, and for "indifference," I_i, as follows:

For all $x, y \in R_+^m$, $x\ P_i\ y$ iff not $y\ D_i\ x$;

For all $x, y \in R_+^m$ $x\ I_i\ y$ iff $x\ D_i\ y$ and $y\ D_i\ x$.

[1] Conventional vector inequality notation is employed. See, e.g., [9].

We further assume:

(A.2) D_i is convex, i.e., for all $x, y \in R_+^m$, if $x P_i y$ then $[\alpha x + (1 - \alpha) y] P_i y$ for all $\alpha \in (0, 1)$;

(A.3) D_i reflects free disposal; i.e., for all $x, y \in R_+^m$, if $x \geqslant y$ then $x D_i y$. (One need not be compensated for receiving more of a commodity that can be freely disposed of.) This assumption is sometimes referred to as weak monotonicity.

We say that mutually advantageous exchange is possible if there exists a coalition of traders or *barter group* $G \subseteq N$ and a partial allocation x_i, $i \in G$, such that

(i) $x_i P_i w_i$, $i \in G$,

(ii) $\sum_G x_i = \sum_G w_i$.

We will employ the notation $\mu(G)$ to denote the number of traders in barter group G and will say that the partial allocation x_i, $i \in G$, involves exchange of k commodities if there exist exactly k distinct values of j for which $x_{ij} \neq w_{ij}$ for some $i \in G$.

III. The Parity Theorems

The main proposition of this section is the following.

THEOREM 1. *Under* (A.1)–(A.3), *if mutually advantageous exchange is possible then it is possible within a barter group containing no more traders than there are commodities.*

Proof. Suppose that mutually advantageous exchange is possible for barter group $G \subseteq N$ via the partial allocation x_i, $i \in G$. We wish to show that mutually advantageous exchange is also possible for barter group $K \subseteq N$ where $\mu(K) \leqslant m$. If either $n = \mu(N) \leqslant m$ or $\mu(G) \leqslant m$ the result is obtained trivially. Accordingly, consider the case $m < \mu(G) \leqslant n$. By definition we have

$$x_i P_i w_i, \qquad i \in G,$$

$$\sum_G x_i = \sum_G w_i.$$

Now define

$$z_i \equiv x_i - w_i \neq 0, \qquad i \in G,$$

and

$$\Gamma \equiv \text{convex hull } \{z_i \in R^m \mid i \in G\}.$$

Γ is a convex polyhedral set in R^m with not more than $\mu(G)$ vertices. Moreover, the fact that

$$\sum_G z_i = \sum_G (1/\mu(G))\, z_i = 0$$

means that the origin belongs to Γ. There are two possibilities. Either (1) the origin belongs to the interior of Γ or (2) the origin belongs to a face or edge of Γ. If the latter is the case, then the origin lies in a linear subspace whose dimensionality does not exceed $m - 1$ and which is spanned by a subset (proper or otherwise) of the z_i's.

Turning first to case (2) it follows from Caratheodory's theorem (see [6, p. 19] for example) that if the origin (or any other point) is expressible as a convex combination of a number of z_i's from a linear space whose dimensionality does not exceed $m - 1$ then the origin is also expressible as a convex combination of not more than m suitably selected z_i's. Thus for some set of indices $K \subset G$ where $\mu(K) \leqslant m$ and a collection of positive α_i's,

$$\sum_K \alpha_i z_i = 0, \qquad \sum_K \alpha_i = 1.$$

Now define

$$\begin{aligned} y_i &= \alpha_i z_i + w_i, && i \in K, \\ &= \alpha_i x_i + (1 - \alpha_i)\, w_i, && i \in K, \end{aligned}$$

from the definition of z_i. By (A.2) we have

$$y_i \, P_i \, w_i, \qquad i \in K.$$

And since

$$\sum_K y_i = \sum_K \alpha_i z_i + \sum_K w_i = \sum_K w_i,$$

mutually advantageous exchange is possible within a barter group containing not more than m members.

Returning to case (1), if the origin belongs to the interior of Γ it follows that there exists a $\delta \in R_+^m$ such that $-\delta$ belongs to a face or edge of Γ. Proceeding as before we are assured of the existence of a set of indices $K' \subset G$ with $\mu(K') \leqslant m$ and a set of positive α_i''s such that

$$\sum_{K'} \alpha_i' z_i = -\delta, \qquad \sum_{K'} \alpha_i' = 1.$$

In this case we define

$$y_i' \equiv \alpha_i'(z_i + \delta) + w_i, \qquad i \in K',$$
$$= \alpha_i'(x_i + \delta) + (1 - \alpha_i')\, w_i, \qquad i \in K',$$

from the definition of z_i. But

$$[x_i + \delta]\, D_i\, x_i, \qquad i \in K' \qquad \text{[by (A.3)]},$$
$$[x_i + \delta]\, P_i\, w_i, \qquad i \in K' \qquad \text{[by (A.1)]},$$

and

$$y_i'\, P_i\, w_i, \qquad i \in K' \qquad \text{[by (A.2)]}.$$

And since

$$\sum_{K'} y_i' = \sum_{K'} \alpha_i' z_i + \delta \sum_{K'} \alpha_i' + \sum_{K'} w_i = \sum_{K'} w_i,$$

mutually advantageous exchange is again possible within a barter group containing not more than m members. This exhausts the possibilities and completes the proof of Theorem 1.

COROLLARY 1. *Under* (A.1)–(A.3), *if mutually advantageous exchange requires at least k traders then it necessarily involves at least k commodities.*

Proof. It is convenient to establish the equivalent proposition that if mutually advantageous exchange involving less than k commodities is possible then it is possible within a barter group containing fewer than k members. This proposition follows trivially from the proof of Theorem 1 and the realization that the initial endowments of the $m - k + 1$ commodities which, as a minimum, are not involved in the exchange can be regarded simply as parameters which influence preferences of traders for the remaining commodities but in no other way affect the analysis.

These results can be strengthened and related to the work of Madden [5] by introducing the following assumptions:

(A.3′) Nonsatiation: For all x, $y \in R_+^m$ and all i, if $x \geqslant y$ then $x\, P_i\, y$. (This assumption is sometimes referred to as strong monotonicity.)

(A.4) Weak continuity: For all x, $y \in R_+^m$ and all i, if $x\, P_i\, y$ and $x \neq 0$ then there exists $\delta_i \in R_+^m$ with $\delta_i \neq 0$ such that $(x - \delta_i)\, P_i\, y$.[2]

[2] That this assumption is weaker than the conventional continuity assumption is evidenced by the fact that it is satisfied by the lexicographic ordering: $x_i\, P_i\, y_i$ iff $\exists\, k < m$ such that $x_{ij} = y_{ij}$ for $j < k$ and $x_{ik} > y_{ik}$; $x_i\, I_i\, y_i$ otherwise. For this ordering $\delta_{ij} = 0$ for $j \neq k$ and $\delta_{ik} > 0$.

An allocation x_i, $i \in N$, is termed *Pareto efficient for group* $G \subseteq N$ if for any other allocation y_i, $i \in N$, $y_j \, P_j \, x_j$ for any $j \in G$ implies $x_{j'} \, P_{j'} \, y_{j'}$ for some $j' \in G$. An allocation x_i, $i \in N$ is *k-way Pareto efficient* if it is Pareto efficient for all groups $G \subseteq N$ such $\mu(G) \leqslant k$. An allocation is *overall Pareto efficient* if it is Pareto efficient for N or, equivalently, if it is n-way Pareto efficient. We may state, then, without proof,

THEOREM 2. *Under* (A.1), (A.2), (A.3'), *and* (A.4), *if an allocation is not overall Pareto efficient then mutually advantageous exchange is possible* (*and thus by Theorem* 1, *possible within a barter group containing no more traders than there are commodities*).

An immediate consequence of Theorem 2 is

COROLLARY 2. *Under* (A.1), (A.2), (A.3'), *and* (A.4), *if an allocation is m-way Pareto efficient then it is overall Pareto efficient.*

IV. A TRADING PROCESS WITH GROUPS OF LIMITED SIZE

We now turn our attention to the asymptotic behavior of a particular trading process. (This process is related to nontatonnement processes studied in [1], [9], and [10].) There are Q different barter groups, $I^1,\ldots, I^Q$, not necessarily pairwise disjoint. In the trading process, coalition I^1 forms and if a Pareto superior reallocation among members of I^1 is possible, the traders within this group effect such a reallocation which is also Pareto efficient for I^1. Hence, as a result of the formation of I^1, no member of I^1 is worse off, and if a reallocation is made, then at least one member becomes better off. The existence of a Pareto efficient allocation for I^1 is assured by assumption (A.1) and the following continuity assumption:

(A.4') For each $y \in R_+^m$ and each $i \in N$, the sets $\{x \mid x \, D_i \, y\}$ and $\{x \mid y \, D_i \, x\}$ are closed.

The next stage in the trading process involves the formation of barter group I^2, which conducts or fails to conduct a trade according to the guidelines just set forth for I^1. The process continues, with the groups $I^3,\ldots, I^Q$ meeting in succession, completing what we term a major cycle. We denote by $M^q(x)$ the set of reallocations of x within I^q which are Pareto efficient for I^q, and at least as desired as x by each member of I^q. That is, $M^q(x) = \{y \in A \mid y$ is Pareto efficient for I^q, $y_i \, D_i \, x_i \; \forall_i \in I^q$, and $y_j = x_j \, \forall j \in N \sim I^q\}$. The composition $M^Q \cdots M^1(x)$, denoted $M(x)$, is

a point-to-set mapping relating an initial endowment x to a set of possible endowments resulting from a major cycle of barter group meetings.

Major cycle follows major cycle, unless some such cycle results in no reallocation of commodities, in which case the process terminates. Once the membership of the Q barter groups is more completely specified, we can inquire whether this process converges, and if so, whether it requires an infinite number of iterations.

Before stating and proving our main convergence theorem, we note that assumptions (A.1) and (A.4′) imply the existence of a (continuous) utility function $f_i(x_i)$ for each trader $i \in N$, having domain R_+^m. (See [2, pp. 143–143] for a discussion of this.) Hence, associated with any allocation $x = (x_1, ..., x_n)$ is a utility vector $(f_1(x_1), ..., f_n(x_n))$. We note further that:

LEMMA 1. *Under* (A.1) *and* (A.4′), *the limit points of the reallocation process described above are allocations among which all traders are indifferent.*

Proof. Let major cycles be indexed by t and let x^t denote the allocation at the end of the tth major cycle. Since $x^t \in A$, a compact subset of R_+^{mn}, the set B of limit points of $\langle x^t \rangle_{t=1}^{\infty}$ is a nonempty subset of A. If B contains but one element, the lemma is trivial. If B contains two or more points, let $\bar{x}$ and $\bar{\bar{x}} \in B$. Let $\langle \bar{x}^t \rangle$ and $\langle \bar{\bar{x}}^t \rangle$ be subsequences of $\langle x^t \rangle$ converging, respectively, to $\bar{x}$ and $\bar{\bar{x}}$. Then, since $f_i(x_i^{t+1}) \geqslant f_i(x_i^t)$, it follows that

$$f_i(\bar{x}_i) \geqslant f_i(\bar{x}_i^t) \qquad \text{for each } t,$$

so

$$f_i(\bar{x}_i) \geqslant \lim f_i(\bar{x}_i^t) = f_i(\bar{\bar{x}}_i).$$

By a symmetric argument,

$$f_i(\bar{\bar{x}}_i) \geqslant f_i(\bar{x}_i),$$

so $\bar{\bar{x}}_i \, I_i \, \bar{x}_i$ for each $i \in N$, as was to be shown.

Two further lemmas will facilitate the exposition of the main convergence theorem. The first of these, whose proof is left to the reader, incorporates a condition of strict convexity.

(A.2′) D_i is strictly convex, i.e., $x_i \, D_i \, y_i$ and $x_i \neq y_i$ imply $[\alpha x_i + (1 - \alpha) y_i] \, P_i \, y$ for all $\alpha \in (0, 1)$.

LEMMA 2. *Under* (A.2′), *if one of two allocations is overall Pareto efficient, not all traders are indifferent between them.*

The next lemma confirms that the mappings M^q have the important property of closedness, or upper semicontinuity.

LEMMA 3. *Under* (A.1), (A.2), (A.3'), *and* (A.4'), *each* M^q *is a closed point-to-set map.*

Proof.[3] It needs to be shown that

(i) $x^s \in A$ converges to x^* as s goes to infinity, and

(ii) $y^s \in M^q(x^s)$, and

(iii) y^s converges to y^* as s goes to infinity

imply

(iv) $y^* \in M^q(x^*)$.

Suppose (i)–(iii) hold. To show that (iv) holds, one needs to demonstrate (a) that y^* is Pareto efficient for I^q, and (b) $(f_1(y_1^*) \cdots f_n(y_n^*)) \geqslant (f_1(x_1^*) \cdots f_n(x_n^*))$, the Pareto noninferiority of y^*. First, suppose that y^* is not Pareto efficient for I^q. Then there exists $y' \in A$ such that $y_i' = y_i^*$ for $i \notin I^q$ and such that $(f_1(y_1') \cdots f_n(y_n')) \geqslant (f_1(y_1^*) \cdots f_n(y_n^*))$. Then, by some small reallocation away from y' among members of I^q, there exists $y^0 \in A$ such that $y_i^0 = y_i^*$ for $i \notin I^q$ and $f_i(y_i^0) > f_i(y_i^*)$ for all $i \in I^q$. (This follows from (A.3') and (A.4'), the strong monotonicity and continuity assumptions.) Now let $\tilde{y}^s$ be a sequence of allocations in A such that $\tilde{y}_i^s = y_i^s = x_i^s$ for $i \notin I^q$ and converging to y^0. By the continuity of the utility functions f_i, for sufficiently large s it must be that $f_i(\tilde{y}_i^s) > f_i(y_i^s)$ for each $i \in I^q$. This contradicts $y^s \in M^q(x^s)$, so y^* must be Pareto efficient for I^q.

To demonstrate Pareto noninferiority, we note first that (ii) assures that $(f_1(y_1^s) \cdots f_n(y_n^s)) \geqslant (f_1(x_1^s) \cdots f_n(x_n^s))$. The continuity of the f_i and the convergence of x^s to x^* and y^s to y^* imply that $(f_1(y_1^*) \cdots (f_n(y_n^*)) \geqslant (f_1(x_1^*) \cdots f_n(x_n^*))$. Hence $y^* \in M^q(x^*)$, and the proof is complete.

The trading process is more completely specified in the following assumption, giving the exact composition of each of the barter groups.

(A.5) Barter groups $I^1, \ldots, I^Q$ consist of the $Q = \binom{n}{m}$ possible groups containing exactly m members each.

The main result of this section is the following convergence theorem.

[3] This generalizes the first part of the proposition in [1, p. 472]. In [5] it is also shown that m-way Pareto efficiency implies overall Pareto efficiency. However, the current result may be regarded as somewhat stronger than Madden's, since he takes utility functions for individuals as a primitive concept rather than their preference orderings, and since a strong continuity assumption is normally required to argue the existence of such functions. In [4], a reverse result is demonstrated: that overall Pareto inefficiency implies $m+1$-way Pareto inefficiency. A nonsatiation assumption is not required for this latter result, nor is any form of continuity.

THEOREM 3. *Under assumptions* (A.1), (A.2), (A.3'), (A.4'), *and* (A.5), *the reallocation process has one or more limit points. All limit points are efficient allocations, and all traders are indifferent among them.*

Proof. The existence of the limit point(s) follows from the compactness of A. The indifference of traders among limit points is a consequence of Lemma 1.

To demonstrate the Pareto efficiency of the limit points, a result due to Zangwill ("Convergence Theorem A" [8, p. 91]) will be used. Zangwill's result is applicable to the trading process discussed here with only formal changes. Those changes derive from the fact that we are here concerned with a vector $(f_1(x_1),...,f_n(x_n))$ of trader utilities, whereas Zangwill's theorem applies to single utility functions. Zangwill's theorem asserts that if the following three conditions (a)–(c) are satisfied, then either the trading process stops at a Pareto-optimal allocation, or the limit of any convergent subsequence of allocations is Pareto-optimal:

(a) All successive allocations are in a compact set.

(b) If x' is not Pareto-optimal, then, for any $y' \in M(x')$, $(f_1(y_1') \cdots f_n(y_n')) \geqslant (f_1(x_1') \cdots f_n(x_n'))$. If x' is Pareto-optimal, then either the process stops or, for any $y' \in M(x')$, $(f_1(y_1') \cdots f_1(y_n')) \geqslant (f_1(x_1') \cdots f_n(x_n'))$.

(c) The point-to-set map M is a closed one.

Certainly (a) holds, since all successive allocations are in the set A, which is compact; (b) also holds, from the specification of the trading process and Theorem 2, which assure us that every major cycle will bring Pareto improvement if any is to be had. Since M is the composition $M^0 \cdots M^1$ of point-to-set maps, each of which are closed by our Lemma 3, it follows from [8, Corollary 4.2.1] that (c) M is closed on A. This completes the proof of the theorem.

If we strengthen (A.2) (convexity of preferences) to (A.2') (strict convexity of preferences) we obtain the result that the trading process has a unique limit point.

COROLLARY 3. *Under* (A.1), (A.2'), (A.3'), (A.4'), *and* (A.5), *the reallocation process converges to a Pareto efficient allocation.*

Proof. By Theorem 3, each limit point of the process is Pareto efficient, and traders are indifferent among them. But Lemma 2 precludes the existence of two or more distinct allocations with these properties. Hence there is but one limit point.

For the trading process with which we have been concerned in this

section, it is required that a barter group effect a Pareto superior and efficient reallocation within itself, if that is possible. Such a reallocation might not make every party to it strictly better off, and hence the trade would not be mutually advantageous in the sense of Sections II and III. One may alter the specification of the trading process to require that when a barter group meets it will effect among its members a reallocation which is mutually advantageous and efficient, or if no such reallocation can be made, no reallocation is made. For this process, Lemmas 1, 2, and 3 are still valid, as are Theorem 3 and Corollary 3.

V. Discussion

We now ask whether the trading process described in the previous section (where the traders form successive coalitions of size m) converges to a Pareto-optimal allocation in a finite number of major cycles. In general, the answer is no. To see this, let $m = 2$ and $n = 3$, and suppose the initial allocation vectors are

$$(2, 0), (2, 0), (0, 2).$$

Further suppose that each trader's preferences can be represented by the utility function $f_i = -(0.5)^{x_{i1}} - (0.5)^{x_{i2}}$. Then $Q = 3$, and we let $I^1 = \{1, 2\}$; $I^2 = \{1, 3\}$; $I^3 = \{2, 3\}$.

For given x_{i1} and x_{i2}, the marginal rate of substitution for trader i is

$$(0.5)^{(x_{i1}-x_{i2})}.$$

For Pareto-optimal allocations $(\tilde{x}_{11}, \tilde{x}_{12})$, $(\tilde{x}_{21}, \tilde{x}_{22})$, and $(\tilde{x}_{31}, \tilde{x}_{32})$, the marginal rate of substitution must be equal for every trader. This means

$$(\tilde{x}_{11} - \tilde{x}_{12}) = (\tilde{x}_{21} - \tilde{x}_{22}) = (\tilde{x}_{31} - \tilde{x}_{32}) = \tilde{k},$$

where $\tilde{k}$ can be calculated from the sum of traders' endowments:

$$\tilde{x}_{11} + \tilde{x}_{21} + \tilde{x}_{31} = 4,$$
$$\tilde{x}_{11} - \tilde{k} + \tilde{x}_{21} - \tilde{k} + \tilde{x}_{31} - \tilde{k} = 2.$$

This gives $\tilde{k} = \frac{2}{3}$.

When two traders i and j meet, they will readjust their commodity holdings to make their marginal rates of substitution equal. Let (x_{i1}, x_{i2}), (x_{j1}, x_{j2}) and (x'_{i1}, x'_{i2}), (x'_{j1}, x'_{j2}) be the pretrade and posttrade holdings, respectively. Then

$$(x'_{i1} - x'_{i2}) = (x'_{j1} - x'_{j2}) = k',$$

where k' may be determined from the equations

$$x'_{i1} + x'_{j1} = x_{i1} + x_{j1},$$

$$x'_{i1} - k' + x'_{j1} - k' = x_{i2} + x_{j2},$$

which gives $k' = [(x_{i1} + x_{j1}) - (x_{i2} + x_{j2})]/2$.

At any point in time, the current allocation can be characterized by a three-component vector, giving for each trader the difference between his holdings of commodity 1 and commodity 2. At the outset, that vector is (2, 2, −2). The trading process can then be characterized by a sequence of such vectors, as follows:

(2, 2, −2)	Initial allocation
(2, 2, −2)	I^1 trade result
(0, 2, 0)	I^2 trade result
(0, 1, 1)	I^3 trade result
END FIRST MAJOR CYCLE	
⋮	
(3/4, 5/8, 5/8)	I^3 trade result
END SECOND MAJOR CYCLE	
⋮	
(42/64, 43/64, 43/64	I^3 trade result
END THIRD MAJOR CYCLE	
⋮	

It is clear that convergence to the limit $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, a Pareto-optimal allocation, occurs only as the number of major cycles goes to infinity.

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Temporary Competitive Equilibrium and the Existence of Self-Fulfilling Expectations*

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1. INTRODUCTION

Recent studies of the temporary competitive equilibrium have focused attention on the way in which economic agents anticipate future prices. The existence of a temporary competitive equilibrium under uncertainty was first proved by Stigum [8] in the context of a model which included stock markets and markets for personal and entrepreneurial debt. Stigum assumed that each agent's subjective probability distribution over future events conditional on current and past observations varied continuously with current observations and that each agent attached subjective probability one to the event in which all future prices are positive [8, pp. 557–558]. These two assumptions have been used in all subsequent studies. In [2], Grandmont introduced the topology of weak convergence of probability measures as a tool for studying agents' expectations and stated the continuity assumption in terms of this topology.

Since most authors have assumed preferences to be strictly monotone, the assumption of the certainty of positive future prices has generally been justified on the grounds that future equilibrium prices must be positive. However, if conditional subjective probabilities are defined at nonpositive current prices then the continuity assumption combined with the assumption that future prices are positive with subjective probability one implies that one can find a current price sufficiently low so that each agent expects next period's price to be higher with probability near one. If next period's equilibrium price is positive but lower than the current

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price, agents' expectations will have been in a sense incorrect. This raises the question of whether the existence of a temporary equilibrium in the current period depends on agents having incorrect expectations about the future.

In this paper the relationship between expectations and eventual equilibria is examined in the context of a multiperiod exchange economy with household production. Each household has a production function which takes tth period inputs to $t+1$st period outputs. The difference between this model and one in which the production sector is modelled separately is that in this model, the investor has complete control over his investment. Thus we avoid the question of what, if anything, a firm with many investors should maximize if profits are uncertain. Our use of production functions is purely for simplicity; it is shown in [5] that the use of production sets would not affect the analysis.

Each period, the consumer chooses current consumption and inputs to maximize the expected value of a multiperiod utility function. Since we are interested only in price expectations, we assume that future prices are the only source of uncertainty facing the household. Ex post, prices are deterministic, subject to the possible multiplicity of equilibria. However, we suppose that consumers do not have perfect knowledge of the equilibrium correspondence but believe prices to be stochastic. Formally, of course, perfect foresight is a special case of stochastic expectations. Each consumer also has an estimation procedure which assigns to each price sequence a value of the parameters of the stochastic price process. The parameters held by agents determine, through their influence on excess demand, a sequence of temporary equilibrium prices. If this price sequence yields as estimates the parameter values held by agents originally, we say that the expectations (parameter values) are self-fulfilling.

It is apparent that self-fulfilling expectations are in the nature of a fixed point. In [4], Grunberg and Modigliani applied the Brouwer Fixed Point Theorem to show that if next period's equilibrium price is a bounded continuous function of its currently predicted value, then a correct prediction exists. To a large extent this paper can be viewed as a revision of [4] in the spirit of modern general equilibrium analysis.

2. Definitions

Notation. For any topological space X, $\mathcal{B}(X)$ denotes the Borel σ-field of subsets of X, and $\mathcal{M}(X)$ denotes the space of all Borel probability measures on X. The space $\mathcal{M}(X)$ is endowed with the topology of weak

convergence. If Y is another topological space, the Borel field $\mathcal{B}(X)$ will often be implicitly identified with the appropriate subfield of $\mathcal{B}(X \times Y)$. The notation $g: X \to Y$ will be used if g is a function whereas if g is a correspondence, we will write $g: X \Rightarrow Y$.

If $\{X_s : 1 \leqslant s \leqslant T\}$ is a countable family of spaces and $1 \leqslant t \leqslant T$, then X^t denotes the product space $\prod_{s=1}^{t} X_s$ with generic element x^t. If $1 \leqslant t < T$, ${}^tX$ denotes the space $\prod_{s=t+1}^{T} X_s$ with generic element ${}^tx$, and if $1 \leqslant t < s \leqslant T$, ${}^tX^s$ denotes the space $\prod_{s'=t+1}^{s} X_{s'}$ with generic element ${}^tx^s$. We will also adopt the following conventions: if $t = 0$, $Y \times X^t = Y$ and $(y, x^t) = y$; if $t = T$, $Y \times {}^tX = Y$ and $(y, {}^tx) = y$; and if $t = s$, $Y \times {}^tX^s = Y$ and $(y, {}^tx^s) = y$. The indices t, s, and T, and only these indices will denote time periods. The indices t and s will always take integer values, whereas T may take the value infinity. Thus if $T = \infty$, the statement $t \leqslant T$ should be read $t < \infty$.

2.2. DEFINITIONS. Let T be the horizon of the economy. There are N consumers, indexed by the subscript i, which will be suppressed in this section, and L commodities, indexed by the subscript l. For each $1 \leqslant t \leqslant T$, let Q_t denote the tth period commodity space. Only markets for current commodities are open in each period so $Q_t = R^L$ for each t. The tth period price space is denoted P_t, where $P_t = \{p \in R^L : p \gg 0$ and $\sum_{l=1}^{L} p_l = 1\}$ for each t. Let C_t denote the consumer's tth period consumption set and let $C = \prod_{t=1}^{T} C_t$. The consumer's input set is denoted A_t, where $A_t = R_+^L$, the nonnegative orthant of R^L, for each t; and the consumer's production possibilities are represented by a production function $f: A_t \to Q_{t+1}$ for each $1 \leqslant t < T$. The endowment of the consumer in period t is denoted ω_t and the tth period budget correspondence, $b_t : P_t \times A_{t-1} \Rightarrow A_t \times C_t$, is given by $b_t(p_t, a_{t-1}) = \{(a_t, c_t) : p_t(a_t + c_t) \leqslant p_t(\omega_t + f(a_{t-1}))\}$ for each t, where it is understood that $f(A_{t-1}) = \{0\}$ if $t = 1$.

Let Θ be a convex subset of R^∞ with the relative topology, and let $\psi: \Theta \to \mathcal{M}(P^T)$. An *expectation* of the consumer is an element $\theta \in \Theta$. For $\theta \in \Theta$, $1 \leqslant t < T$, and $p^t \in P^t$, let $\psi_t : \Theta \times P^t \to \mathcal{M}({}^tP)$ be given by $\psi_t(\theta, p^t) = \psi_t(\cdot \mid \theta, p^t)$, the regular conditional distribution on ${}^tP$ determined by $\psi(\theta)$ and p^t [7, pp. 146–147].

2.3. *Remarks.* An element $\theta \in \Theta$ is called an expectation because ψ associates with θ a subjective probability measure over price sequences. More intuitively, θ can be considered a parametric description of a distribution in $\mathcal{M}(P^T)$. This parameterization is introduced purely for convenience and involves no loss of generality since $\mathcal{M}(P^T)$ is homeomorphic to a convex subset of R^∞ [7, pp. 43–44]. The expectations set Θ and the function ψ will be allowed to differ across consumers.

2.4. *Future production-consumption plans.* The consumer's demand correspondence will be derived using the technique presented in [6] (see also [5]). For each t, let $h_t : A_t \twoheadrightarrow {}^tP \times {}^tA \times {}^tC$ be given by $h_t(a_t) = \{({}^tp, {}^ta, {}^tc): (a_s, c_s) \in b_s(p_s, a_{s-1})$ for all $s > t\}$. Let $1 \leqslant t < T$ and let $(\theta, p^t, a_t) \in \Theta \times P^t \times A_t$. If $\phi \in \mathcal{M}({}^tP \times {}^tA \times {}^tC)$ and

(1) for each $E \in \mathcal{B}({}^tP)$, $\phi(E) = \psi_t(\theta, p^t)(E)$, and

(2) for each $t < s < T$, the σ-fields $\mathcal{B}({}^tA^s \times {}^tC^s)$ and $\mathcal{B}({}^sP)$ are conditionally independent relative to $\mathcal{B}({}^tP^s)$, that is, for any $E \in \mathcal{B}({}^tA^s \times {}^tC^s)$ and any $F \in \mathcal{B}({}^sP)$, $\phi(E \cap F \mid \mathcal{B}({}^tP^s)) = \phi(E \mid \mathcal{B}({}^tP^s))\, \phi(F \mid \mathcal{B}({}^tP^s))$ [1, p. 284],

then ϕ is called a future production-consumption plan. If in addition,

(3) $\operatorname{supp} \phi \subset h_t(a_t)$

then ϕ is said to be feasible. Let $F_t(\theta, p^t, a_t)$ denote the set of feasible future production-consumption plans determined by (θ, p^t, a_t).

2.5. *Remarks.* If $\phi \in \mathcal{M}({}^tP \times {}^tA \times {}^tC)$ the conditional distribution of inputs and consumption given prices is a randomized strategy. Condition (3) states that all price-input-consumption sequences in the support of ϕ must be consistent with the consumer's budget correspondence. Condition (1) states that the marginal distribution of future prices agrees with $\psi_t(\theta, p^t)$; and (2) is an information constraint, requiring that input-consumption decisions made in any period depend only on prices observed up to that period.

2.6. *Current demand.* The consumer's preferences over consumption sequences are represented by a von Neuman–Morgenstern utility index $u: C \to R$. For each t, let $v_t : \Theta \times P^t \times A_t \times C^t \to R$ be given by

$$v_t(\theta, p^t, a_t, c^t) = \sup \left\{ \int_{{}^tP \times {}^tA \times {}^tC} u(c^t, \cdot)\, d\phi,\ \phi \in F_t(\theta, p^t, a_t) \right\},$$

where it is understood that $v_t(\theta, p^t, a_t, c^t) = u(c^t)$ if $t = T$. For each t, let $\xi_t : \Theta \times P^t \times A_{t-1} \times C^{t-1} \twoheadrightarrow A_t \times C_t$ be given by $\xi_t(\theta, p^t, a_{t-1}, c^{t-1}) = \{(a_t, c_t): (a_t, c_t)$ maximizes $v_t(\theta, p^t, a_t', (c^{t-1}, c_t'))$ subject to $(a_t', c_t') \in b_t(p_t, a_{t-1})\}$. For each t, ξ_t is the consumer's tth period current demand correspondence.

2.7. DEFINITIONS. An estimation procedure is a nonempty valued convex-valued u.s.c. correspondence $e: P^T \twoheadrightarrow \Theta$. If $p^T \in P^T$ and $\theta \in e(p^T)$ we will say that p^T fulfills the expectation θ.

2.8. *Remarks.* One of the simplest estimation procedures is that which associates with each price sequence its empirical distribution. This procedure is appropriate under the assumption that prices are independently and identically distributed in each period. Let $P = \{p \in R^L : p \gg 0\}$ and let $P_t = P$ for all $1 \leqslant t \leqslant T$, where $T < \infty$. Let $\Theta = \mathcal{M}(P)$, and let $\psi: \Theta \to \mathcal{M}(P^T)$ be given by

$$\psi(\theta)(E_1 \times \cdots \times E_t \times \cdots \times E_T) = \theta(E_1) \cdots \theta(E_t) \cdots \theta(E_T)$$

for $\theta \in \mathcal{M}(P)$ and $\{E_t\}_{t=1}^T \subset \mathcal{B}(P)$. Then let e be given by

$$e(p^T)(E) = (1/T)\,\#\{t: p_t \in E\} \qquad \text{for} \quad p^T \in P^T \quad \text{and} \quad E \in \mathcal{B}(P).$$

It is easily seen that e is continuous.

If prices are not assumed to be independently distributed over time, it may be appropriate to model the price process as an autoregressive process. Let $\{p_{-\tau}\}_{\tau=0}^k \subset P$. Consider the process described by the stochastic difference equation

$$\log p_t = \sum_{\tau=1}^{k} \theta_\tau \log p_{t-\tau} + U_t$$

where $\{U_t\}_{t=1}^T$ is a sequence of independent normal random variables with mean zero and covariance matrix $\sum$, and θ_τ is an $L \times L$ matrix for each $1 \leqslant \tau \leqslant k$. Let Θ be compact and convex, and let each $\theta \in \Theta$ denote a collection of matrices $\{\theta_\tau\}_{\tau=1}^k$. For $1 \leqslant j \leqslant L$ and $1 \leqslant \tau \leqslant k$ let $\theta_{\tau j}$ be the jth row of θ_τ. Then let the estimation procedure e be given by $e(p^T) = \{\hat{\theta}: \hat{\theta}$ minimizes $\sum_{j=1}^L \sum_{t=1}^T (\log p_{tj} - \sum_{\tau=1}^k \theta_{\tau j} \log p_{t-\tau})^2\}$ for $p^T \in P^T$. It is easily seen that e is convex-valued and u.s.c. If Θ were not bounded, $e(p^T)$ might not be bounded for some price sequences p^T.

An estimate of Σ can be determined in the following way. For $(\theta, p^T) \in \Theta \times P^T$, let

$$\hat{\Sigma}(\theta, p^T) = \sum_{t=1}^{T} \left(\log p_t - \sum_{\tau=1}^{k} \theta_\tau \log p_{t-\tau}\right)' \left(\log p_t - \sum_{\tau=1}^{k} \theta_\tau \log p_{t-\tau}\right).$$

Let $\Theta' = \Theta \times P^T$ and let $e': P^T \twoheadrightarrow \Theta'$ be given by $e'(p^T) = (e(p^T), p^T)$ for $p^T \in P^T$. Then let ψ associate with each $(\theta, p^T) \in \Theta'$ the element of $\mathcal{M}(P^T)$ determined by the parameters $(\theta, \hat{\Sigma}(\theta, p^T))$. Then e' is u.s.c. and convex-valued.

The requirements in 2.7 are also generally satisfied by Bayesian estimation techniques. Let Θ be a closed convex set in a finite-dimensional vector space and let $\pi \in \mathcal{M}(\Theta)$ be an a priori probability measure on Θ. Let μ be the appropriate Lebesgue measure on P^T and let $l: \Theta \times P^T \to R$ be a bounded, nonnegative function such that

(a) for each $\theta \in \Theta$, $l(\theta, \cdot)$ is continuous;

(b) for each $\theta \in \Theta$ and each $E \in \mathscr{B}(P^T)$, $\psi(\theta)(E) = \int_E l(\theta, \cdot)\, d\mu$; and

(c) for each $p^T \in P^T$, $\int_\Theta l(\cdot, p^T)\, d\pi > 0$.

Let $\gamma\colon P^T \to \mathscr{M}(\Theta)$ be given by

$$\gamma(p^T)(E) = \frac{\int_E l(\cdot, p^T)\, d\pi}{\int_\Theta l(\cdot, p^T)\, d\pi} \quad \text{for} \quad p^T \in P^T \quad \text{and} \quad E \in \mathscr{B}(\Theta).$$

Then $\gamma(p^T)$ is the a posteriori probability measure on Θ given p^T. An estimation procedure can then be defined by letting $e(p^T)$ be the mean of $\gamma(p^T)$ for each $p^T \in P^T$. Condition (a) implies that for each $E \in \mathscr{B}(\Theta)$ the function $\gamma(\cdot)(E)$ is continuous, which insures that $\gamma(\cdot)$ is continuous. Therefore e is continuous.

The a posteriori probability measure $\gamma(p^T)$ can also be used to revise the subjective probability measure on P^T. For $p^T \in P^T$, let $\delta(p^T) \in \mathscr{M}(P^T)$ be given by $\delta(p^T)(E) = \int_\Theta \psi(\cdot)(E)\, d\gamma(p^T)$, for $E \in \mathscr{B}(P^T)$. The continuity of $\delta(\cdot)$ is implied by the continuity of ψ and $\gamma(\cdot)$. Then $\delta(p^T)$ is an a posteriori probability measure on P^T given by p^T.

As the above examples indicate, the most restrictive feature of Definition 2.7 is the condition that e be nonempty valued. This condition will not generally be satisfied by nontrivial estimation procedures unless $T < \infty$. The requirement that e be convex-valued is also restrictive, but can often be finessed, as in the estimation of Σ described above. The estimation procedure in the second example also illustrates the limitations imposed by the condition that e be compact-valued.

The requirement that e be u.s.c. rules out estimation procedures which are based on hypothesis tests. Such procedures are commonly used to determine the order of an autoregressive process. For example, let e_k be the estimation procedure described in the second example when the order of the process is k, and let e_{k-1} be the estimation procedure when the order is $k-1$. Then let the procedure e be defined by setting $e(p^T) = e_k(p^T)$ if the zero matrix is outside the confidence region for θ_k at the 5% significance level and $e(p^T) = e_{k-1}(p^T)$ otherwise. The e is not u.s.c.

3. Equilibrium

3.1. *Notation.* The ith consumer is characterized by the consumption sets $\{C_t^i\}_{t=-1}^T$, the input sets $\{A_t^i\}_{t=-1}^T$ (recall that $A_t^i = \mathbb{R}_+^L$ for all i, t), the production function f^i, the endowments $\{\omega_t^i\}_{t=-1}^T$, the budget correspondences $\{b_t^i\}_{t=-1}^T$, a von Neuman-Morgenstern utility index u^i, an

expectations set Θ^i, a function ψ^i, and an estimation procedure e^i. If X^i is a set pertaining to the ith consumer then the tth product of X^i will be written $X^{(t)t}$. Also, let $X = \prod_{i=1}^{N} X^i$. Thus $\Theta = \prod_{i=1}^{N} \Theta^i$, $C_t = \prod_{i=1}^{N} C_t^i$, $C^t = \prod_{s=1}^{t} \prod_{i=1}^{N} C_s^i$; etc.

3.2. *Equilibrium.* A *sequence of temporary equilibria*, or more briefly, an *equilibrium*, is a vector (θ, p^T, a^T, c^T) of expectations, prices, inputs, and consumptions such that

(i) $(a_t^i, c_t^i) \in \xi_t^i(\theta^i, p^t, a_{t-1}^i, c^{(i)t-1})$ for all i and all t;

(ii) $\sum_{i=1}^{N} (a_t^i + c_t^i - f^i(a_{t-1}^i) - \omega_t^i) \leqslant 0$ for all t; and

(iii) $p_t \sum_{i=1}^{N} (a_t^i + c_t^i - f^i(a_{t-1}^i) - \omega_t^i) = 0$ for all t.

If, in addition,

(iv) $\theta^i \in e^i(p^T)$ for all i,

then (θ, p^T, a^T, c^T) is said to be an *equilibrium with self-fulfilling expectations.*

3.3. DEFINITIONS. The tth period aggregate endowment, ω_t, is defined by $\omega_t = \sum_{i=1}^{N} \omega_t^i$ for each t. Let $Q_1 = \{q_1 \in R_+^L : q_1 \leqslant \omega_1\}$, and let $Q_2 = \{q_2 \in R_+^L : q_2 \leqslant \omega_2 + \sum_{i=1}^{N} f^i(a_1^i)$ for some $a \in A$, such that $\sum_{i=1}^{N} a_1^i \in Q_1\}$. Proceeding inductively, let $Q_t = \{q_t \in R_+^L : q_t \leqslant \omega_t + \sum_{i=1}^{N} f^i(a_{t-1}^i)$ for some $a_{t-1} \in A_{t-1}$ such that $\sum_{i=1}^{N} a_{t-1}^i \in Q_{t-1}\}$. The set Q_t is the tth period attainable set.

3.4. ASSUMPTIONS. The following assumptions are made for all i and all t.

(A.1) C_t^i is a closed convex subset of R_+^L and $0 \in C_t^i$;

(A.2) $\omega_t^i \geqslant 0$ and $\omega_t^i \neq 0$;

(A.3) f^i is continuous, concave, nonnegative, and $f^i(0) = 0$;

(A.4) ψ_t^i has a continuous version;

(A.5) u^i is bounded, continuous, concave, strictly increasing, and $u^i(0) = 0$;

(A.6) $\omega_t \gg 0$; and

(A.7) $Q_t \subset \operatorname{int} C_t^i$, where int denotes interior relative to R_+^L.

3.5. LEMMA. *For all i and all t, ξ_t^i is u.s.c. and convex valued.*

Proof. The result follows from assumptions A.1–A.5 and [6, Theorems 3.2, 3.5].

3.6. *Remarks.* Assumptions A.1–A.5 are more or less standard. Assumption A.6 implies that production does not create commodities that would not otherwise be present in the economy. The necessity of this assumption arises from the fact that the expected profitability of a product based on current and past prices may be insufficient to warrant its production. Then in the next period, if the aggregate endowment of that commodity is zero, an equilibrium need not exist. Since preferences are strictly monotone, A.7 implies nonsatiation on the attainable set. It should be noted that since $\hat{Q}_t$ depends on the endowments and production possibilities of all consumers, A.7 entails an interdependence of consumer characteristics. Nonsatiation is often obtained by setting all consumption sets equal to R_+^L. This specification is not adopted here because our strongest result is obtained when consumption sets are compact.

Since self-fulfilling expectations are modeled as a fixed point, existence can be established by a standard argument once the following boundary condition is obtained.

3.7. *Boundary condition.* If $\{\theta_n, p_n^T, a_n^T, c_n^T\}_{n=1}^{\infty} \subset \Theta \times P^T \times A^T \times C^T$ such that

(i) $\theta_n^i \in e^i(p_n^T)$ for all i and all n;

(ii) $(a_{(t)n}^i, c_{(t)n}^i) \in \xi_t^i(\theta_n^i, p_n^t, a_{(t-1)n}^i, c_n^{(i)t-1})$ for all i, all t, and all n;

and

(iii) for all t and all $p_t \in P_t$

$$\limsup_{n\to\infty} p_t\left(\sum_{i=1}^{N}(a_{(t)n}^i + c_{(t)n}^i - f^t(a_{(t-1)n}^i) - \omega_t^i\right) \leqq 0,$$

then for all t, $\mathrm{cl}\{p_{(t)n}\}$ is a compact subset of P_t.

3.8. LEMMA. *If* 3.7 *is satisfied then an equilibrium with self-fulfilling expectations exists.*

Proof. The proof is straightforward and is omitted.

3.9. LEMMA. *Let* $1 \leqslant t \leqslant T$, *let* $1 \leqslant i \leqslant N$, *let*

$$\{\theta_n^i, p_n^t, a_{(t-1)n}^i, c_n^{(i)t-1}\}_{n=1}^{\infty} \subset \Theta^i \times P^t \times A_{t-1}^t \times C^{(i)t-1},$$

and let

$$\{a_{(t)n}^i, c_{(t)n}^i\}_{n=1}^{\infty} \subset A_t \times C_t \text{ such that } (a_{(t)n}^i, c_{(t)n}^i) \in b_t^i(p_{(t)n}, a_{(t-1)n}^i)$$

for all n. Suppose that (suppressing the superscript i for convenience)

(i) $(c_n^{t-1}, c_{(t)n}) \to (c^{t-1}, c_t) \in C_t$;

(ii) *there exists a subset* $E \subset {}^tP$, *a compact subset* $K \subset {}^tC$, *and* $\epsilon > 0$ *such that for all n:* $\psi_t(\theta_n, p_n{}^t)(E) > \epsilon$ *and if* ${}^tp \in E$ *and* $({}^ta, {}^tc, {}^tp) \in h_t(a_{(t)n})$ *then* ${}^tc \in K$; *and*

(iii) *there exists a sequence* $\{\lambda_m\}_{m=1}^{\infty} \subset (0, 1)$ *such that* $\lambda_m \to 1$, *and a sequence* $\{\tilde{c}_{(t)m,n}\}_{m,n=1}^{\infty} \subset C_t$ *such that for all m,* $\lim_{n\to\infty} \tilde{c}_{(t)m,n} = \tilde{c}_t \neq 0$, *and for all m and all n* $(\lambda_m a_{(t)n}, \lambda_m c_{(t)n} + \tilde{c}_{(t)m,n}) \in b_t(p_{(t)n}, a_{(t-1)n})$. *Then for sufficiently large n,* $(a_{(t)n}, c_{(t)n}) \notin \xi_t(\theta_n, p_n{}^t, a_{(t-1)n}, c_n^{t-1})$.

Proof. If $T < \infty$, the case $t = T$ is a straightforward application of the concavity and strict monotonicity of u. Let T be arbitrary and let $t < T$. For each $s > t$ and each n, take a measurable selection from the u.s.c. correspondence $\xi_s(\theta_n, (p_n{}^t, \cdot), \cdot, (c_n^{t-1}, c_{(t)n}, \cdot))$: ${}^tP^s \times A_{s-1} \times {}^tC^{s-1} \to A_s \times C_s$, and let ϕ^n be the future production-consumption plan determined by these selections and $(\theta_n, p_n{}^t, a_n{}^t, c_n{}^t)$. Suppose by way of contradiction that $(a_{(t)n}, c_{(t)n}) \in \xi_t(\theta_n, p_n{}^t, a_{(t-1)n}, c_n^{t-1})$ for all n. Then by [6, Theorem 3.3] ϕ^n maximizes

$$\int_{{}^tP\times{}^tA\times{}^tC} u(c_n{}^t, \cdot)\, d\phi \text{ subject to } \phi \in F_t(\theta_n, p_n{}^t, a_n{}^t) \text{ for each } n.$$

For each m and each n, let $\phi_{n,m}$ be the future production-consumption plan determined by the above selections and $(\theta_n, p_n{}^t, \lambda_m a_{(t)n}, [c_n^{t-1}, \lambda_m c_{(t)n} + \tilde{c}_{(t)m,n}])$. Let δ^n and $\delta_{n,m}$ be the marginal distributions on ${}^tC$ associated with ϕ^n and $\phi_{n,m}$, respectively. Then

$$\begin{aligned}\int_{{}^tP\times{}^tA\times{}^tC} u(c_n^{t-1}, c_{(t)n}, \cdot)\, d\phi^n &= \int u(c_n^{t-1}, c_{(t)n}, \cdot)\, d\delta^n \\ &\geqslant \int u(c_n^{t-1}, \lambda_m c_{(t)n} + \tilde{c}_{(t)m,n}, \cdot)\, d\delta_{m,n} \\ &> \lambda_m \int u(c_n^{t-1}, c_{(t)n} + \tilde{c}_{(t)m,n}, \cdot)\, d\delta^n\end{aligned}$$

for all m and all n, where the last inequality follows from A.3 and A.5.

Since K is compact, there exists $\epsilon'' > 0$, and $N(m) > 0$ for each m, such that

$$u(c_n^{t-1}, c_{(t)n} + \tilde{c}_{(t)m,n}, {}^tc) > u(c_n^{t-1}, c_{(t)n}, {}^tc) + \epsilon''$$

for all ${}^tc \in K$ and all $n \geqslant N(m)$. Since $\delta_n(K) > \epsilon$ for all n,

$$\int_K u(c_n^{t-1}, c_{(t)n} + \tilde{c}_{(t)n,m}, \cdot)\, d\delta_n > \int_K u(c_n^{t-1}, c_{(t)n}, \cdot)\, d\delta_n + \epsilon \cdot \epsilon''$$

for all $n \geqslant N(m)$. Therefore

$$\int_{{}^tC\setminus K} [u(c_n^{t-1}, c_{(t)n}, \cdot) - \lambda_m u(c_n^{t-1}, c_{(t)n} + \tilde{c}_{(t)m,n}, \cdot)]\, d\delta_n$$

$$> -\int_K [u(c_n^{t-1}, c_{(t)n}, \cdot) - \lambda_m u(c_n^{t-1}, c_{(t)n} + \tilde{c}_{(t)m,n}, \cdot)]\, d\delta_n$$

$$> \lambda_m \epsilon'' \cdot \epsilon - (1 - \lambda_m) \int_K u(c_n^{t-1}, c_{(t)n}, \cdot)\, d\delta_n$$

and thus

$$\int_{{}^tC} (1 - \lambda_m)\, u(c_n^{t-1}, c_{(t)n}, \cdot)\, d\delta_n > \lambda_m \epsilon \cdot \epsilon'' \quad \text{for all} \quad n \geqslant N(m).$$

However, since $\lambda_m \to 1$, this contradicts the boundedness of u and the proof is complete.

3.10. *Remarks.* Lemma 3.9 will be used to establish the boundary condition. Roughly speaking, the lemma states that if the consumer's planned future consumption is in a compact set K with probability $\epsilon > 0$ then he will be willing to reduce his current inputs by a small fraction (determined by K and ϵ), reducing planned future consumption by the same fraction, in order to obtain a given increase in current consumption. The intuition behind this result will be further clarified by examples in Sections 3.12 and 3.16.

3.11. THEOREM. *If ${}^tC_i$ is compact for all $1 \leqslant i \leqslant N$ and all $1 < t \leqslant T$, then an equilibrium with self-fulfilling expectations exists.*

Proof. It suffices to show that if $\{\theta_n, p_n{}^T, a_n{}^T, c_n{}^T\}_{n=1}^{\infty} \subset \Theta \times P^T \times A^T \times C^T$ such that 3.7(i)–(iii) are satisfied then $\liminf_{n\to\infty} p_{(ti)n} > 0$ for all t, l. Suppose that to the contrary, $\lim_{n\to\infty} p_{(ti)n} = \bar{p}$ where $\bar{p}_{tl} = 0$ for some $1 \leqslant l \leqslant L$, taking a subsequence if necessary. By 3.7(iii), $\{c_{(t)n}^i\}$ has a cluster point in $\bar{Q}_t \subset \operatorname{int} {}^tC_t{}^i$ for each i. Since $\bar{p}_{tl'} > 0$ for some l' and $\omega_{tl'}^{i'} > 0$ for some i', one can for consumer i' construct a sequence $\{\lambda_m\}$ and a sequence $\{\tilde{c}_{(t)m,n}^{i'}\}$ satisfying 3.9(iii). Letting $E = {}^tP$ and $K = {}^tC$ in 3.9(ii), we obtain from Lemma 3.9 a contradiction to 3.7(i).

3.12. *Remarks.* It should be emphasized that the compactness of consumption sets is not used to bound current demand but rather to insure that if commodity 2 becomes infinitely expensive relative to commodity 1, a consumer endowed with commodity 2 will demand more of commodity 1 than the economy can supply. Suppose that the consumer is storing his endowment in the belief that the rate of exchange will be even more

favorable in the future. If his future consumption set is compact, his planned future consumption of commodity 1 is bounded and thus the expected gain from storing an infinitesimal amount of commodity 2 will ultimately be dominated by the gain from exchanging it for a large amount of commodity 1 in the current period. However, if his future consumption of commodity 1 is not bounded, the expected gain from storage may dominate the gain from current trade whenever the price of commodity 1 is positive. This possibility is illustrated by the following example.

Suppose the consumer has a two period planning horizon, that there are two commodities in each period, and that commodity 1 is perishable and commodity 2 can be stored without cost. Let $C_1 = C_2 = R_+^2$ and let the consumer's utility function u: $C^2 \to R$ be given by

$$u(c_{11}, c_{12}, c_{21}, c_{22}) = \log[c_{11} + c_{12} + 3(c_{21} + c_{22}) + 1].$$

Suppose that the consumer is endowed with one unit of each commodity in each period and that he expects the second period price to be equal to the first period price with probability one. Then in the first period, at any positive price, the consumer will store his endowment of commodity 2 and trade his endowment of commodity 1 for p_{12}/p_{11} units of commodity 2, also for storage. Thus there is no positive first period price at which the consumer's excess demand for commodity 2 is nonpositive.

However, no matter what the outcome of the first period, the only second period equilibrium price for this one consumer economy is the price $(p_{21}, p_{22}) = (\frac{1}{2}, \frac{1}{2})$. Thus the consumer's expected second period price is unrelated to the eventual second period price. It is thus natural to ask whether there is some relationship between expectations and outcomes, other than perfect foresight, which would prevent the kind of difficulties illustrated above. Theorem 3.14 below gives one answer to this question.

3.13. THEOREM. *Suppose that for each i and each $t < T$ there is some $\epsilon > 0$ and a compact set $E \subset {}^tP$ such that $\psi_t{}^i({}^t\theta^i, {}^tp^i)(E) > \epsilon$ for all $p^T \in P^T$ and all ${}^t\theta^i \in e^i(p^T)$. Then an equilibrium with self-fulfilling expectations exists.*

Proof. Let $\{\theta_n, p_n{}^T, a_n{}^T, c_n{}^T\}_{n=1}^{\infty} \subset \Theta \times P^T \times A^T \times C^T$ such that 3.7(i)–(iii) are satisfied. Let $t < T$ and $1 \leqslant i \leqslant N$ and let $K = \text{cl} \bigcup_{n=1}^{\infty} \{{}^tc^i$: for some ${}^ta^i$ and some ${}^tp \in E$, $({}^tp, {}^ta^i, {}^tc^i) \in h_t{}^i(a^i_{(t)n})\}$, where E is given in the hypothesis. Then K is compact and the proof can be completed by aplying Lemma 3.9.

3.14. THEOREM. *Suppose that $T < \infty$ and that for each $1 \leqslant i \leqslant N$ and each $1 \leqslant t \leqslant T$, and each ${}^tp \in {}^tP$ there exists a number $\epsilon({}^tp) > 0$*

and a compact neighborhood of ${}^tp$, $E({}^tp) \subset {}^tP$ such that for all $p^t \in P^t$ and all $\theta^t \in e^t({}^tp, p^t)$, $\psi_t{}^t(\theta^t, p^t)(\text{int } E({}^tp)) > \epsilon({}^tp)$. Then an equilibrium with self-fulfilling expectations exists.

Proof. Let $\{\theta_n, p_n{}^T, a_n{}^T, c_n{}^T\}_{n=1}^{\infty} \subset \Theta \times P^T \times A^T \times C^T$ such that 3.7(i)–(iii) are satisfied. Then a standard argument shows that $\text{cl}\{p_{(T)n}\}$ is a compact subset of P_T. Suppose by way of contradiction that $p_{(T-1)n} \to p_{T-1} \in \text{cl } P_{T-1} \backslash P_{T-1}$. Taking a subsequence if necessary, let $p_{(T)n} \to p_T$. By the Portmanteau Theorem $\psi_{T-1}^t(\theta_n{}^t, p_n^{T-1})(E(p^T)) > \epsilon(p_t)/2$ for all n sufficiently large. As in the proof of 3.13 one shows that $\text{cl}\{p_{(T-1)n}\}$ is a compact subset of P_{T-1}, and the proof can be completed inductively.

3.15. COROLLARY. *Suppose that $T < \infty$ and that for each $1 \leqslant i \leqslant N$, e^i is the estimation procedure which assigns to each $p^T \in P^T$ the empirical distribution of p^T. Then an equilibrium with self-fulfilling expectations exists.*

3.16. *Remarks.* Theorems 3.13 and 3.14 depend crucially on the boundedness of the utility functions $\{u^i\}_{i=1}^N$. In the example in 3.12, suppose that the consumer's utility function is given by

$$u(c_{11}, c_{12}, c_{21}, c_{22}) = c_{11} + c_{12} + 3(c_{21} + c_{22}).$$

Suppose also that his estimation procedure assigns to each price sequence its empirical distribution. As before, the only second period equilibrium is $p_2{}^0 = (\frac{1}{2}, \frac{1}{2})$. Therefore for any $p_1 \in P_1$, if $\theta^i = e^i(p_1, p_2{}^0)$, then $\psi_1(\theta^i, p_1)$ is the distribution which assigns probability $\frac{1}{2}$ to the event $\{p_2 = p_1\}$ and probability $\frac{1}{2}$ to the event $\{p_2 = p_2{}^0\}$. Then for any $p_1 \in P_1$, the marginal expected utility of trading the first period endowment of commodity 1 for commodity 2 and storing the latter until next period is

$$\frac{1}{2}\left[3\left(\frac{p_{11}}{p_{12}} \cdot \frac{p_{12}}{p_{11}}\right)\right] + \frac{1}{2}\left[3\left(\frac{p_{11}}{p_{12}} \cdot \frac{p_{22}^0}{p_{21}^0}\right)\right] = \frac{3}{2} + \frac{3}{2}\frac{p_{11}}{p_{12}}.$$

The marginal expected utulity of consuming commodity 1 in the first period is $\frac{1}{2} + \frac{1}{2} = 1$, so the consumer will always prefer to trade commodity 1 for commodity 2 in the first period. Thus an equilibrium with self-fulfilling expectations does not exist in this case. This is so because although the marginal utility of trading commodity 1 for commodity 2 in the event $\{p_2 = p_2{}^0\}$ goes to zero as p_{11} goes to zero, the marginal utility in the event $\{p_2 = p_1\}$ remains constant at 3 as p_{11} goes to zero, even though the planned future consumption of commodity 2 in this event diverges to infinity. If the consumer's utility function is

$$u(c_{11}, c_{12}, c_{21}, c_{22}) = \log[c_{11} + c_{12} + 3(c_{21} + c_{22}) + 1],$$

then the marginal utility of trading commodity 1 for commodity 2 and storing the latter in the event $\{p_2 = p_1\}$ is

$$3\left(\frac{p_{11}}{p_{12}} \cdot \frac{p_{12}}{p_{11}}\right)\left\{c_{11} + c_{12} + 3\left[1 + \frac{p_{12}}{p_{11}}(1 + a)\right] + 1\right\}^{-1}$$

whenever $p_{11} \leqslant p_{12}$, where a is the amount of commodity 2 stored in the first period. Thus the marginal utility in this event goes to zero with p_{11}. The log transformation though monotonic has the effect of increasing the consumer's risk aversion and makes the certainty of an arbitrarily large current consumption of commodity 1 more desirable than the uncertain prospect of an arbitrarily large future consumption of commodity 1.

It is well known that sequences of temporary equilibria need not be ex post Pareto optimal, and the following example of a two period, two consumer economy indicates that the fulfillment of expectations does not eliminate this inefficiency. As before, there are two commodities in each period, commodity 2 can be stored without cost from the first period to the second, and commodity 1 is perishable. Suppose that each consumer's estimation procedure assigns to each price sequence its emprirical distribution. Let the endowment of the first consumer be given by $(\omega_1^1, \omega_2^1) = ((0, 1), (0, 1))$, let $C_1^1 = C_2^1 = R_+^2$, and let $u^1: C_1^1 \times C_2^1 \to R$ be given by

$$u^1(c_{11}^1, c_{12}^1, c_{21}^1, c_{22}^1) = (3/4)\, c_{11}^1 + (5/4)\, c_{12}^1 + c_{21}^1 + c_{22}^1.$$

Let a^1 denote the amount of commodity 2 stored by the first consumer in the first period. Then in the second period the first consumer's problem is to maximize $c_{21}^1 + c_{22}^1$ subject to $p_{21}c_{21}^1 + p_{22}c_{22}^1 \leqslant p_{22}(1 + a^1)$. The consumer's second period demand correspondence is then given by

$$\begin{aligned}
\xi_2^1(p_{21}, p_{22}, a^1) &= (0, (1 + a^1)) && \text{if } p_{21} > p_{22} \\
&= [\ ] && \text{if } p_{21} = p_{22} \\
&= \left(\frac{p_{22}}{p_{21}}(1 + a^1), 0\right) && \text{if } p_{21} < p_{22},
\end{aligned}$$

where [ ] denotes the interval $[(0, (1 + a^1)), (p_{22}/p_{21}(1 + a^1), 0)]$.

Let the endowment of the second consumer be given by $(\omega_1^2, \omega_2^2) = ((1, 0), (1, 0))$, let $C_1^1 = C_2^1 = R_+^2$ and let $u^2: C_1^2 \times C_2^2 \to R$ be given by

$$u^2(c_{11}^2, c_{12}^2, c_{21}^2, c_{22}^2) = c_{11}^2 + c_{12}^2 + (\tfrac{1}{2})\, c_{21}^2 + c_{22}^2.$$

Let a^2 denote the amount of commodity 2 stored by the second consumer

in the first period. Then the second consumer's second period demand correspondence is given by

$$\begin{aligned}\xi_2^2(p_{21}, p_{22}, a^2) &= (0, (p_{21} + p_{22}a^2)/p_{22}) && \text{if } 2p_{21} > p_{22}\\ &= [\ ] && \text{if } 2p_{21} = p_{22}\\ &= ((p_{21} + p_{22}a^2)/p_{21}, 0) && \text{if } 2p_{21} < p_{22}.\end{aligned}$$

A triple $(p_2, c_2^1, c_2^2) \in P_2 \times C_2^1 \times C_2^2$ will be a second period equilibrium if and only if $(c_{21}^1, c_{22}^1) + (c_{21}^2, c_{22}^2) = (1, 1 + a^1 + a^2)$, $(c_{21}^1, c_{22}^1) \in \xi_2^1(p_{21}, p_{22}, a^1)$, and $(c_{21}^2, c_{22}^2) \in \xi_2^2(p_{21}, p_{22}, a^2)$. It can easily be shown that the unique second period equilibrium is $(\bar{p}_2, \bar{c}_2^1, \bar{c}_2^2) = ((\frac{1}{2}, \frac{1}{2}), (1, a^1), (0, 1 + a^2))$. In this example, the second period equilibrium price is independent of previous investment although this is clearly not the case in general.

For $p_1 \in P_1$, let $\theta^1 = e^1(p_1, \bar{p}_2)$. Then the first consumer's expected utility of c_{11}^1, c_{12}^1, a^1 is given by

$$\begin{aligned}v_1^1(c_{11}^1, c_{12}^1, \phi^1(\theta^1, a^1)) = \frac{1}{2}\left\{\left(\frac{3}{4}\right)c_{11}^1 + \left(\frac{5}{4}\right)c_{12}^1 + \frac{p_{12}(1 + a^1)}{\min\{p_{11}, p_{12}\}}\right\}\\ + \frac{1}{2}\left\{\left(\frac{3}{4}\right)c_{11}^1 + \left(\frac{5}{4}\right)c_{12}^1 + (1 + a)\right\},\end{aligned}$$

where $\phi^1(\theta^1, a^1)$ is the future strategy determined by θ^1, a^1, and the consumer's second period demand correspondence. The consumer's first period problem is to maximize $v_1^1(c_{11}^1, c_{12}^1, \phi^1(\theta^1, a^1))$ subject to $p_{11}c_{11}^1 + p_{12}(c_{12}^1 + a^1) \leqslant p_{12}$. Let x^1 denote the first consumer's total first period demand for commodity 2, that is $x^1 = c_{12}^1 + a^1$. Then the consumer will allocate x^1 to current consumption and storage according to the correspondence $(x^1, p_1) \Rightarrow (c_{12}^1, a^1)$ given by

$$\begin{aligned}(c_{12}^1, a^1) = (x^1, 0) \quad &\text{if } \frac{5}{4} > \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[p_{11}, p_{12}]}\\ = [\ ] \quad &\text{if } \frac{5}{4} = \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[p_{11}, p_{12}]}\\ = (0, x^1) \quad &\text{if } \frac{5}{4} < \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[p_{11}, p_{12}]}.\end{aligned}$$

The consumer's total demand for commodities 1 and 2 in the first period is given by

$$(c_{11}^1, x^1) = \left(\frac{p_{12}}{p_{11}}, 0\right) \quad \text{if } \left(\frac{1}{p_{11}}\right)\left(\frac{3}{4}\right) > \frac{1}{p_{12}} \max\left\{\frac{5}{4}, \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[p_{11}, p_{12}]}\right\}$$

$$= [\,] \quad \text{if } \left(\frac{1}{p_{11}}\right)\left(\frac{3}{4}\right) = \frac{1}{p_{12}} \max\left\{\frac{5}{4}, \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[p_{11}, p_{12}]}\right\}$$

$$= (0, 1) \quad \text{if } \left(\frac{1}{p_{11}}\right)\left(\frac{3}{4}\right) < \frac{1}{p_{12}} \max\left\{\frac{5}{4}, \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[p_{11}, p_{12}]}\right\}.$$

Applying the same derivations for the second consumer, we have

$$(c_{12}^2, a^2) = (x^2, 0) \quad \text{if } 1 > \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[2p_{11}, p_{12}]}$$

$$= [\,] \quad \text{if } 1 = \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[2p_{11}, p_{12}]}$$

$$= (0, x^2) \quad \text{if } 1 < \frac{1}{2} + \left(\frac{1}{2}\right)\frac{p_{12}}{\min[2p_{11}, p_{12}]}$$

$$(c_{11}^2, x^2) = (1, 0) \quad \text{if } \frac{1}{p_{11}} > \frac{1}{p_{12}} \max\left(1, \frac{1}{2} + \frac{1}{2}\frac{p_{12}}{\min[2p_{11}, p_{12}]}\right)$$

$$= [\,] \quad \text{if } \frac{1}{p_{11}} = \frac{1}{p_{12}} \max\left(1, \frac{1}{2} + \frac{1}{2}\frac{p_{12}}{\min[2p_{11}, p_{12}]}\right)$$

$$= \left(0, \frac{p_{11}}{p_{12}}\right) \quad \text{if } \frac{1}{p_{11}} < \frac{1}{p_{12}} \max\left(1, \frac{1}{2} + \frac{1}{2}\frac{p_{12}}{\min[2p_{11}, p_{12}]}\right).$$

Equilibrium in the first period requires that $(c_{11}^1, x^1) + (c_{11}^2, x^2) = (1, 1)$. The set of first period equilibria can be seen to consist of three regions:

(1) $p_{11} \leqslant p_{12} < (\frac{3}{2})\, p_{11}$, $(c_{11}^1, c_{12}^1, a^1) = (0, 1, 0)$,

$(c_{11}^2, c_{12}^2, a^2) = (1, 0, 0)$;

(2) $p_{12} = (\frac{3}{2})\, p_{11}$, $(c_{11}^1, c_{12}^1, a^1) \in [(0, 1, 0), (0, 0, 1)]$,

$(c_{11}^2, c_{12}^2, a^2) = (1, 0, 0)$; and

(3) $(\frac{3}{2})\, p_{11} < p_{12} \leqslant 2p_{11}$, $(c_{11}^1, c_{12}^1, a^1) = (0, 0, 1)$,

$(c_{11}^2, c_{12}^2, a^2) = (1, 0, 0)$.

Let $p_{12} = p_{11}$. Then the equilibrium consumption sequence $((c_1^1, c_2^1), (c_1^2, c_2^2))$ corresponding to the equilibrium price sequence $(p_1, \bar{p}_2)$ is $(c_1^1, c_2^1) = ((0, 1), (1, 0))$, $(c_1^2, c_2^2) = ((1, 0), (0, 1))$. The first consumer's utility of this allocation is $u^1((0, 1), (1, 0)) = 2\frac{1}{2}$ and the second

consumer's utility is $u^2((0, 1), (0, 1)) = 2$. If $p_{12} = 2p_{11}$, the equilibrium consumption sequence corresponding to $(p_1, \bar{p}_2)$ is the same for the second consumer but $(c_1^1, c_2^1) = ((0, 0), (1, 1))$ and $u^1((0, 0), (1, 1)) = 2$. Thus we have one equilibrium which is Pareto-inferior to another. Even worse, if $p_{12} = (\frac{3}{2})p_{11}$, we have a continuum of equilibrium allocations which is Pareto ordered. This example indicates that the fulfillment of expectations need not lead to ex post Pareto optimality. Another interesting feature of this example is that if $p_{12} > (\frac{3}{2})p_{11}$, the equilibrium allocation gives the first consumer less utility than does his original endowment.

4. Conclusion

The results presented above demonstrate that in a multiperiod exchange economy with household production, the existence of equilibria is not incompatible with the fulfillment of agent's expectations. In [5], similar results are obtained for an economy with markets for personal debt (short-term bounds). However, the model is essentially a static one, which accounts for our inability to deal satisfactorily with infinite horizon economies.

For example, Corollary 3.15 states that if T is finite there exists a probability measure θ such that if all consumers believe that prices are i.i.d. with the distribution θ, then there is a sequence of equilibrium prices for which θ is the empirical distribution. However, if T is infinite, we cannot even guarantee that a sequence of equilibrium prices has an empirical distribution. Thus the extension of 3.15 to the infinite horizon case is beyond the scope of the methods presented above.

A more dynamic extension of 3.15 is the following. Suppose agents believe prices to be i.i.d and in each period each agent's subjective distribution is the empirical distribution of prices observed up to that time. Does there exist a sequence of equilibria for which the empirical distributions converge? More generally, does there exist a nontrivial estimation procedure which yields convergent estimates? We intend to make these questions a subject of further research.

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Intransitivities in Multidimensional Voting Models and Some Implications for Agenda Control*

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1. INTRODUCTION

The problem of intransitivities in social choice has been the subject of much investigation since Arrow's pioneering work in this area. In the context of social choice over multidimensional policy spaces, Plott [10] has shown the severity of the restrictions which are needed in order to generate an equilibrium policy outcome. Little attention has been paid, however, to the properties of the intransitivities when these strong equilibrium conditions are not met. One exception is Tullock [13], who has argued that Arrow's result is irrelevant in this context because the cycle set will be a fairly small area in the space. But Tullock's argument is not rigorous, and no other work has proceeded any further along this line.

In this paper, we show a rather surprising result, namely, that in the case where all voters evaluate policy in terms of a Euclidian metric, if there is no equilibrium outcome, then the intransitivities extend to the whole policy space in such a way that all points are in the same cycle set. The implications of this result are that it is theoretically possible to design voting procedures which, starting from any given point, will end up at any other point in the space of alternatives, even at Pareto dominated ones. A constructive proof is given below which does precisely this in the Euclidian case. While we only consider the case of Euclidian metrics here, there does not seem to be any reason why the results herein would not extend to more general types of utility functions.

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2. Assumptions and Definitions

We assume a set $N = \{1, 2, \ldots, n\}$ of *voters*, and assume that the *policy space* X is Euclidian m space, i.e., $X = R^m$. For each voter $i \in N$, we assume there is a utility function $U_i : X \to R$ which for present purposes is assumed to be a monotone decreasing function of Euclidian distaince; i.e., for all $i \in N$, $\exists x_i \in R^m$ s.t.

$$U_i(x) = \Phi_i \| x - x_i \|. \tag{2.1}$$

Here $\| \cdot \|$ represents the standard Euclidian norm, and $\Phi_i : R \to R$ is any strictly monotone decreasing function. We use the notation

$$\begin{aligned} x \succ_i y &\Leftrightarrow U_i(x) > U_i(y), \\ x \succcurlyeq_i y &\Leftrightarrow U_i(x) \geq U_i(y). \end{aligned} \tag{2.2}$$

Given the nature of the utility functions it follows that

$$x \succ_i y \Leftrightarrow \| x - x_i \| < \| y - x_i \|. \tag{2.3}$$

We use the notation $| B |$ to represent the number of elements in a set $B \subseteq N$, and use the shorthand $| x \succcurlyeq_i y | = |\{i \in N \mid x \succcurlyeq_i y\}|$. Then, we can define a majority preference relation over R^m as follows. For any x, $y \in R^m$

$$x \succcurlyeq y \Leftrightarrow | x \succcurlyeq_i y | \geq n/2. \tag{2.4}$$

Defining the strong majority relation in the usual way (i.e., $x \succ y \Leftrightarrow x \succcurlyeq y$ and $\sim(y \succcurlyeq x)$), it follows that

$$x \succ y \Leftrightarrow | x \succ_i y | > n/2. \tag{2.5}$$

If all voters evaluate policy in terms of Euclidian distance, the conditions for equilibria can be stated in terms of the existence of a total median. We develop this formally:

For any $y \in R^m$, $c \in R$ we can define a hyperplane as follows:

$$H_{y,c} = \{x \mid x' \cdot y = c\}. \tag{2.6}$$

This partitions R^n into three sets, $H_{y,c}$, $H^+_{y,c}$, and $H^-_{y,c}$, where

$$\begin{aligned} H^+_{y,c} &= \{x \mid x' \cdot y > c\}, \\ H^-_{y,c} &= \{x \mid x' \cdot y < c\}. \end{aligned} \tag{2.7}$$

Now, for any $S \subseteq R^m$, we write $| S | = |\{i \mid x_i \in S\}|$. Then $H_{y,c}$ is said to be

a *median hyperplane* $\Leftrightarrow |H^+_{y,c}| \leqslant n/2$ and $|H^-_{y,c}| \leqslant n/2$. We let $\mathbf{M}$ be be the set of median hyperplanes. It is proved in [6] that for all $y \in R^m$, there is at least one $H_{y,c} \in \mathbf{M}$, although this may not be unique.

DEFINITION 1. *A vector $x^* \in X$ is a total median iff for all $y \in R^m$, $\exists H_{y,c} \in \mathbf{M}$ such that $x^* \in H_{y,c}$. It is a strong total median if in addition, for all y, $H_{y,c} \in \mathbf{M}$ is unique.*

A total median is not necessarily unique, but a strong total median is unique. Notice that whenever there are an odd number of voters, any total median is unique, and is also strong. For even numbers of voters it is possible to have a unique total median which is not strong, as would be the case if four voters were arranged with their ideal points at the corners of a square.

DEFINITION 2. *A vector $x^* \in X$ is a majority Condorcet point iff $x^* \succcurlyeq y$ for all $y \in X$.*

Davis, Degroot, and Hinich [1] prove the following theorem, which establishes necessary and sufficient conditions for the existence of a majority Condorcet point and for transitive social ordering in the Euclidian model.

THEOREM 1. *If all U_i are as in* (2.1), *then $x^* \in X$ is a Condorcet point iff it is a total median. Further, if x^* is a strong total median, the social order is transitive on X, with $x \succcurlyeq y \Leftrightarrow \| x - x^* \| \leqslant \| y - x^* \|$.*

Proof. See [1, Theorems 1 and 4, and Corollary 2]. Q.E.D.

Figure 1 illustrates the necessity of the strong total median to guarantee transitivity of the social ordering. Here there is a unique total median at $x^* = \sum_{i=1}^{4} x_i/4$, but it is not a strong total median. In this example, we have $z \sim x$, $x \sim y$, yet $y \succ z$, violating transitivity of the social ordering.

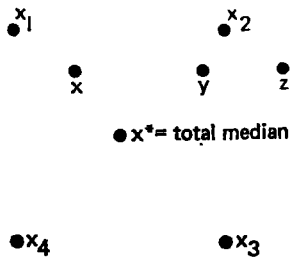


FIGURE 1

With the exception of the above type of problem, generated by even numbers of voters, which gives rise to intransitive indifference, Theorem 1 shows that the existence of Condorcet points and the existence of transitivity of the social ordering both coincide. For odd numbers of voters, the two properties completely coincide. This result is not too surprising, but given the severity of the restrictions needed to guarantee transitivity (namely, existence of a strong total median) it is of considerable interest to explore the nature of the intransitivities when these symmetry conditions are not met.

3. The Extent of Intransitivities

In this section, we show that when transitivity breaks down, it *completely* breaks down, engulfing the whole space in a single cycle set. The slightest deviation from the conditions for a Condorcet point (for example, a slight movement of *one* voter's ideal point) brings about this possibility:

THEOREM 2. *Assume* $m \geqslant 2$, $n \geqslant 3$, *and all voters have utility functions as in* (2.1). *If there is no total median, then for any* $x, y \in X$, *it is possible to find a sequence of alternatives,* $\{\theta_0, \ldots, \theta_N\}$ *with* $\theta_0 = x$, $\theta_N = y$, *such that* $\theta_{i+1} \succ \theta_i$ *for* $0 \leqslant i \leqslant N - 1$.

Proof. For each $y \in R^m$, with $\| y \| = 1$, define $C_y \subseteq R$ to be the set of c satisfying $\{x \mid x \cdot y = c\} \in \mathbf{M}$. It is easily shown that C_y is a closed interval. So, letting $c_y = \inf C_y$, it follows that $c_y \in C_y$, and hence, we define H_y, for any y, as

$$H_y = \{x \mid x' \cdot y = c_y\} \in \mathbf{M}. \tag{3.1}$$

Now it can be shown that a total median exists iff there is an $x^* \in R^m$ with

$$x^* \in \bigcap_{\|y\|=1} H_y^+, \tag{3.2}$$

where $H_y^+ = H_y \cup H_y^+ = \{x \mid x' \cdot y \geqslant c_y\}$.

Since there is no total median, it follows that there is no common solution to the above system of inequalities. By Helley's theorem, it follows that we can find a set of $m + 1$ vectors, $y_0, \ldots, y_m$, with no common solution to

$$x^{*\prime} \cdot y_i \geqslant c_{y_i} = c_i. \tag{3.3}$$

Out of this set, we pick a subset of vectors with no common solution

(without loss of generality assume they are the first $p+1$ vectors, $y_0, \ldots, y_p$), such that for any j, $0 \leqslant j \leqslant p$, there *is* a common solution to

$$x^{*\prime} \cdot y_i \geqslant c_i \qquad \text{for} \quad i \neq j,\ 0 \leqslant i \leqslant p. \tag{3.4}$$

For each $0 \leqslant j \leqslant p$ we set z_j to be a solution to

$$z_j' \cdot y_i = c_i \qquad \text{for all } i \neq j,\ 0 \leqslant i \leqslant p, \tag{3.5}$$

and set $z = (1/(p+1)) \sum_{j=0}^{p} z_j$; we assume without loss of generality that the origin of the vector space is at z (i.e., $z = 0$, the 0 vector). Then it follows that $c_i > 0$, for all $0 \leqslant i \leqslant p$, because

$$0 = z' \cdot y_i = \sum_{j=0}^{p} z_j' \cdot y_i = pc_i + z_i' \cdot y_i < (p+1)\, c_i\,. \tag{3.6}$$

Further, for any $x \in R^m$, note that

$$x' \cdot y_i \leqslant 0 \qquad \text{for some } 0 \leqslant i \leqslant p. \tag{3.7}$$

Otherwise for some large $\alpha \in R$, αx is a common solution for $\alpha x' \cdot y_i \geqslant c_i$ for $0 \leqslant i \leqslant p$, a contradiction. Setting $H_i = H_{y_i}$, Fig. 2 illustrates a possible configuration of the y_i and H_i for the two-dimensional case.

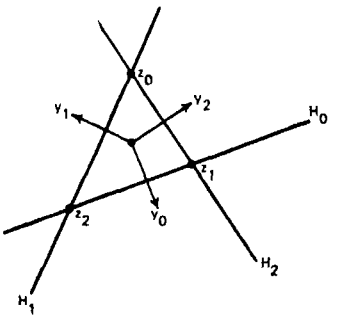


FIGURE 2

Now, for any θ_k, we construct θ_{k+1} as follows: From (3.7), it follows that for some i, $\theta_k' \cdot y_i \leqslant 0$. Pick any such i. Then, we define θ_{k+1} as follows:

$$\theta_{k+1} = \theta_k + [c_i - 2y_i' \cdot \theta_k]\, y_i\,. \tag{3.8}$$

Figure 3 illustrates this for the two-dimensional case. Now,

$$\begin{aligned} \| \theta_k \|^2 &= \|(y_i' \cdot \theta_k)\, y_i + (\theta_k - (y_i' \cdot \theta_k)\, y_i)\|^2 \\ &= \|(y_i' \cdot \theta_k)\, y_i \|^2 + \| \theta_k - (y_i' \cdot \theta_k)\, y_i \|^2 \\ &= (y_i' \cdot \theta_k)^2 + \| \theta_k - (y_i' \cdot \theta_k)\, y_i \|^2, \end{aligned} \tag{3.9}$$

and similarly

$$\| \theta_{k+1} \|^2 = (y_i' \cdot \theta_{k+1})^2 + \| \theta_{k+1} - (y_i' \cdot \theta_{k+1})\, y_i \|^2, \tag{3.10}$$

but, from (3.8),

$$\begin{aligned} &\theta_{k+1} - (y_i' \cdot \theta_{k+1})\, y_i \\ &\quad = \theta_k + [c_i - 2y_i' \cdot \theta_k]\, y_i - y_i' \cdot (\theta_k + (c_i - 2y_i' \cdot \theta_k)\, y_i)\, y_i \\ &\quad = \theta_k + [c_i - 2y_i' \cdot \theta_k]\, y_i - [c_i - 2y_i' \cdot \theta_k]\, y_i - (y_i' \cdot \theta_k)\, y_i \\ &\quad = \theta_k - (y_i' \cdot \theta_k)\, y_i . \end{aligned} \tag{3.11}$$

So, substituting (3.11) in (3.10), we get

$$\begin{aligned} \| \theta_{k+1} \|^2 &= (y_i' \cdot \theta_{k+1})^2 + \| \theta_k - (y_i' \cdot \theta_k)\, y_i \|^2 \\ &= (y_i' \cdot \theta_{k+1})^2 - (y_i' \cdot \theta_k)^2 + \| \theta_k \|^2 \\ &= \| \theta_k \|^2 + \Delta, \end{aligned} \tag{3.12}$$

where $\Delta = (y_i' \cdot \theta_{k+1})^2 - (y_i' \cdot \theta_k)^2$. But, now, using (3.8),

$$\begin{aligned} \Delta &= [y_i' \cdot (\theta_k + (c_i - 2y_i' \cdot \theta_k)\, y_i)]^2 - (y_i' \cdot \theta_k)^2 \\ &= [y_i' \cdot \theta_k + (c_i - 2y_i' \cdot \theta_k)]^2 - (y_i' \cdot \theta_k)^2 \\ &= [c_i - y_i' \cdot \theta_k]^2 - (y_i' \cdot \theta_k)^2 \\ &= c_i^2 - 2c_i y_i' \cdot \theta_k \geqslant c_i^2 \end{aligned} \tag{3.13}$$

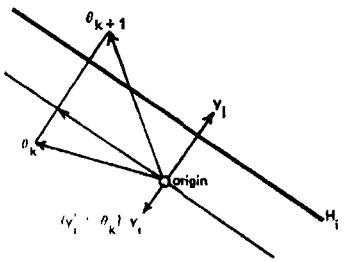


FIGURE 3

since $y_i' \cdot \theta_k \leq 0$ and $c_i > 0$. Hence,

$$\| \theta_{k+1} \|^2 \geq \| \theta_k \|^2 + c_i^2. \tag{3.14}$$

It is obvious, then, by successive application of the above algorithm, we can get θ_j as far from the origin as we want.

Next we prove that $\theta_{k+1} \succ \theta_k$. To see this, note that for any $j \in N$,

$$\begin{aligned}
\theta_{k+1} \succ_j \theta_k &\Leftrightarrow \| x_j - \theta_{k+1} \| < \| x_j - \theta_k \| \\
&\Leftrightarrow x_j \cdot (\theta_{k+1} - \theta_k) > ((\theta_{k+1} + \theta_k)'/2)(\theta_{k+1} - \theta_k) \\
&\Leftrightarrow x_j' \cdot y_i > ((\theta_{k+1} + \theta_k)/2)' \cdot y_i \\
&\Leftrightarrow x_j' \cdot y_i > c_i/2.
\end{aligned}$$

But now, since $H_i = \{x \mid x' \cdot y_i = c_i\} \in \mathbf{M}$ and, by assumption, $\{x \mid x' \cdot y_i = c_i/2\} \notin \mathbf{M}$, it follows that $|\{x \mid x' \cdot y_i > c_i/2\}| > n/2$ hence, $|\theta_{k+1} \succ_j \theta_k| > n/2$ and it follows that $\theta_{k+1} \succ \theta_k$, as we wanted to show.

Thus, we have a sequence $\{\theta_1, \theta_2, \ldots\}$ such that

$$\theta_{k+1} \succ \theta_k \tag{3.15}$$

and such that

$$\| \theta_k \| \to \infty \qquad \text{as} \quad k \to \infty. \tag{3.16}$$

But now, we must show that for any x, y, we can construct a sequence satisfying (3.15) and (3.16), such that $\theta_0 = x$, and $\theta_N = y$. There is no problem with θ_0, but we must show we can get $\theta_N = y$.

To show this, we simply take $B = \{x \mid \| x \| < \rho\}$ to be a sphere of radius ρ satisfying $|B| > n/2$ and $y \in B$. Then, we set $B^* = \{x \mid \| x \| > 3\rho\}$.

It follows, for any $\theta \in B^*$, that $y \succ \theta$, since for any $x_i \in B$, $\| \theta - x_i \| > 2\rho$, and $\| y - x_i \| < 2\rho$.

Hence, we pick a sequence $\{\theta_0, \ldots, \theta_{N-1}\}$ satisfying (3.15) and (3.16) with $\theta_0 = x$, $\theta_{N-1} \in B^*$. Then we set $\theta_N = y$, and from the above argument, $\theta_N \succ \theta_{N-1}$. But then $\{\theta_0, \ldots, \theta_N\}$ is a sequence of proposals satisfying

$$\begin{aligned}
&\theta_0 = x, \\
&\theta_N = y, \\
&\theta_{i+1} \succ \theta_i, \qquad 0 \leq i \leq N - 1,
\end{aligned} \tag{3.17}$$

and we are done. Q.E.D.

In Fig. 4, we illustrate the above algorithm for a simple example with five voters in two dimensions. Here we construct a cycle which arrives at a Pareto dominated point y, from a Pareto optimal point x. Note that the algorithm given is not necessarily the most efficient way of getting from x to y. In particular, as illustrated here, it is seldom necessary to actually

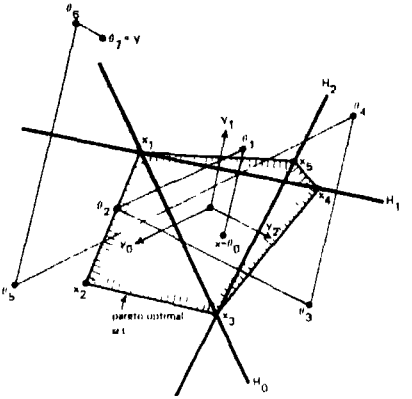


FIGURE 4

get θ_{N-1} in B^*. Frequently one will obtain a θ_k prior to this stage which will beat y.

4. CONCLUSION

The theorem of the previous section shows that, at least for the Euclidian case, either the majority rule social order is completely transitive, or it is involved in a single cycle set. This result is of course dependent on the assumption of Euclidian utility functions. It seems probable, however, that the results would extend to a much larger class of utility functions. In particular it is conjectured that the same type of result would hold if each utility function were separable, i.e., of the form

$$U_i(x) = \sum_{j=1}^{m} U_{ij}(x^j),$$

where x^j is the jth component of the vector x, and where U_{ij} is any real valued function.

In cases where majority rule is not transitive, attempts have been made in the literature to isolate subsets of alternatives which are either more stable or are in some sense normatively better than other points in the space. Some of these attempts have been based on various definitions of "top" cycle sets. Kadane [3] shows the vector of medians is always in such a set in a multidimensional model, and in a more general framework, the idea of top cycles serves as a basis for Schwartz's GOCHA set [11, 12] (called $O(a, s)$ in [11]). In the Euclidean example of this paper, the top

cycle set includes the whole of R^m. If the results here extend to more general utility functions, it would suggest that such generalized equilibrium notions may not be too powerful in infinite alternative spaces.

The existence of a single cycle set implies that it is *possible* for majority rule to wander anywhere in the space of alternatives. The *likelihood* of this occurring probably is strongly dependent on the nature of the institutional mechanisms which generate the agenda. In the context of two-party competition, McKelvey and Ordeshook [6] prove, in the Euclidian case, that mixed strategy solutions are limited to the set of "partial medians," and recently Kramer [5] has shown that in a sequence of elections, where each candidate attempts to maximize plurality against the position of the previous winning candidate, that candidates converge towards the "minimax" set. Both the minimax set and the set of partial medians always exist, and tend to be small and centrally located subsets of the Pareto optimals. For the above institutional mechanisms then, the existence of a single cycle set would be largely irrelevant, and the conclusions of Tullock [13] basically confirmed.

When there is the possibility of *control* of the agenda, either exogenously or by some member of the voting body, the existence of a single cycle set *would* be of considerable importance, as can be illustrated for the Euclidian case. From [8, 10], it follows that the existence of a Condorcet point is equivalent to a type of weak symmetry between the voters. Weak symmetry occurs when it is possible to find a point, x^*, such that voters can be divided into pairs with ideal points in opposite directions from x^*. Thus, if voters i and j are paired, we must have $(x_i - x^*) = -\alpha(x_j - x^*)$ for some $\alpha > 0$, as in Fig. 5. Any remaining voter (at most one) must be at the point x^*. This condition of weak symmetry is equivalent to the existence of a total median at x^*. With an odd number of voters it is equivalent to existence of a strong total median. Given the severity of the above conditions, the chances are very slim that such a point will exist in any particular situation. Even if a strong total median exists, it is possible for any *one* voter, by changing his preferences, to eliminate this total median. Thus, in the illustration of Fig. 5, Voter 8 could misrepresent

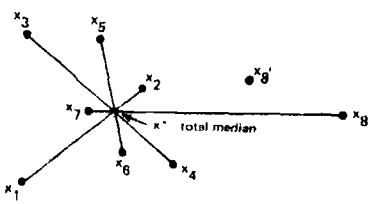


FIGURE 5

his preferences by voting *as if* he had a Euclidian utility function centered at x_8' instead of at x_8. Now there is no total median, and we are in a state of flux as described in Theorem 2.

It follows from the above consideration that if any one voter, say the "Chairman," has complete control over the agenda (in the sense that he can choose, at each stage of the voting, any proposal $\theta_t \in R^n$ to be considered next) that he can construct an agenda which will arrive at any point in space, in particular at his ideal point. Even if there is a majority rule equilibrium, as in Fig. 5, the Chairman (say Voter 8) could construct an agenda which terminates at his actual ideal point (x_8) by first misrepresenting his preferences to create the intransitivities and then applying Theorem 2 to choose the appropriate agenda. This type of manipulation is possible *regardless* of the preferences of the other voters and regardless of whether the "sincere" social ordering is transitive.

The possibility outlined here for controlling the social outcome through control of the agenda depends on several assumptions which are implicit in the above scenario but which should be made more explicit. First, the Chairman must have perfect information of the other voters' preferences in order to design such an agenda. In light of the above analysis, it would obviously not be in the other voters' interests to supply such information. Second, it depends on individuals being able to make fine distinctions between alternatives without becoming indifferent. The algorithm of Theorem 2 depends on finding new alternatives which some pivotal voters just barely prefer to the previous motion. If voters cannot make such fine distinctions, this could impose some limits on the space of intransitivities such that Theorem 2 would no longer hold. Finally, the result depends on other voters voting sincerely and without collusion. If the other voters see what is occurring and know what agenda is being used they might, even without collusion, vote against their preferences at some stage (i.e., vote sophisticatedly) in order to outwit our clever Chairman. Gibbard [2] and Pattanaik [9] show that such consideration cannot be ruled out in general and Kramer [4] analyzes such behavior in a multidimensional context, proving the existence of an equilibrium to the above model if sophistication is taken into account. If collusion occurs, then one must model the above as an n-person game without sidepayments (see [6]), and for all practical purposes, the chairman loses his power since any coalition can ensure any particular alternative in a given agenda by voting appropriately as a bloc at each stage of the agenda. Nevertheless, subject to the qualifications made above, the result of this paper, if it can be generalized, suggests that control of the agenda may be a powerful tool in a "naive" voting body.

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Notes, Comments, and Letters to the Editor

Risk, Stochastic Preference, and the Value of Information: A Comment*

In [2, Theorem 3] Gould proves that, with a utility function $h(\alpha, x)$ linear in the random variable α (x is the control variable), the value of information increases as risk, defined in the Rothschild–Stiglitz sense [4], increases. In this comment, we discuss Gould's result and we give more results on the relationship between risk and the value of information.

The decision maker is assumed to choose x to maximize his expected utility:

$$\int_0^1 h(\alpha, x)\, dF(\alpha, r) \tag{1}$$

where $F(\alpha, r)$ belongs to a family of distribution functions (indexed by r) of the random variable α which, without loss of generality, take their values in $[0, 1]$.

Let $\tilde{x}(r)$ be the optimal decision of problem (1). Following Gould [2], we assume that the decision maker can obtain perfect information, i.e., the true value of α, before taking his decision. Then, let $x^*(\alpha)$ be the optimal decision for each α.

The value of information is then defined as

$$V = \int_0^1 h(\alpha, x^*(\alpha))\, dF(\alpha, r) - \int_0^1 h(\alpha, \tilde{x}(r))\, dF(\alpha, r). \tag{2}$$

When, intuitively, we "think" that the value of information increases as risk increases, I believe that we have in mind the case where the increase in risk makes the expected utility of the agent without information *decrease more* than the expected utility of the agent with information.

But, under Gould's assumptions, the second member of the r.h.s. of (2) is fixed, i.e., an increase in risk does not change the expected utility (without information). The first term of the r.h.s. of (2) is then proved to be increasing. But shouldn't we expect in most economic problems that the expected utility (when information is bought) would decrease when risk increases?

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In the next paragraphs, we explore this conjecture and its counterpart when the first term of the r.h.s. of (2) is kept fixed.

DEFINITION 1. A change in risk (i.e., a change in r) is said to be type-1 compensated for $h(\cdot, \cdot)$ iff

$$\int_0^1 h(\alpha, \tilde{x}(r))\, dF(\alpha, r) \text{ is constant.}$$

DEFINITION 2. The family of distribution functions $\{F(\alpha, r)\}$ satisfies the stochastic dominance property (see [3]) iff

(a) $\int_0^y F_r(\alpha, r)\, d\alpha \gtrless 0$ for all $y \in [0, 1]$,

(b) $F(\alpha, r_1) \neq F(\alpha, r_2)$ if $r_1 \neq r_2$.

THEOREM 1. *If the utility function is concave in* (α, x) *(strictly in* x*) and increasing in* α*, then the value of information decreases as* r *increases for a type-1 compensated change in risk when the stochastic dominance property holds. Moreover, as* r *increases, risk increases in a Diamond-Stiglitz* [1] *sense.*

Proof. By definition of a type-1 compensated change in risk

$$\int_0^1 h(\alpha, \tilde{x}(r))\, dF(\alpha, r) \text{ is constant.}$$

Therefore, we shall consider only $\int_0^1 h(\alpha, x^*(\alpha))\, dF(\alpha, r)$. The derivatives of $h(\alpha, x^*(\alpha))$ are (see [2])

$$dh/d\alpha = h_\alpha > 0,$$

$$d^2h/d\alpha^2 = h_{\alpha\alpha} - (h_{\alpha x}^2/h_{xx}) = (h_{\alpha\alpha}h_{xx} - h_{\alpha x}^2)/h_{xx} \leqslant 0.$$

Thus, h is concave and increasing in α. Differentiating $\int_0^1 h(\alpha, x^*(\alpha))\, dF(\alpha, r)$ with respect to r and integrating twice by part, we obtain

$$\begin{aligned}
&\int_0^1 h(\alpha, x^*(\alpha))\, dF_r(\alpha, r) \\
&\quad = -\int_0^1 (dh/d\alpha)(\alpha, x^*(\alpha))\, F_r(\alpha, r)\, d\alpha \\
&\quad = -[dh/d\alpha \cdot \int_0^\alpha F_r(s, r)\, ds]_0^1 + \int_0^1 (d^2h/d\alpha^2) \int_0^\alpha F_r(s, r)\, ds\, d\alpha \\
&\quad = -(dh/d\alpha) \int_0^1 F_r(s, r)\, ds + \int_0^1 (d^2h/d\alpha^2) \left(\int_0^\alpha F_r(s, r)\, ds \right) d\alpha. \quad (3)
\end{aligned}$$

Since $dh/d\alpha > 0$, $d^2h/d\alpha^2 \leqslant 0$, $\int_0^y F_r(s, r)\, ds \geqslant 0$ for all $y \in [0, 1]$, (3) is negative, giving the first part of the theorem.

We know that a Diamond–Stiglitz increase in risk is here characterized by (see [1])

$$\int_0^y h_\alpha(\alpha, \tilde{x}(r))\, F_r(\alpha, r)\, d\alpha \geqslant 0 \qquad \text{for all} \quad y \in [0, 1], \tag{4}$$

$$\int_0^1 h_\alpha(\alpha, \tilde{x}(r))\, F_r(\alpha, r)\, d\alpha = 0. \tag{5}$$

Equation (5) is easily derived from the definition of a compensated increase in risk: we differentiate $\int_0^1 h(\alpha, \tilde{x}(r))\, dF(\alpha, r) =$ constant with respect to r.

$$d\tilde{x}/dr \int_0^1 h_x(\alpha, \tilde{x}(r))\, dF(\alpha, r) + \int_0^1 h(\alpha, \tilde{x}(r))\, dF_r(\alpha, r) = 0. \tag{6}$$

From the definition of $\tilde{x}(r)$ we know that $\int_0^1 h_x(\alpha, \tilde{x}(r))\, dF(\alpha, r) = 0$.

Then, integrating by parts (6), we obtain

$$-\int_0^1 F_r(\alpha, r)\, h_\alpha(\alpha, \tilde{x}(r))\, d\alpha = 0.$$

Equation (4) can be derived as follows: From

$$\int_0^y F_r(\alpha, r)\, d\alpha \geqslant 0 \qquad \text{for all} \quad y \in [0, 1],$$

we know that F_r is first positive and therefore we have the following picture (Fig. 1) since h_α is positive and decreasing. Consequently, we have

$$\begin{aligned}
&\int_0^y h_\alpha(\alpha, \tilde{x}(r))\, F_r(\alpha, r)\, d\alpha \\
&\quad = \int_0^{y_1} h_\alpha(\alpha, \tilde{x}(r))\, F_r(\alpha, r)\, d\alpha + \int_{y_1}^y h_\alpha(\alpha, \tilde{x}(r))\, F_r(\alpha, r)\, d\alpha \\
&\quad \geqslant h_\alpha(y_1, \tilde{x}(r)) \int_0^{y_1} F_r(\alpha, r)\, d\alpha + h_\alpha(y_1, \tilde{x}(r)) \int_{y_1}^y F_r(\alpha, r)\, d\alpha \\
&\quad = h_\alpha(y_1, \tilde{x}(r)) \int_0^y F_r(\alpha, r)\, d\alpha \geqslant 0,
\end{aligned}$$

and this can be done for any interval. Q.E.D.

Remark. Indeed, Gould [2] needs the linearity in α *and* the strict concavity in x of the utility function to obtain his result. With Theorem 1,

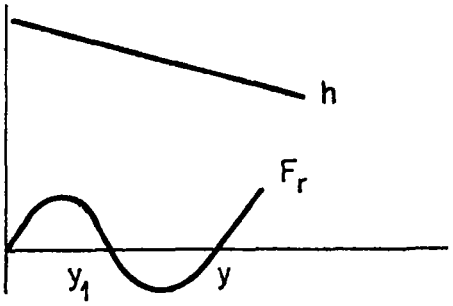


FIGURE 1

we can conclude that in his theorem strict increases in the value of information occur only for cases where $h(\alpha, x)$ is linear in α, strictly concave in x, but *not* concave in both variables. When $h(\alpha, x)$ is also concave in both variables, the value of information is unchanged.

Another natural polar case to consider corresponds to a change in risk which keeps constant the expected utility (with information). Our intuition would then suggest that, in most economic problems, the value of information increases as risk increases.

DEFINITION 3. A change in risk (i.e., a change in r) is said to be type-2 compensated for $h(\cdot, \cdot)$ iff

$$\int_0^1 h(\alpha, x^*(\alpha))\, dF(\alpha, r) \text{ is constant.}$$

THEOREM 2. *If the utility function is concave in α, strictly-concave in x and increasing in α, then the value of information increases as r increases for a type-2 compensated change in risk when the stochastic dominance property holds. Moreover, as r increases, risk increases in a Diamond–Stiglitz sense.*

Proof. By definition of a type-2 compensated change in risk,

$$\int_0^1 h(\alpha, x^*(\alpha))\, dF(\alpha, r) \text{ is constant.}$$

Therefore, we shall consider only $-\int_0^1 h(\alpha, \tilde{x}(r))\, dF(\alpha, r)$. Differentiating with respect to r gives

$$(d/dr)\left[\int_0^1 h(\alpha, \tilde{x}(r))\, dF(\alpha, r)\right]$$
$$= \int_0^1 h(\alpha, \tilde{x}(r))\, dF_r(\alpha, r) + \int_0^1 (\partial h/\partial \tilde{x}) \cdot (d\tilde{x}/dr)\, dF(\alpha, r).$$

But $\int_0^1 h(\alpha, \tilde{x}(r))\, dF_r(\alpha, r) = -\int_0^1 F_r(\alpha, r)\, h_\alpha(\alpha, \tilde{x}(r))\, d\alpha \leqslant 0$ by an argument used in the proof of Theorem 1 ($h_\alpha > 0$, $h_{\alpha\alpha} \leqslant 0$, $\int_0^y F_r(\alpha, r)\, d\alpha \geqslant 0$ for all $y \in [0, 1]$).

Moreover, we have

$$\int_0^1 (\partial h/\partial \tilde{x})(d\tilde{x}/dr)\, dF(\alpha, r) = (d\tilde{x}/dr) \int_0^1 (\partial h/\partial \tilde{x})\, dF(\alpha, r) = 0$$

from the first-order condition.

Thus, $(d/dr)[-\int_0^1 h(\alpha, \tilde{x}(r))\, dF(\alpha, r)] \geqslant 0$. Here, a characterization of an increase in risk in a Diamond–Stiglitz sense, is

$$\int_0^y (dh/d\alpha)(\alpha, x^*(\alpha))\, F_r(\alpha, r)\, d\alpha \geqslant 0 \qquad \text{for all} \quad y \in [0, 1], \tag{7}$$

$$\int_0^1 (dh/d\alpha)(\alpha, x^*(\alpha))\, F_r(\alpha, r)\, d\alpha = 0. \tag{8}$$

Equations (7) and (8) are obtained as in Theorem 1. Q.E.D.

As pointed out in the introduction, the most general (but difficult) case occurs when both the expected values of the agent with information and without information change with a modification in riskiness. From above we have

$$dV/dr = \int_0^1 [h(\alpha, x^*(\alpha)) - h(\alpha, \tilde{x}(r))]\, dF_r(\alpha, r).$$

But, then the sign of dV/dr is defined by the complex interaction of the change in F and the change in $h(\alpha, x^*(\alpha)) - h(\alpha, \tilde{x}(r))$.

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Independence of Nonfeasible Alternatives, and Independence of Nonoptimal Alternatives

1. INTRODUCTION

The intuitive notion of independence of irrelevant alternatives has ambiguous meaning in the literature on collective decision theory. In the context of social choice function the independence condition means that the choice set does change if an alternative which does not belong to this set is deleted. Such condition is used also in Nash's bargaining theory and in the theory of individual decision making under uncertainty [5]. This intuitive meaning we attempt to capture by the condition of independence of nonoptimal alternatives (henceforth referred to as INOA). The second meaning of the independence of irrelevant alternatives condition (IIA) has been used by Arrow [1] in the context of social welfare functions. According to this meaning, the social ranking of feasible alternatives is not changed when individuals change their ranking of nonfeasible alternatives. We term such a condition, when applied to social choice function, independence of nonfeasible alternatives (henceforth referred to as INFA).

The ambiguity surrounding the use of the condition of IIA is partly due to the lack of a formal framework within which its two meanings could be compared. This confusion led Arrow [2] to reverse his opinion about the question of whether rank-voting satisfies the condition of IIA. We show here that the INFA condition captures the meaning of Arrow's formal statement of IIA, while the INOA condition captures the meaning of the rank-voting example which Arrow gave to justify IIA. It should be noted that the distinction between the two conditions of independence has been pointed out by Gibbard [4] who termed them fixity and independence, respectively.

The next section introduces a concept of social choice function upon which the two distinct independence conditions can be imposed. It is followed by an inquiry into the interdependence among the independence conditions and the condition of cheat-proofness. The interest in the latter condition, which has been the subject of recent work on social choice theory (see [3, 6, 7]) is that it lends plausibility to the condition IIA for social welfare functions.

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2. The Mathematical Model

Let A denote any finite set of no less than three alternatives, and let N denote the finite set of individuals in the society ($N = \{1,..., n\}$). Any binary, anti-symmetric, transitive, reflexive, and total relation on A is referred to as a *ranking*. Let Γ_A denote the set of all rankings. Given A, a profile (on A) is, as usual, a vector $(R_1,..., R_n)$, where $R_i (\forall i \in N)$ belongs to Γ_A.

A conditional profile, which is the variable entering our social choice function, is, by definition, a $(n+2)$-list $(B \mid A, R_1,..., R_n)$ where B is included in A, and $R_1,..., R_n$, rank the elements of A, $(\forall i, R_i \in \Gamma_A)$. Let Π denote the set of all conditional profiles.

The problem facing society is to select an alternative in B, given the preferences of members of the community on A. In this note a social choice function (for short SCF), say F, assigns to every conditional profile, an alternative in B. Formally, $F : \Pi \to A$, such that $F(B \mid A, R_1,..., R_n) \in B$, $\forall (B \mid A, R_1,..., R_n) \in \Pi$.

Using the above notations and definitions, we turn now to define the conditions that may be imposed on a SCF, $F : \Pi \to A$.

INFA. A SCF, F, is said to be independent of nonfeasible alternatives if for all pairs of profiles and all subsets of A, $F(B \mid A, R_1,..., R_n) = F(B \mid A, R_1',..., R_n')$, whenever $(\forall i)$ R_i and R_i' are identical on B.

INOA. A SCF, F, is said to be independent of nonoptimal alternatives if for all profiles and all pairs of subsets of A, $F(B \mid A, R_1,..., R_n) = F(B' \mid A, R_1,..., R_n)$, whenever $F(B \mid A, R_1,..., R_n) \in B'$, and $B' \subset B$.

The last condition means that the outcome is not affected by deleting alternatives which were not chosen and which belong to the feasible set.

CP. A SCF, F, is said to be manipulable at the conditional profile $(B \mid A, R_1,..., R_n)$ by an individual j, if there is a ranking R_j' of A such that $F(B \mid A, R_1,..., R_{j-1}, R_j', R_{j+1},..., R_n) P_j F(B \mid A, R_1,..., R_n)$ where P_j denotes the strict preference relation induced by R_j. F is said to be cheat-proof if it is nowhere manipulable.

The conditions of unanimity (weak Pareto) and nondictotorship, in this paper have the usual meaning, Finally, note that the condition of unrestricted domain is implicitly imposed on F, in this model, by defining all of Π to be the domain of F.

3. The Relations among INFA, INOA, and CP

Proposition 1. *A cheat-proof* SCF *is independent of nonfeasible alternatives* (CP *implies* INFA)

This proposition is essentially [6, Lemma 6], within the context of a somewhat different model.

Proof. Suppose that the proposition is false, i.e., there exist two conditional profiles such that

$$F(B \mid A, R_1, \ldots, R_n) = a, F(B \mid A, R_1', \ldots, R_n') = b, b \neq a,$$

and $(\forall i)$, R_i and R_i' are identical on B. Hence, there exists an individual, say j, such that $F(B \mid A, R_1, \ldots, R_j, R_{j+1}', \ldots, R_n') = a$ and $F(B \mid A, R_1, \ldots, R_{j-1}, R_j', \ldots, R_n') = c \neq a$. If cP_ja, then j can manipulate the outcome by switching to R_j'. If aP_jc, then individual j can manipulate the outcome by switching from R_j' to R_j. Thus, F is not cheat-proof—a contradiction. Q.E.D.

Remark 1. If the domain of F is not restricted to anti-symmetric rankings, then F may be both cheat-proof and dependent upon nonfeasible alternatives. For example, let individual 1 be a dictator of SCF, and whenever his top alternative is not unique, a choice among his top alternatives is made according to some arbitrary, imposed anti-symmetric ranking as follows. If there is an alternative in $A - B$ which is indifferent in the ranking of the dictator to another alternative in A, the first alternative in the imposed ranking among the dictator's top alternatives is selected. Otherwise, the second alternative in the imposed ranking among the dictator's top alternatives is selected.

Remark 2. Note that CP does not imply INOA. For example, suppose that F is dictatorial, in a restricted sense, i.e., the dictator varies with the feasible set.

PROPOSITION 2. *A* SCF *which satisfies unanimity, independence of nonfeasible alternatives and independence of nonoptimal alternatives is cheat-proof.* (INFA & IONA & *unanimity imply* CP.)

Proof. We apply Arrow's theorem to alternatives in A. Define a social welfare function on profiles on A, say, $F^*: \Gamma_A{}^N \to \Gamma_A$, in the following way: The social ranking R, associated with a given profile $(R_1, \ldots, R_n)$ on A, ranks x before y if and only if, $F(\{x, y\} \mid A, R_1, \ldots, R_n) = x$, or if $x = y$.

Independence of irrelevant alternatives, as formalized by Arrow for social welfare function, is implied by INFA. Unanimity of F^* is a direct consequence of unanimity of F. The social ranking R is total and reflexive by definition. To prove that R is transitive, suppose, per absurdum, that there are three distinct alternatives, x, y, z, such that

$$F(\{x, y\} \mid A, R_1, \ldots, R_n) = x, F(\{y, z\} \mid A, R_1, \ldots, R_n) = y,$$

and $F(\{x, z\} | A, R_1, \ldots, R_n) = z$. Set, without loss of generality,

$$F(\{x, y, z\} | A, R_1, \ldots, R_n) = x.$$

Then, deletion of y from the feasible set, and application of INOA lead to a contradiction.

By Arrow's theorem, there is a dictator, say i, for F^*. We show that i is also a dictator for F, which implies cheat-proofness. (i is a dictator for F if his top feasible alternative is selected by F.) Let $F(B | A, R_1, \ldots, R_n) = x$. We show that for any $x \neq y \in B$, $xR_i y$. Deleting all the alternatives in B except x and y and application of INOA results in $F(\{x, y\} | A, R_1, \ldots, R_n) = x$, i.e., xRy. But the social ranking is that of the dictator; hence, $xR_i y$.

Q.E.D.

From the proof it is clear that Proposition 2 combines two propositions: (i) INFA & INOA & unanimity imply dictatorship; (ii) dictatorship implies CP.

Remark 3. As before, allowing indifference among distinct alternatives contradicts the implication of the theorem. For example, let F be such that when the dictator, say i, is indifferent among his top feasible alternatives, then F selects one of these alternatives according to the reverse ranking of individual j. If this procedure does not yield a unique solution, then some given a priori ranking is used to resolve the ambiguity. Clearly, though F satisfies unanimity, INOA, and INFA, the jth individual may be in a position to manipulate F.

PROPOSITION 3. *The conditions of* INFA *and* INOA *are independent. (Neither of them implies the other.)*

Proof. The fact that INFA does not imply INOA is well established. For example, see Arrow [2]. For instance, any commonly used voting function, say rank-voting, satisfies INFA but violates INOA.

We now give an example of a SCF, F, which satisfies INOA, unanimity, and nondictatorship, but not INFA. For any conditional profile, let F select the top element in the ranking of B by individual i, if and only if this element is among the top three alternatives in the ranking of A by i. Otherwise F selects the top feasible element in the ranking of individual j ($j \neq i$). Q.E.D.

In the last example F does not satisfy CP. This, in conjunction with the example in Remark 2, can be summarized as

PROPOSITION 4. CP *and* INOA *are independent. (Neither of them imply the other.)*

Note that Propositions 3 and 4 are valid even when the set of SCF's is restricted to those functions which satisfy the unanimity condition.

Finally we state the relations between Arrow's IIA condition, and the INFA and INOA conditions. Let a social welfare function, F^*: $\Gamma_A{}^N \to \Gamma_A$, be given. Define a SCF, F: $\Pi \to A$ as follows: $F(B \mid A, R_1, \ldots, R_n)$ is the $F^*(R_1, \ldots, R_n)$ − maximal element in B. Then

PROPOSITION 5. *The* SCF, *F, defined above satisfies* INOA. *If F* satisfies Arrow's* IIA, *then F satisfies* INFA.

The proof is obvious. Note also that from the proof of Proposition 2, we also get the converse.

PROPOSITION 6. *A* SCF *F, satisfying* INOA, *induces a social welfare function,* F^*: $\Gamma_A{}^N \to \Gamma_A$, *if the* SCF *F satisfies also* INFA, *then the induced F* satisfies Arrow's* IIA.

4. SUMMARY

The relationships among the conditions of INFA, INOA, and CP can be summarized with the use of a simple diagram. Assuming unanimity, the relations are as in Fig. 1.

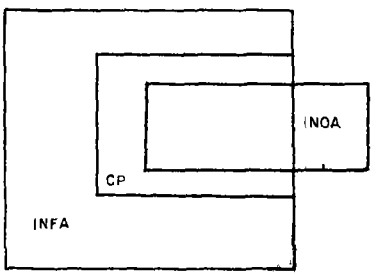


FIGURE 1

Note that the relation between CP and INFA (Proposition 1) is independent of the unanimity requirement. On the other hand, the conclusion of Proposition 2 fails if the condition of unanimity is eliminated. For example, let SCF F select in any feasible set the least preferred alternative of individual i. Clearly, though F satisfies INOA and INFA, it is manipulable by individual i, whenever B contains more than one alternative.

ACKNOWLEDGMENTS

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A Note on Shepherd's Duality Theorem

A simple proof of one of the main results in Shepherd's duality theory is given without reference to the Kuhn–Tucker theory. The proof involves only manipulation of definitions.

The purpose of this short note is to give a simple proof of one of the main results in Shepherd's duality theory. In the note, we follow the notation of Jacobsen [1] and give a direct proof of his main result, Proposition 0, without any reference to the Kuhn–Tucker theory which he uses. In fact, the proof involves little more than the manipulation of the definitions. In the next paragraph, the basic definitions and properties are given. The reader is referred to [1] and [2] for details.

Let X and Y denote the nonnegative orthants in n-space and m-space, respectively. The production structure L is defined to be a map from Y to $A(x)$, the subsets of X. The sets $L(u)$ are assumed to have properties L.1–L.7 of [1]. In particular, we note that $L(u)$ is a closed, convex set. The function g: $Y \times X \to R_+ U\{\infty\} +$ is defined as

$$g(u, x) = \min\{\lambda \mid \lambda \geqslant 0, \lambda x \in L(u)\}.$$

Then $\psi(u, x) = [g(u, x)]^{-1}$ is called the distance function of the production correspondence. The main property of ψ we require is that for $u \geqslant 0$, $\psi(u, \lambda x) = \lambda\psi(u, x)$ for all $\lambda > 0$, i.e., ψ is positively homogeneous of degree one in x. (See [2] for proof.) Finally, define $Q(u, p) = \inf_x \{p \cdot x \mid x \in L(u)\} = \inf_x \{p \cdot x \mid \psi(u \cdot x) \geqslant 1, x \geqslant 0\}$, $\psi^*(u, p) = \inf_{x \geqslant 0} \{p \cdot x - \psi(u, x)\}$ where ψ^* is the conjugate function of ψ with respect to x. Since ψ is positively homogeneous of degree one, it is obvious that $\psi^*(u, p)$ is zero or $-\infty$.

PROPOSITION.

$$\begin{aligned} \psi^*(u, p) &= 0 && \text{if } Q(u, p) \geqslant 1, \\ &= -\infty && \text{if } Q(u, p) < 1. \end{aligned}$$

Proof. Assume $Q(u, p) \geqslant 1$. Suppose $p \cdot x_0 < \psi(u, x_0)$ for some $x_0 \in X$.

Case I. Assume $0 < \psi(u, x_0) < \infty$ and let $\lambda_0 = 1/\psi(u, x_0)$, $x_1 = \lambda_0 x_0$.

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Now $p \cdot x_1 = \lambda_0 p \cdot x_0 < \lambda_0 \psi(u, x_0) = \psi(u, x_1) = 1$. But $Q(u, p) \geqslant 1 \Rightarrow p \cdot x \geqslant 1$ whenever $\psi(u, x) \geqslant 1$, and the point x_1 contradicts this.

Case II. Assume $\psi(u, x_0) = \infty$. Then $p \cdot x_0 < \infty$. If $p \cdot x_0 < 1$, we have a contradiction as above. If $1 \leqslant p \cdot x_0$, let $\lambda_0 = 1/p \cdot x_0 > 0$, $\lambda_1 = \lambda_0 - \epsilon > 0$ for some $\epsilon > 0$ and $x_1 = \lambda' x_0$. Now $p \cdot x_1 = p \cdot \lambda' x_0 < p \cdot \lambda_0 x_0 = 1$ and $\psi(u, x_1) = \lambda' \psi(u, x_0) = \infty$. Hence, we have a contradiction as above.

Case III. Assume $\psi(u, x_0) = 0$. Since $p \cdot x_0 \geqslant 0 \; \forall p, x_0 \in X$, we cannot have $p \cdot x_0 < \psi(u, x_0)$. Therefore, we have shown that $Q(u, p) \geqslant 1$, $p \cdot x \geqslant \psi(u, x) \; \forall x \in X$, $\psi^*(u, p) = 0$. Assume $Q(u, p) < 1$. Then there exists x_0 such that $p \cdot x_0 < 1$ and $\psi(u, x_0) \geqslant 1$.

Therefore, $\psi^*(u, p) \leqslant p \cdot x_0 - \psi(u, x_0) < 0$, which implies $\psi^*(u, p) = -\infty$.

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The Factor–Price Frontier and Duality with Many Primary Factors

1. Samuelson [20] has stated the following theorem without proof:

FACTOR–PRICE FRONTIER THEOREM. *If all goods can be produced over time, at constant returns to scale, with no joint production, out of themselves and out of one or more primary factors, then there exist n quasi-concave montonic factor-price trade-off frontiers relating* (i) *the rate of interest or profit and* (ii) *the real wages (or rents or prices) of those primary factors in terms of any one of the good's price. i.e.,*

$$1 + r = h_i(W_1/P_i\,,\ W_2/P_i\,,\ldots) \qquad (i = 1,\ldots, n), \tag{1}$$

where an increase in all or any subset of $(W_1/P_i\,,\ W_2/P_i\,,\ldots)$ *will lower r and where for fixed r the trade-off between* $(W_1/P_i\,,\ W_2/P_i\,,\ldots)$ *is a convex function.*

The first part of the theorem generalizes the factor–price frontier theorem to models with many primary factors, a result also stated for the case of two primary factors by Burmeister [4, pp. 436–437]. The frontiers

$$1 + r = h_i(W_1/P_i\,,\ W_2/P_i\,,\ldots), \qquad i = 1,\ldots, n,$$

are derived from the equations determining steady-state equilibrium prices as functions of the vector of primary factor prices, w, and the interest rate, r. To prove the existence of the latter functions whatever their properties, one need only use the given primary factor prices to construct a "composite primary commodity," reducing the problem to the one-primary-factor case for which the result is known. Samuelson [19, pp. 409–410] suggests this trick, and it has been used by Arrow and Hahn [2, pp. 237–238], although not for cases with $r > 0$. Complete proofs of this "dynamic" nonsubstitution theorem for one primary factor are found in [7, 15, 16]; proofs of the original "static" nonsubstitution theorem for the $r = 0$ case are contained in [1, 12, 14, 17]. An alternative proof (not using the composite commodity device) when there are many primary factors and when $r \geqslant 0$ is given by Burmeister .

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The last sentence of Samuelson's theorem is much more subtle and, upon reflection, of fundamental importance. Below we provide a very general proof of Samuelson's proposition which does not rely on anything but the existence of cost functions in a nonjoint production economy with constant returns to scale. In particular, we do not require strict quasi-concavity or differentiability of the cost or production functions so that the result is of interest.

2. We have m primary factors and n produced commodities with unit input coefficients

$$b = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & & b_{mn} \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Primary factor prices $w = (w_1, \ldots, w_m) > 0$ and r are given exogenously, and steady-state equilibrium commodity prices are then determined, provided $0 \leqslant r < R$ where R is the maximal profit rate the technology can sustain in steady-state equilibrium. (We assume that the economy is productive implying that $R > 0$.) The reader may interpret our steady-state equilibria relationships as arising from an underlying dynamic technology formulated in either discrete or continuous time.

In all that follows we take a fixed r satisfying $0 \leqslant r < R$. (R may be infinite.) Then for any given $w > 0$, (a^*, b^*) is selected from the set of technologically feasible $\{a, b\}$ to minimize

$$\begin{aligned} p &= wb[I - (r + \delta)a]^{-1} \\ &= wb^*[I - (r + \delta)\, a^*]^{-1} \end{aligned} \tag{1}$$

where δ_i is the depreciation rate for the ith commodity and

$$(r + \delta) = \operatorname{diag}(r + \delta_1, \ldots, r + \delta_n).$$

Note that (1) satisfies the "price = cost" condition

$$p = wb^* + p(r + \delta)\, a^*,$$

and standard arguments assure that

$$[I - (r + \delta)\, a^*]^{-1}$$

exists and is nonnegative for $r \in [0, R)$. Although the cost-minimizing (a^*, b^*) need not be unique, the corresponding p is unique for prescribed

w. For all feasible a, $[I - (r + \delta) a]^{-1}$ exists and is nonnegative provided $r \in [0, R)$.

Again remembering that r is fixed ($0 \leqslant r < R$), for any $w > 0$ we define the matrix

$$F(w) = b^*[I - (r + \delta) a^*]^{-1} = F(w; a, b). \tag{2}$$

Hence from (1), $p = wF(w)$. Now since the technology is free of joint production, one easily can see that, for given $w > 0$, *every* component of the vector p is simultaneously minimized by selecting a feasible production technique (a^*, b^*). Accordingly, we have the following fundamental property of nonjoint production technologies,

$$p = wF(w) \leqslant wF(\alpha) \qquad \text{for all } \alpha \neq w. \tag{3}$$

In particular, select

$$\alpha = \lambda w + (1 - \lambda) \bar{w}, \qquad 0 \leqslant \lambda \leqslant 1, w \neq \bar{w}. \tag{4}$$

Then we have

$$p = wF(w) \leqslant wF(\alpha), \tag{5}$$

$$\bar{p} = \bar{w}F(\bar{w}) \leqslant \bar{w}F(\alpha), \tag{6}$$

and taking a convex combination of (5) and (6), we find that

$$\lambda p + (1 - \lambda) \bar{p} = \lambda wF(w) + (1 - \lambda) \bar{w}F(\bar{w}) \leqslant \lambda wF(\alpha) + (1 - \lambda) \bar{w}F(\alpha). \tag{7}$$

The latter inequality is crucial in the following proof.

3. We now prove the last sentence in Samuelson's theorem.

THEOREM 1 (Generalized Nonsubstitution Theorem). *Consider steady-state equilibrium of the economy described above, and let r be fixed, $0 \leqslant r < R$. The vector function*

$$p = \psi(w; r), \qquad w > 0, \tag{8}$$

is homogeneous of degree one and concave in w. Moreover, if the underlying nonjoint production technology is neoclassical with strictly quasi-concave production and cost functions, then $\psi(w; r)$ will be strictly quasi-concave in w.

Proof.

Let $q = (q_1, ..., q_n)$ be the vector of factor prices for produced commodities, where in steady-state equilibrium $q = p(r + \delta) = wF(w)(r + \delta)$ and where the matrix $(r + \delta) = \text{diag}\,(r + \delta_1, ..., r + \delta_n)$. (Samuelson's theorem is stated for discrete time in the special circulating capital case with $\delta_i = 1$.) The vector of cost functions is written as

$$\phi(q, w) = \phi[p(r + \delta), w] \tag{9}$$

where $\phi(\cdot)$ is nondecreasing in q. Thus, using (7) and dropping the fixed r from ψ, we have

$$\psi(\lambda w + \mu\bar{w}) \geqslant \phi[\lambda wF(w)(r + \delta) + \mu\bar{w}F(\bar{w})(r + \delta), \lambda w + \mu\bar{w}] \tag{10}$$

where $0 \leqslant \lambda \leqslant 1, \mu \equiv 1 - \lambda$, $\alpha \doteq \lambda w + \mu\bar{w}$, and $w \neq \bar{w}$. Now from the concavity of $\phi(\cdot)$, we have

$$\begin{aligned} \text{r.h.s. of (10)} &\geqslant \lambda\phi[wF(w)(r + \delta), w] + \mu\phi[\bar{w}F(\bar{w})(r + \delta), \bar{w}] \\ &= \lambda\psi(w) + \mu\psi(\bar{w}). \end{aligned} \tag{11}$$

Combining (10) and (11) establishes concavity:

$$\psi[\lambda w + (1 - \lambda)\,\bar{w}] \geqslant \lambda\psi(w) + (1 - \lambda)\,\psi(\bar{w}), \qquad 0 \leqslant \lambda \leqslant 1, w \neq \bar{w}. \tag{12}$$

Homogeneity of degree one follows immediately since $\phi(\cdot)$ and $wF(w)$ have the same property. Q.E.D.

4. The vector function (8) is monotonic in r (monotonicity in r for fixed w follows immediately from (2)), and hence defines Samuelson's n factor–price frontiers of the form

$$r = H^i(w_1/p_i, ..., w_m/p_i), \qquad i = 1, ..., n, \tag{13}$$

where $-H^i(\cdot)$ is quasi-concave. These factor–price frontiers have an interesting duality property when $r = 0$ which, for the case of a single primary factor, was first observed by Samuelson [8 pp. 202–203, especially Footnote 2, p. 202]. We now generalize these duality results to the case of many primary factors with supplies $x_1, ..., x_m$. (The results proved here are special cases of general duality relationships that are fundamental in competitive economic systems; see, for example, [10, especially pp. 290–293]). Differentiability of the cost functions is now postulated.

THEOREM 2. *At* $r = 0$

$$\left(\frac{\partial p_1}{\partial w_j},\ldots,\frac{\partial p_n}{\partial w_j}\right)\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} = x_j, \qquad j = 1,\ldots,m,$$
$$\left(\frac{\partial p_1}{\partial r},\ldots,\frac{\partial p_n}{\partial r}\right)\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} = \sum_{i=1}^{n} p_i k_i, \tag{14}$$

or

$$\frac{\partial p}{\partial w}\, c = x,$$
$$\frac{\partial p}{\partial r}\, c = pk \qquad \text{at} \quad r = 0. \tag{15}$$

Moreover, if there is one final output with $c_1 > 0$, $c_i = 0$, $i = 2,\ldots,n$, *then the real factor-price frontier (in terms of commodity one, final output) may be written as*

$$1 = \psi^1(w_1/p_1,\ldots, w_m/p_1; r) \tag{16}$$

with

$$\frac{\partial(w_i/p_1)}{\partial(w_j/p_1)} = -\frac{\psi_j^1}{\psi_i^1} = -\frac{\partial p_1/\partial w_j}{\partial p_1/\partial w_i} = -\frac{x_j}{x_i},$$
$$\frac{\partial(w_i/p_1)}{\partial r} = -\frac{\psi_r^1}{\psi_i^1} = -\frac{\partial p_1/\partial r}{\partial p_1/\partial w_i} = -\frac{\sum_{h=1}^{n} p_h k_h}{x_i} \qquad \text{at} \quad r = 0. \tag{17}$$

Thus, the elasticities

$$\eta_{ij} = -\frac{w_j}{w_i}\frac{\partial(w_i/p_1)}{\partial(w_j/p_1)} = \frac{w_j x_j}{w_i x_i},$$
$$\eta_{ir} = -\frac{r}{w_i}\frac{\partial(w_i/p_1)}{\partial r} = \frac{r\sum_{h=1}^{n} p_h k_h}{w_i x_i} = 0 \tag{18}$$

measure relative factor shares at $r = 0$.

Proof. From (9)

$$\frac{\partial p}{\partial w} = \begin{pmatrix}\frac{\partial p_1}{\partial w_1} & \cdots & \frac{\partial p_n}{\partial w_1}\\ \vdots & & \vdots\\ \frac{\partial p_1}{\partial w_m} & \cdots & \frac{\partial p_n}{\partial w_m}\end{pmatrix} = \frac{\partial q}{\partial w}\,\phi_q + \phi_w \tag{19}$$

where $\phi_q = a$, $\phi_w = b$, while $q = p(r + \delta)$ in steady-state equilibrium. Thus we calculate

$$\partial p/\partial w = b[I - (r + \delta)a]^{-1},$$
$$\partial p/\partial r = pa[I - (r + \delta)a]^{-1}. \tag{20}$$

Define

$$\text{total output vector } z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix},$$

$$\text{capital stock vector } k = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix},$$

$$\text{primary factor vector } x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix},$$

and

$$\text{consumption vector } c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix};$$

the depreciation matrix is $\delta = \text{diag}\,(\delta_1, \ldots, \delta_n)$. In steady-state equilibrium

$$z = \delta k + c, \tag{21}$$

while full employment of capital and primary factors necessitates

$$k = az, \tag{22}$$

and

$$x = bz, \tag{23}$$

respectively. Combining (21), (22), and (23), we find

$$z = [I - \delta a]^{-1} c, \tag{24}$$

$$\begin{aligned} bz &= b[I - \delta a]^{-1} c = x, \\ az &= a[I - \delta a]^{-1} c = k. \end{aligned} \tag{25}$$

Thus (20) and (25) imply (15), and then (17) and (18) are immediate.

Q.E.D.

The above results generalize to golden-rule configurations of exponential golden-age growth paths where all m primary factors grow at the common rate g and where $r = g \geqslant 0$. Likewise other factor–price frontier duality results hold when $r = g$; see [3, 7, 23]. The exclusion of joint production is crucial for equilibria to be located *on* factor–price frontiers. In models which admit joint production, equilibria may lie *on or above* an analogous minimum frontier, although duality results remain valid at $r = g$. Such joint production problems have been studied by Burmeister and Kuga [8].

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Author Index

B

BENVENISTE, LAWRENCE M., 325, 338
BROCK, WILLIAM A., 164
BURMEISTER, EDWIN, 496

C

CAMPBELL, DONALD E., 259
CASS, DAVID, 1, 31

D

DIXIT, AVINASH, 242

F

FIELD, C. A., 494

G

[illegible], R. E., 114
[illegible]M, DANIEL A., [illegible]3

H

HARRIS, Milton, 298

JENNERGREN, L. PETER, 443
JORDAN, J. S., 455

K

KARNI, EDI, 488
KHAN, M. ALI, 273
KOLM, SERGE-CHRISTOPHE, 416

L

LAFFONT, JEAN-JACQUES, [illegible]
LAU, LAWRENCE J., [illegible]
LIPPMAN, STEVEN A., 365

M

MANTEL, ROLF R., 197
MCCALL, JOHN J., 365
MCKELVEY, RICHARD D., 472

N

NAKAYAMA, MIKIO, 347

O

OKUNO, MASAHIRO, 402

P

PETERSON, DAVID W., 443

R

ROCKAFELLAR, R. TYRRELL, 71, 191

S

SAMPSON, ANTHONY A., 315
SCHECHTMAN, JACK, 218
SCHEINKMAN, JOSÉ A., 11, 164
SCHMEIDLER, DAVID, 488
SHELL, KARL, 1, 31

W

WEINTRAUB, E. ROY, 443
WEITZMAN, MARTIN L., 351
WILLIG, ROBERT D., 391

Y

YAARI, MENAHEM E., 20[illegible]

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